Exploring conformal invariance with hierarchical models

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2d Conformal Field Theory, a success story:

Ising model: Construction of the CFT from first principles, i.e., microscopic description, and proof of conformal invariance. Work of Smirnov, Dubedat, Chelkak, Hongler, Izyurov,... Universality, inclusion of next-to-nearest neighbor interactions etc. Work of Giuliani, Mastropietro, Greenblatt, Antinucci,...

Liouville CFT: Work of Duplantier, Sheffield, David, Kupiainen, Rhodes, Vargas,...

multiple SLE based models: Work of Kytölä, Peltola, Flores, Kleban, Wu,...

...

This talk exclusively focuses on CFT in dimension $d \geq 3$, in Euclidean signature, and on simplified toy models which may help in exploring this (mathematical) terra incognita.

The Möbius group of global conformal transformations:

By a theorem of Liouville, for $d \geq 3$, conformal transformations of $\mathbb{R}^d$ reduce to global conformal maps which form a group: the Möbius group $\mathcal{M}(\mathbb{R}^d)$.

Let $\widehat{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\} \simeq \mathbb{S}^d$. One-point compactification identified with sphere via stereographic projection.

Definition 1: The Möbius group $\mathcal{M}(\mathbb{R}^d)$ is the group of bijective transformations of $\widehat{\mathbb{R}^d}$ generated by isometries, dilations and the unit sphere inversion $J(x) = |x|^{-2}x$.

Equivalent definition 2: The Möbius group $\mathcal{M}(\mathbb{R}^d)$ is the invariance group of the absolute cross-ratio

$$CR(x_1, x_2, x_3, x_4) = \frac{|x_1 - x_3|\ |x_2 - x_4|}{|x_1 - x_4|\ |x_2 - x_3|}\ .$$

Namely, it is the group of bijections $\widehat{\mathbb{R}^d} \to \widehat{\mathbb{R}^d}$ which preserve the cross-ratio for all quadruples of distinct points in $\widehat{\mathbb{R}^d}$.

A touristic view of AdS/CFT:

Conformal ball model: $\widehat{\mathbb{R}^d} \simeq \mathbb{S}^d$ seen as boundary of $\mathbb{B}^{d+1}$ with metric $ds = \frac{2|dx|}{1-|x|^2}$.

Half-space model: $\mathbb{R}^d$ seen as boundary of $\mathbb{H}^{d+1} = \mathbb{R}^d \times (0, \infty)$ with metric $ds = \frac{|dx|}{x_{d+1}}$.

Key geometric fact: The is a bijection $f \in \mathcal{M}(\mathbb{R}^d) \leftrightarrow$ hyperbolic isometry of the interior $\mathbb{B}^{d+1}$ or $\mathbb{H}^{d+1}$, the Euclidean AdS space.

The conformal map f simply is the extention by continuity of the hyperbolic isometry to the boundary.

A scalar field $\mathcal{O}$ of scaling dimension Δ in a CFT on $\mathbb{R}^d$ has pointwise correlations which satisfy

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \left(\prod_{i=1}^{n} |J_f(x_i)|^{\frac{\Delta}{d}} \right) \times \langle \mathcal{O}(f(x_1)) \cdots \mathcal{O}(f(x_n)) \rangle$$

for all $f \in \mathcal{M}(\mathbb{R}^d)$ and all collection of distinct points in $\mathbb{R}^d \backslash \{f^{-1}(\infty)\}$. Here, $J_f(x)$ denotes the Jacobian of f at x. The AdS/CFT correspondence, discovered by Maldacena 1997 and made more precise by Gubser, Klebanov, Polyakov and Witten 1998, postulates a relation of the form:

$$\left\langle \; e^{\int_{\mathbb{R}^d} j(x)\mathcal{O}(x) d^d x} \; \right\rangle_{\mathrm{CFT}} = e^{-S[\phi_{\mathrm{ext}}]}$$

where $S[\phi]$ is an action for a field $\phi(x, x_{d+1})$ on AdS space and ϕ_{ext} makes it extremal for a boundary condition $\phi(x, x_{d+1}) \sim (x_{d+1})^{d-\Delta} j(x)$ when $x_{d+1} \to 0$.

