RENORMALIZATION GROUP TRAJECTORIES BETWEEN TWO FIXED POINTS

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- Main reference: A. A. CMP 07'

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- Outline:
  1. Global dynamics of Wilson's RG
  2. Rigorous results (selection)
  3. The BMS model
  4. Good infinite-volume coordinates
  5. Idea of the proof
  6. Functional analysis, norms
  7. Perspectives

1. Global Dynamics of Wilson's Renormalization Group:

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QFT functional integrals: a challenge for mathematicians

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$$\int_{\mathcal{F}} D\phi \cdots e^{-\int_{\mathbb{R}^d}[\frac{1}{2}(\nabla\phi)^2(x)+\mu\phi(x)^2+g\phi(x)^4]dx}$$

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QFT functional integrals: a challenge for mathematicians

e. g., the ϕ^4 model

$$\int_{\mathcal{F}} D\phi \cdots e^{-\int_{\mathbb{R}^d}[\frac{1}{2}(\nabla\phi)^2(x)+\mu\phi(x)^2+g\phi(x)^4]dx}$$

$\mathcal{F}$ infinite-dimensional space of functions $\mathbb{R}^d \to \mathbb{R}$
$D\phi$ Lebesgue measure on $\mathcal{F}$

- Construction by scaling limit of lattice theories on $(a\mathbb{Z})^d \subset \mathbb{R}^d \Longleftrightarrow$ cut-off $\frac{1}{a}$ on momenta in Fourier space
- rescale to unit lattice
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- approximants to continuum theory = points in the space of all possible unit cut-off theories
- RG = dynamical system on this space
- $d\nu$ measure on random ϕ with $\hat{\phi}(p) = 0$ if $|p| > 1$
- introduce magnification ratio $L > 1$
- split $\phi = \zeta + \phi_{\mathrm{low}}$

$$\zeta \iff L^{-1} < |p| \leq 1$$
$$\phi_{\mathrm{low}} \iff |p| \leq L^{-1}$$

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Local features

Global features ? e. g. heteroclinic trajectories between fixed points

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Local:

- RG exponents for Gaussian fixed point, $\exists$ nontrivial IR fp in "$4-\epsilon$" dimensions, local stable/unstable manifolds, for HM: Bleher-Sinai CMP 73', 75', Collet-Eckmann CMP 77', LNP 78', Gawędzki-Kupiainen CMP 83', JSP 84', Pereira JMP 93'
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Global:

- uniqueness of IR fp in LPA for $3 \leq d < 4$: Lima CMP 87'
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- BMS model, construction of discrete heteroclinic trajectories joining Gaussian UV fp to nontrivial IR fp: A. A. CMP 07'

3. The BMS Model:

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Scalar field $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$Z = \underbrace{\int D\phi \; e^{-\frac{1}{2}\langle \phi, (-\Delta)^{\frac{3+\epsilon}{4}} \phi \rangle_{L^2(\mathbb{R}^3)}}}_{\text{Gaussian measure}} - \overbrace{\int dx (\mu {:}\phi^2(x){:} + g {:}\phi^4(x){:})}^{\text{potential } V(\phi)}$$

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- propagator $(-\Delta)^{-\frac{3+\epsilon}{4}}(x,y) \sim \frac{1}{|x-y|^{2[\phi]}}$
- $[\phi] = \frac{3-\epsilon}{4}$
- propagator $\sim \int_0^\infty \frac{dl}{l} l^{-2[\phi]} \ u\left(\frac{x-y}{l}\right)$
- u finite range, smooth, and nonnegative in x and p
- unit cut-off $C(x-y) = \int_1^\infty \frac{dl}{l} l^{-2[\phi]} \ u\left(\frac{x-y}{l}\right)$

- split $C(x-y) = \Gamma(x-y) + C_{L^{-1}}(x-y)$
  with $C_{L^{-1}}(x-y) = L^{-2[\phi]} C(L^{-1}(x-y))$ and

$$\Gamma(x-y) = \int_1^L \frac{dl}{l} l^{-2[\phi]} \; u\left(\frac{x-y}{l}\right)$$