AdS/CFT or holographic correspondence not yet known explicitly, i.e., exact $S[\phi]$ still mysterious. However, physicists have been experimenting with toy actions of the form:

$$\int_{\mathbb{R}^d\times(0,\infty)} d^d x \, dx_{d+1} \, \sqrt{\det g_{\mu\nu}} \, \left\{ \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2 + \cdots \right\}$$

where m^2 is related to Δ and is allowed to be (not too) negative.

This gives an expansion for connected CFT correlations in terms of tree-level Feynman diagrams (Witten diagrams). The simplest "Mercedes logo" 3-point Witten diagram reproduces the correct CFT prediction

$$\frac{O(1)}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}$$

for $\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle$ by a calculation of Freedman, Mathur, Matusis and Rastelli 1999.

A (kinda) trivial example: a free CFT

Borel probability measure μ on $S'(\mathbb{R}^d)$ given by centered Gaussian measure with covariance

$$C(f,g) = \int_{\mathbb{R}^d} \frac{d^d\xi}{(2\pi)^d} \frac{\overline{\widehat{f}(\xi)}\widehat{g}(\xi)}{|\xi|^{d-2\Delta}}$$

with $\Delta \in (0, d/2)$.
Formally, given by path integral $\int D\phi \ \ldots \ e^{-S(\phi)}$ with action

$$S(\phi) = \frac{1}{2}\langle \phi, (-\Delta)^{\alpha}\phi\rangle_{L^2}$$

with $\alpha = \frac{d}{2} - \Delta$.
Moments (distributions on $\mathbb{R}^{nd}$) are $L^{1,loc}$ and given by integration against n-point functions $\langle\phi(x_1)\ldots\phi(x_n)\rangle$

$$\langle \phi(x_1)\phi(x_2) \rangle = \frac{\kappa}{|x_1 - x_2|^{2\Delta}}$$

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \langle \phi(x_1)\phi(x_2) \rangle \ \langle \phi(x_3)\phi(x_4) \rangle$$
$$+ \langle \phi(x_1)\phi(x_3) \rangle \ \langle \phi(x_2)\phi(x_4) \rangle + \langle \phi(x_1)\phi(x_4) \rangle \ \langle \phi(x_2)\phi(x_3) \rangle$$

etc. by Isserlis-Wick formula.

Remark 1: The CFT is unitary and satisfies Osterwalder-Schrader positivity for $\Delta \geq \frac{d-2}{2}$.

Remark 2: This is not a trivial example from the point of view of the conformal bootstrap and the AdS/CFT correspondence.

Proof of conformal invariance: Showing

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \left(\prod_{i=1}^{n} |J_f(x_i)|^{\frac{\Delta}{d}}\right) \times \langle \phi(f(x_1))\cdots\phi(f(x_n))\rangle$$

reduces to the $n = 2$ case. For f a Euclidean isometry of $\mathbb{R}^d$ or a dilation: trivial. Only need to check the case of unit sphere inversion $f = J$.

The Jacobian matrix is $|x|^{-2}(\delta_{ij} - 2x_i x_j |x|^{-2})$ and the local rescaling factor is $|J_J(x)|^{\frac{1}{d}} = |x|^{-2}$. Result then follows from elementary identity

$$|J(x) - J(y)| = \frac{|x - y|}{|x|\ |y|}$$

QED.

The good news:

All of the above makes sense for the hierarchical model, i.e., p-adic analogue.
See in particular:

- Melzer, IJMP 1989.
- Lerner, Missarov, LMP 1991.
- Gubser et al. "p-Adic AdS/CFT", CMP 2017.
- Gubser et al. "$O(N)$ and $O(N)$ and $O(N)$", JHEP 2017.