- convolution $d\mu_C = d\mu_\Gamma \star d\mu_{C_{L^{-1}}}$

$$Z = \int d\mu_C(\phi) \; \mathcal{Z}(\phi) = \int d\mu_{C_{L^{-1}}}(\psi) d\mu_\Gamma(\zeta) \; \mathcal{Z}(\psi + \zeta)$$

$$= \int d\mu_C(\phi) \; (\mathcal{R}\mathcal{Z})(\phi)$$

$(\mathcal{R}\mathcal{Z})(\phi) = \int d\mu_\Gamma(\zeta) \; \mathcal{Z}(\phi_{L^{-1}} + \zeta)$ and $\phi_{L^{-1}}(x) = L^{-[\phi]} \phi(L^{-1}x)$

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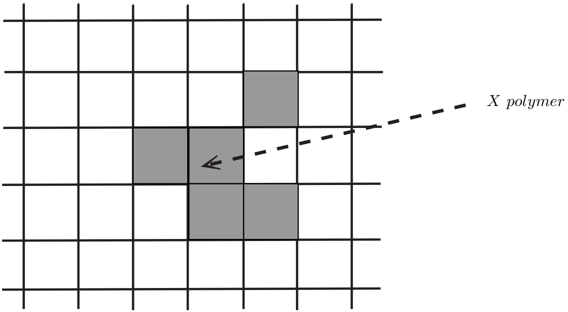
RG map: $\mathcal{Z} \longrightarrow \mathcal{R}\mathcal{Z}$

4. Good Infinite-Volume Coordinates:

Brydges et al.

$\mathbb{Z}^3 \subset \mathbb{R}^3 \implies$ cell decomposition



In finite box Λ

$$\mathcal{Z}(\Lambda, \phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{X_1,\ldots,X_n \\ \text{disjoint in } \Lambda}} \exp\left[- \int_{\Lambda \setminus (\cup X_i)} dx \{ g : \phi^4(x) :_C + \mu : \phi^2(x) :_C \} \right] \times K(X_1, \phi|_{X_1}) \cdots K(X_n, \phi|_{X_n})$$

- $\mathcal{Z} \longleftrightarrow (g, \mu, K)$
- $K = (K(X, \cdot))_{X \text{ polymer}}$ collection of local functionals

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- need to extract second order perturbation theory:

$$K(X,\phi) = g^2[\text{explicit complicated formula}]e^{-V(X,\phi)} + R(X,\phi)$$

- R of order g^3

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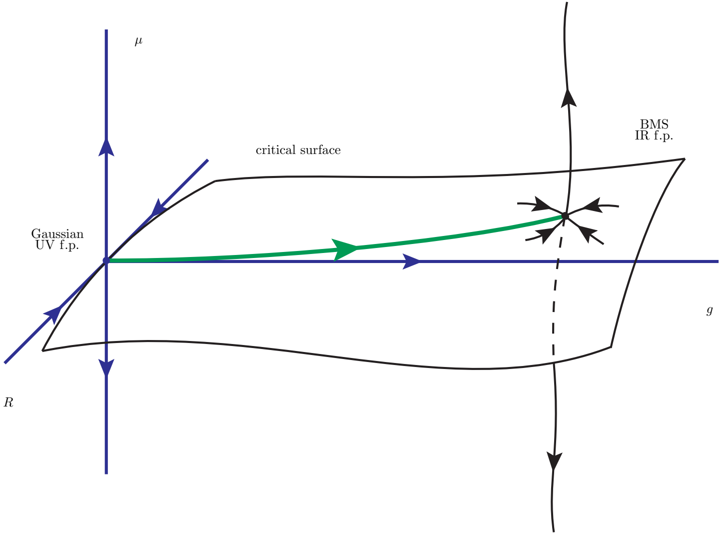
- R of order g^3
- RG map: $(g,\mu,R)\longrightarrow(g',\mu',R')$

Sketch of the RG phase portrait:



RG map in (g, μ, R) coordinates:

$$
\begin{aligned}
g' &= L^{\epsilon} g - L^{2\epsilon} a(L, \epsilon) g^2 + \xi_g(g, \mu, R) \\
\mu' &= L^{\frac{3+\epsilon}{2}} \mu + \xi_\mu(g, \mu, R) \\
R' &= \mathcal{L}^{(g,\mu)}(R) + \xi_R(g, \mu, R)
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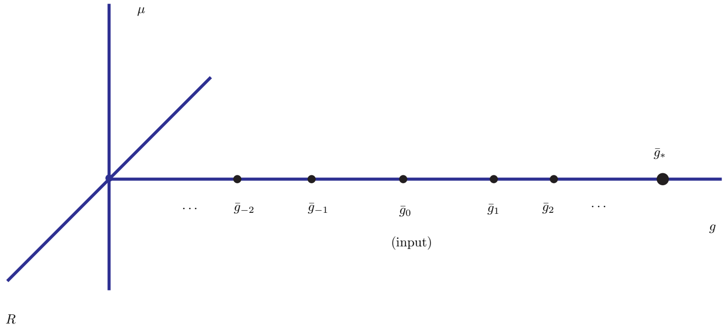
Simplified RG map:

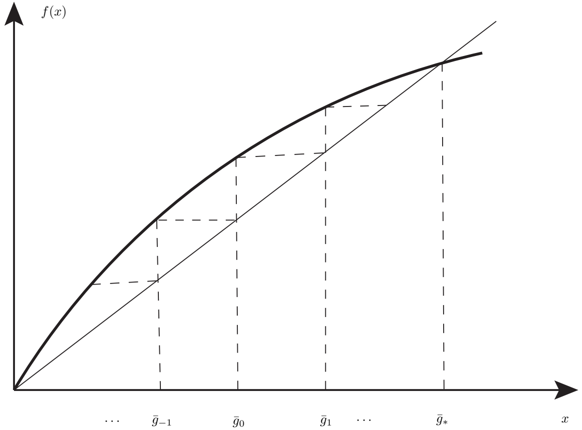
$$\begin{aligned} g' &= L^{\epsilon} g - L^{2\epsilon} a(L,\epsilon) g^2 \\ \mu' &= L^{\frac{3+\epsilon}{2}} \mu \\ R' &= \mathcal{L}^{(g,\mu)}(R) \end{aligned}$$

Approximate fixed point at $(\bar{g}_*, 0, 0)$ with

$$\bar{g}_* = \frac{L^{\epsilon} - 1}{L^{2\epsilon} a} \sim \epsilon$$



Fake trajectory $(\bar{g}_n)_{n\in\mathbb{Z}}$ obtained by simple 1d, yet nonlinear, iteration by $f(x) = L^{\epsilon}x - L^{2\epsilon}ax^2$



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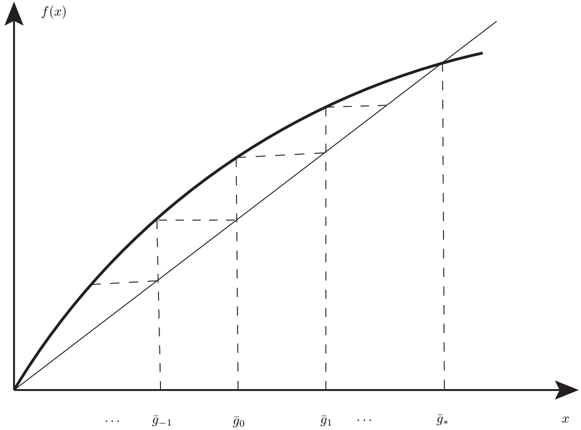
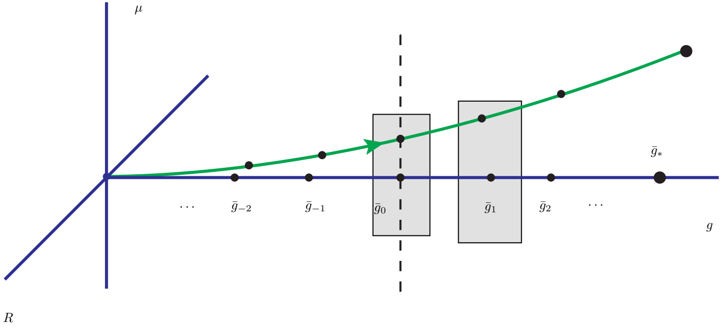


Note that $f'(0) = L^{\epsilon} > 1$ and $f'(\bar{g}_*) = 2 - L^{\epsilon} < 1$

True trajectories $(g_n, \mu_n, R_n)_{n \in \mathbb{Z}}$ constructed as perturbations of fake trajectories



Theorem

(A. A. CMP 07')