The calculations of the last reference for scaling dimensions of Φ and Φ^2, for $N = 1$ in hierarchical case were made nonperturbatively rigorous in:

"Rigorous quantum field theory functional integrals over the p-adics I: anomalous dimensions", arXiv 2013, by A.A., Ajay Chandra (Imperial College), Gianluca Guadagni (UVa).

The hierarchical or p-adic continuum:

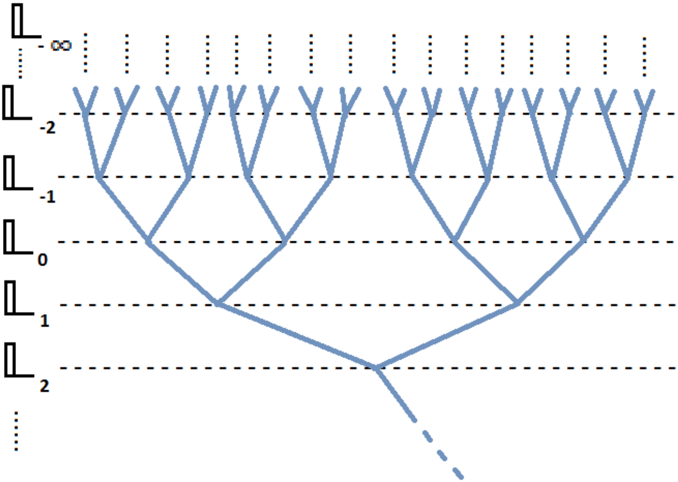
Let p be a fixed integer ≥ 2.
For $k \in \mathbb{Z}$, let $\mathbb{L}_k$ be the set of cubes in $\mathbb{R}^d$ of the form

$$a + p^k \left[-\frac{1}{2}, \frac{1}{2}\right)^d$$

with $a \in p^k\mathbb{Z}^d$. For fixed k, these cubes form a partition of $\mathbb{R}^d$. If p is odd, and one considers all of the cubes in $\cup_{k\in\mathbb{Z}}\mathbb{L}_k$ then these cubes are nested.

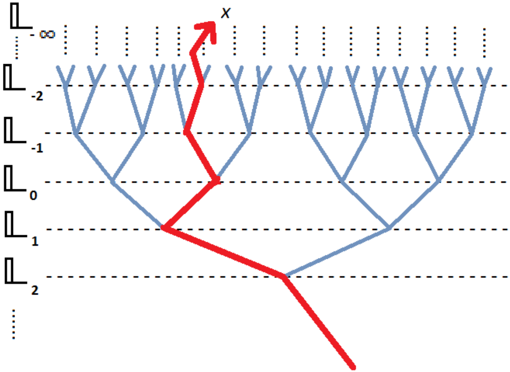
Hence $\mathbb{T} := \cup_{k\in\mathbb{Z}}\mathbb{L}_k$ naturally has the structure of a doubly infinite tree which is organized into layers or generations $\mathbb{L}_k$.

Now forget about $\mathbb{R}^d$ and just remember the tree. No harm in allowing p even now. Secret further (purely esthetic) hypothesis: restrict to p a prime number.



Picture for $d = 1$, $p = 2$

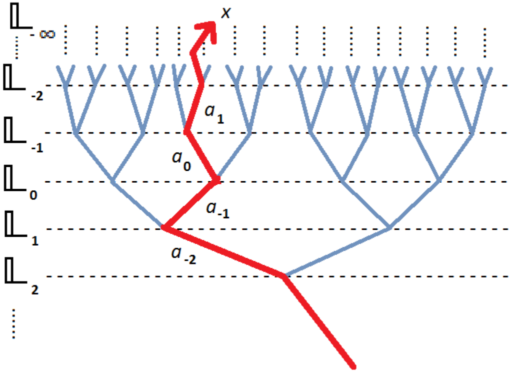
The hierarchical continuum $\mathbb{Q}_p^d$:= leafs at infinity "$\mathbb{L}_{-\infty}$". More precisely, these leafs at infinity are the infinite bottom-up paths in the tree. $\mathbb{T}$, with the graph distance, will play the role of hyperbolic space $\mathbb{H}^{d+1}$ of AdS bulk space.



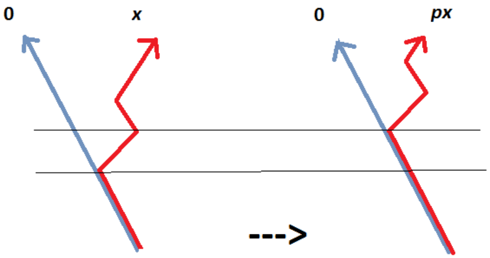
A path representing an element $x \in \mathbb{Q}_p^d$

A point $x =\in \mathbb{Q}_p^d$ is encoded by a sequence $(a_n)_{n\in\mathbb{Z}}$, $a_n \in \{0, 1, \ldots, p-1\}^d$.

a_n represents the local coordinates for a cube of $\mathbb{L}_{-n-1}$ inside a cube of $\mathbb{L}_{-n}$.



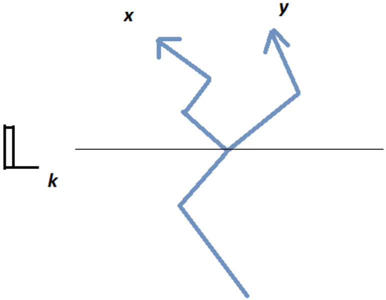
Moreover, rescaling is defined as follows.
If $x = (a_n)_{n\in\mathbb{Z}}$ then $px := (a_{n-1})_{n\in\mathbb{Z}}$, i.e., upward shift.



Likewise $p^{-1}x$ is downward shift, and so on for the definition of $p^k x$, $k \in \mathbb{Z}$.

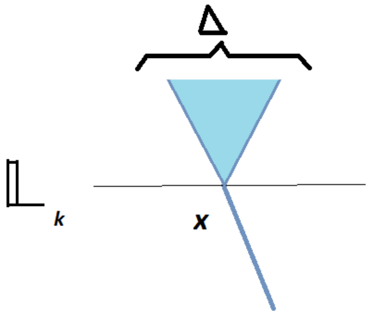
Distance:

If $x, y \in \mathbb{Q}_p^d$, their distance is defined as $|x - y|_p := p^k$ where k is the depth where the two paths merge. Just a notation.



Keep in mind that $|px - py|_p = p^{-1}|x - y|_p$.

Closed balls Δ of radius p^k correspond to the nodes $\mathbf{x} \in \mathbb{L}_k$



Lebesgue measure:

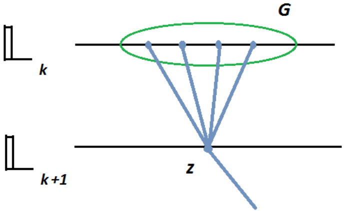
Metric space $\mathbb{Q}_p^d$ $\rightarrow$ Borel σ-algebra $\rightarrow$ Lebesgue (or additive Haar) measure d^dx which gives a volume p^{dk} to closed balls of radius p^k.

The hierarchical unit lattice:
Truncate the tree at level zero and take $\mathbb{L} := \mathbb{L}_0$. Using the identification of nodes with balls, define the hierarchical distance as

$$d(\mathbf{x}, \mathbf{y}) = \inf\{|x - y|_p \mid x \in \mathbf{x},\ y \in \mathbf{y}\} .$$

This is the setting used since Dyson in the statistical mechanics literature on hierarchical models.

The massless Gaussian measure:



To every group of offsprings G of a vertex $\mathbf{z} \in \mathbb{L}_{k+1}$ associate a centered Gaussian random vector $(\zeta_{\mathbf{x}})_{\mathbf{x}\in G}$ with $p^d \times p^d$ covariance matrix made of $1 - p^{-d}$'s on the diagonal and $-p^{-d}$'s everywhere else. We impose that Gaussian vectors corresponding to different layers or different groups are independent. We have $\sum_{\mathbf{x}\in G} \zeta_{\mathbf{x}} = 0$ a.s.

The ancestor function: for $k < k'$, $\mathbf{x} \in \mathbb{L}_k$, let $\mathrm{anc}_{k'}(\mathbf{x})$ denote the ancestor in $\mathbb{L}_{k'}$.
Ditto for $\mathrm{anc}_{k'}(x)$ when $x \in \mathbb{Q}_p^d$.
The massless Gaussian field $\phi(x)$, $x \in \mathbb{Q}_p^d$ of scaling dimention $[\phi]$ is given by

$$\phi(x) = \sum_{k\in\mathbb{Z}} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

$$\langle \phi(x)\phi(y) \rangle = \frac{c}{|x-y|^{2[\phi]}}$$

This is heuristic since ϕ is not well-defined in a pointwise manner. We need random Schwartz(-Bruhat) distributions on $\mathbb{Q}_p^d$.

I will now drop the p from $|\cdot|_p$.

Test functions:

$f : \mathbb{Q}_p^d \to \mathbb{R}$ is smooth if it is locally constant.
Define $S(\mathbb{Q}_p^d)$ as the space of compactly supported smooth functions.
Take locally convex topology generated by the set of all semi-norms on $S(\mathbb{Q}_p^d)$.

Distributions:

$S'(\mathbb{Q}_p^d)$ is the dual space with strong topology (happens to be same as weak-$*$).

$$S(\mathbb{Q}_p^d) \simeq \oplus_{\mathbb{N}} \mathbb{R} \ .$$

Thus

$$S'(\mathbb{Q}_p^d) \simeq \mathbb{R}^{\mathbb{N}}$$

with product topology. $\Omega := S'(\mathbb{Q}_p^d)$ is a Polish space.

The p-adic CFT toy model:

$d = 3$, $[\phi] = \frac{3-\epsilon}{4}$, $L = p^{\ell}$ zooming-out factor for RG

$r \in \mathbb{Z}$ UV cut-off, $r \to -\infty$

$s \in \mathbb{Z}$ IR cut-off, $s \to \infty$

The regularized Gaussian measure μ_{C_r} is the law of

$$\phi_r(x) = \sum_{k=\ell r}^{\infty} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

Sample fields are true fonctions that are locally constant on scale L^r. These measures are scaled copies of each other.

If the law of $\phi(\cdot)$ is μ_{C_0}, then that of $L^{-r[\phi]}\phi(L^r \cdot)$ is μ_{C_r}.

The free CFT first as a warm up:
The centered Gaussian measure $\mu_{C_{-\infty}}$ on $S'(\mathbb{Q}_p^3)$ is the weak limit, when $r \to -\infty$, of the Borel probability measures μ_{C_r} which are supported on smooth fields $\phi(x)$ where smooth means locally constant. The scale of constancy is L^r, analogue of lattice mesh.
The two point function, for $x \neq y \in \mathbb{Q}_p^3$ is

$$\langle \phi(x)\phi(y) \rangle_{\mu_{C_{-\infty}}} = \frac{\kappa}{|x-y|^{2[\phi]}} \ .$$

and higher correlation functions are given by the Isserlis-Wick formula.

Proposition: $\mu_{C_{-\infty}}$ is a CFT.

But, but,. . . , what does conformal invariance mean for these funny hierarchical models???

The p-adic Möbius group :

From old work by Lerner and Missarov (early 1990's, i.e., before AdS/CFT !).

Define the one-point compactification $\widehat{\mathbb{Q}_p^d} = \mathbb{Q}_p^d \cup \{\infty\}$.

Definition 1: The p-adic Möbius group $\mathcal{M}(\mathbb{Q}_p^d)$ is the group of bijective transformations of $\widehat{\mathbb{Q}_p^d}$ generated by (ultrametric) isometries, dilations $x \mapsto p^k x$, $k \in \mathbb{Z}$ and inversion $J(x) = |x|_p^2 x$. Here $|x|_p := |x - 0|_p$ where 0 is a preferred point in $\mathbb{Q}_p^d$, e.g., all-left path in the tree.

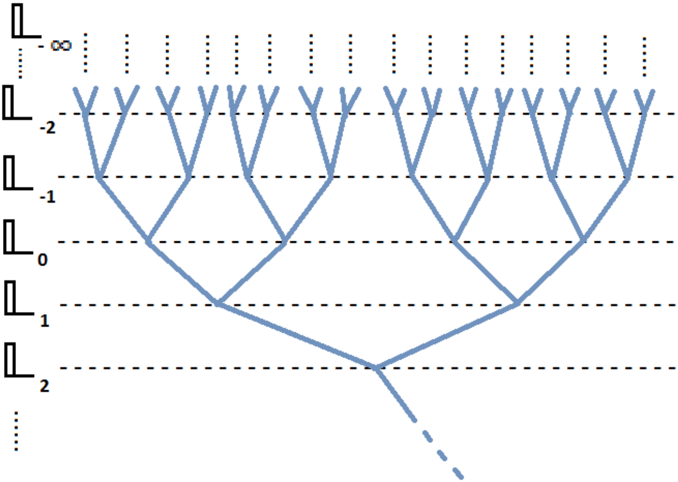
Equivalent definition 2: Can also define the absolute cross-ratio for the ultrametric distance. $\mathcal{M}(\mathbb{Q}_p^d)$ then is the group of transformations of $\widehat{\mathbb{Q}_p^d} = \mathbb{Q}_p^d \cup \{\infty\}$ which preserve this cross-ratio.

Mumford-Manin-Drinfeld Lemma

$$CR(x_1, x_2, x_3, x_4) := \frac{|x_1 - x_3|\ |x_2 - x_4|}{|x_1 - x_4|\ |x_2 - x_3|} = p^{-\delta(x_1 \to x_2; x_3 \to x_4)} ,$$

where $\delta(x_1 \to x_2; x_3 \to x_4)$ is the number of common edges for the two bi-infinite paths $x_1 \to x_2$ and $x_3 \to x_4$, counted positively if orientations agree and negatively otherwise.

From lemma, one can deduce a bijective correpondence:
$f \in \mathcal{M}(\mathbb{Q}_p^3) \leftrightarrow$ hyperbolic isometry of the interior $\mathbb{T}$.



The tree, once again.

Conformal invariance at the level of n-point functions is defined exactly as before but with the local rescaling factor

$$|J_f(x)|^{\frac{1}{d}}$$

now replaced by the Radon-Nikodym derivative formula

$$\left(\frac{d((f^{-1})_* m)}{dm}(x)\right)^{\frac{1}{d}}$$

where m is the previous Lebesgue measure on $\mathbb{Q}_p^d$.

Now on to the proof of conformal invariance for the free measure $\mu_{C_{-\infty}}$ on $S'(\mathbb{Q}_p^d)$.

Proof of conformal invariance: Showing

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \left(\prod_{i=1}^{n}|J_f(x_i)|^{\frac{\Delta}{d}}\right)\times\langle\phi(f(x_1))\cdots\phi(f(x_n))\rangle$$

reduces to the $n=2$ case. For f an isometry of $\mathbb{Q}_p^d$ or a dilation: trivial. Only need to check the case of unit sphere inversion $f=J$.

The local rescaling factor is $|J_J(x)|^{\frac{1}{d}} = |x|^{-2}$. Result then follows from elementary identity

$$|J(x)-J(y)| = \frac{|x-y|}{|x|\ |y|}$$

QED.

Fix the dimensionless parameters g, μ and let $g_r = L^{-(3-4[\phi])r} g$ and $\mu_r = L^{-(3-2[\phi])r} \mu$. Same as strict scaling limit of fixed critical probability measure on unit lattice. Bare/dimensionful couplings g_r, μ_r go to ∞.

Let $\Lambda_s = \overline{B}(0, L^s)$, IR (or volume) cut-off.

Let

$$V_{r,s}(\phi) = \int_{\Lambda_s} \{g_r :\phi^4:_r (x) + \mu_r :\phi^2:_r (x)\} d^3x$$

where $:\phi^k:_r$ is Wick ordering using $d\mu_{C_r}$.
Define the probability measure

$$d\nu_{r,s}(\phi) = \frac{1}{\mathcal{Z}_{r,s}} e^{-V_{r,s}(\phi)} d\mu_{C_r}(\phi) \ .$$

Let $\phi_{r,s}$ be the random distribution in $S'(\mathbb{Q}_p^3)$ sampled according to $\nu_{r,s}$ and define the squared field $N_r[\phi_{r,s}^2]$ which is a deterministic function(al) of $\phi_{r,s}$, with values in $S'(\mathbb{Q}_p^3)$, given by

$$N_r[\phi_{r,s}^2](j) = (Z_2)^r \int_{\mathbb{Q}_p^3} \{Y_2 :\phi_{r,s}^2:_r (x) - Y_0 L^{-2r[\phi]}\}\ j(x)\ d^3x$$

for suitable parameters Z_2, Y_0, Y_2. We also need a Y_1.

Our main result concerns the limit law of the pair $(Y_1\phi_{r,s}, N_r[\phi_{r,s}^2])$ in $S'(\mathbb{Q}_p^3) \times S'(\mathbb{Q}_p^3)$ when $r \to -\infty$, $s \to \infty$ (in any order).

For the precise statement we need the approximate fixed point value

$$\bar{g}_* = \frac{p^\epsilon - 1}{36 L^\epsilon (1 - p^{-3})}\ .$$

Theorems:

Theorem 1: A.A.-Chandra-Guadagni 2013

$\exists\rho > 0$, $\exists L_0$, $\forall L \geq L_0$, $\exists\epsilon_0 > 0$, $\forall\epsilon \in (0,\epsilon_0]$, $\exists[\phi^2] > 2[\phi]$, $\exists$ fonctions $\mu(g)$, $Y_0(g)$, $Y_2(g)$ on $(\bar{g}_* - \rho\epsilon^{\frac{3}{2}}, \bar{g}_* + \rho\epsilon^{\frac{3}{2}})$ such that if one lets $\mu = \mu(g)$, $Y_0 = Y_0(g)$, $Y_2 = Y_2(g)$ and $Z_2 = L^{-([\phi^2]-2[\phi])}$ then the joint law of $(Y_1\phi_{r,s}, N_r[\phi^2_{r,s}])$ converge weakly and in the sense of moments to that of a pair $(\phi, N[\phi^2])$ such that:

1. $\forall k \in \mathbb{Z}$, $(L^{-k[\phi]}\phi(L^k\cdot), L^{-k[\phi^2]}N[\phi^2](L^k\cdot)) \stackrel{d}{=} (\phi, N[\phi^2])$.
2. $\langle\phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} < 0$ i.e., ϕ is non-Gaussian. Here, $\mathbf{1}_{\mathbb{Z}_p^3}$ denotes the indicator function of $\overline{B}(0,1)$.
3. $\langle N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3}), N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} = 1$.
4. $\langle\phi(\mathbf{1}_{\mathbb{Z}_p^3})^2\rangle = 1$.

The mixed correlation functions satisfy, in the sense of distributions,

$$\langle\ \phi(L^{-k}x_1)\cdots\phi(L^{-k}x_n)\ N[\phi^2](L^{-k}y_1)\cdots N[\phi^2](L^{-k}y_m)\ \rangle$$

$$= L^{-(n[\phi]+m[\phi^2])k}\ \langle\ \phi(x_1)\cdots\phi(x_n)\ N[\phi^2](y_1)\cdots N[\phi^2](y_m)\ \rangle$$

For our hierarchical version of the 3D fractional ϕ^4 model we also proved $[\phi^2]-2[\phi]=\frac{1}{3}\epsilon+o(\epsilon)$.

This was predicted by Wilson in "Renormalization of a scalar field theory in strong coupling", PRD 1972.

This is also what is expected for the Euclidean model on $\mathbb{R}^3$.

Not too far, if one boldly extrapolates to $\epsilon = 1$, from the most precise available estimates concerning the short range 3D Ising model: $[\phi^2]-2[\phi]=0.376327\ldots$ (JHEP 2016 by Kos, Poland, Simmons-Duffin and Vichi, using conformal bootstrap).

We also proved the law $\nu_{\phi\times\phi^2}$ of $(\phi, N[\phi^2])$, is independent of g in the interval $(\bar{g}_* - \rho\epsilon^{\frac{3}{2}}, \bar{g}_* + \rho\epsilon^{\frac{3}{2}})$. This also holds if one also adds ϕ^6, ϕ^8,... terms in the potential, with small couplings. We proved strong local universality for a non-Gaussian scaling limit.

Theorem 2: A.A.-Chandra-Guadagni 2013

$\nu_{\phi\times\phi^2}$ is fully scale invariant, i.e., invariant under the action of the scaling group $p^{\mathbb{Z}}$ instead of the subgroup $L^{\mathbb{Z}}$. Moreover, $\mu(g)$ and $[\phi^2]$ are independent of the arbitrary factor L.

The two-point correlations are given in the sense of distributions by

$$\langle\phi(x)\phi(y)\rangle = \frac{c_1}{|x-y|^{2[\phi]}}$$

$$\langle N[\phi^2](x)\ N[\phi^2](y)\rangle = \frac{c_2}{|x-y|^{2[\phi^2]}}$$

Note that $2[\phi^2] = 3 - \frac{1}{3}\epsilon + o(\epsilon) \to$ still $L^{1,\mathrm{loc}}$!

Theorem 3: A.A., May 2015

Use ψ_i to denote the scaling limits ϕ or $N[\phi^2]$. Then, for all mixed correlation $\exists$ a smooth (i.e., locally constant) fonction $\langle \psi_1(z_1) \cdots \psi_n(z_n) \rangle$ on $(\mathbb{Q}_p^3)^n \backslash \mathrm{Diag}$ which is locally integrable (on the big diagonal Diag) and such that

$$\mathbb{E}\ \psi_1(f_1) \cdots \psi_n(f_n) = \int_{(\mathbb{Q}_p^3)^n \backslash \mathrm{Diag}} \langle \psi_1(z_1) \cdots \psi_n(z_n) \rangle\ f_1(z_1) \cdots f_n(z_n)\ d^3 z_1 \cdots d^3 z_n$$

for all test functions $f_1, \ldots, f_n \in S(\mathbb{Q}_p^3)$.

This hinges on showing the BNNFB (basic nearest neighbor factorized bound) of A.A., "A Second-Quantized Kolmogorov-Chentsov Theorem via the Operator Product Expansion", CMP 2020. The BNNFB is

$$| \langle \psi_1(z_1) \cdots \psi_n(z_n) \rangle | \leq O(1) \times \prod_{i=1}^{n} \frac{1}{|z_i - \mathrm{n.n.}|^{[\psi_i]}}$$

when $z_1, \ldots, z_n$ are confined to a compact set.

This follows from the use of the SDRG (space-dependent renormalization group) to derive an explicit representation of pointwise correlations in terms of very close analogues of tree Witten diagrams. Hence, the emergent connection to the AdS/CFT correspondence.

Thank you for your attention.