In the regime where $\epsilon > 0$ *is small enough, for any* $\omega_0 \in]0, \frac{1}{2}[$, *there exists a complete trajectory* $(g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ *for the RG map such that* $\lim_{n\to-\infty}(g_n, \mu_n, R_n) = (0, 0, 0)$ *the Gaussian ultraviolet fixed point, and* $\lim_{n\to+\infty}(g_n, \mu_n, R_n) = (g_*, \mu_*, R_*)$ *the BMS nontrivial infrared fixed point, and determined by the 'initial condition' at unit scale*

$$g_0 = \omega_0 \bar{g}_*$$

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- rewrite RG in terms of deviation variables $(\delta g_n, \mu_n, R_n)$

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\begin{aligned}
\delta g_{n+1} &= f'(\bar{g}_n)\delta g_n + \left[-L^{2\epsilon} a\ \delta g_n^2\ + \xi_g(\bar{g}_n + \delta g_n, \mu_n, R_n)\right]\ , \\
\mu_{n+1} &= L^{\frac{3+\epsilon}{2}} \mu_n + \xi_\mu(\bar{g}_n + \delta g_n, \mu_n, R_n)\ , \\
R_{n+1} &= \mathcal{L}^{(\bar{g}_n + \delta g_n, \mu_n)}(R_n) + \xi_R(\bar{g}_n + \delta g_n, \mu_n, R_n)
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Boundary conditions:

- Infrared: μ_n does not blow up when $n \to +\infty$
- Ultraviolet: R_n does not blow up when $n \to -\infty$
- Anthropic?: $\delta g_0 = 0$

Forward and backward integral equations towards the boundary conditions

$\forall n > 0,$

$$\delta g_n = f'(\bar{g}_{n-1})\delta g_{n-1} + \left[-L^{2\epsilon} a \ \delta g_{n-1}^2 \right.$$
$$\left. + \xi_g(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}, R_{n-1})\right]$$

$\forall n < 0,$

$$\delta g_n = \frac{1}{f'(\bar{g}_n)}\delta g_{n+1} - \frac{1}{f'(\bar{g}_n)}\left[-L^{2\epsilon} a \ \delta g_n^2 \ + \xi_g(\bar{g}_n + \delta g_n, \mu_n, R_n)\right]$$

$\forall n \in \mathbb{Z},$

$$\mu_n = L^{-\left(\frac{3+\epsilon}{2}\right)}\mu_{n+1} - L^{-\left(\frac{3+\epsilon}{2}\right)}\xi_\mu(\bar{g}_n + \delta g_n, \mu_n, R_n)$$

$\forall n \in \mathbb{Z},$

$$R_n = \mathcal{L}^{(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1})}(R_{n-1}) + \xi_R(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}, R_{n-1})$$

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then iterate until hit the b.c.

Fixed point equation in space of sequences:

$\forall n > 0,$

$$\delta g_n = \sum_{0 \le p < n} \left(\prod_{p < j < n} f'(\bar{g}_j) \right) \left[-L^{2\epsilon} a \; \delta g_p^2 \; + \xi_g(\bar{g}_p + \delta g_p, \mu_p, R_p) \right]$$

$\forall n < 0,$

$$\delta g_n = - \sum_{n \le p < 0} \left(\prod_{n \le j \le p} \frac{1}{f'(\bar{g}_j)} \right) \left[-L^{2\epsilon} a \; \delta g_p^2 \; + \xi_g(\bar{g}_p + \delta g_p, \mu_p, R_p) \right]$$

$\forall n \in \mathbb{Z},$

$$\mu_n = - \sum_{p \ge n} L^{-\left(\frac{3+\epsilon}{2}\right)(p-n+1)} \; \xi_\mu(\bar{g}_p + \delta g_p, \mu_p, R_p)$$

$\forall n \in \mathbb{Z},$

$$R_n = \sum_{p < n} \mathcal{L}^{(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1})} \circ \mathcal{L}^{(\bar{g}_{n-2} + \delta g_{n-2}, \mu_{n-2})} \circ \cdots$$

$$\cdots \circ \mathcal{L}^{(\bar{g}_{p+1} + \delta g_{p+1}, \mu_{p+1})} \left(\xi_R(\bar{g}_p + \delta g_p, \mu_p, R_p) \right)$$

Use this to define a map on a space of sequences $(\delta g_n, \mu_n, R_n)_{n \in \mathbb{Z}}$

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$\forall n > 0,$

$$\delta g_n' = \sum_{0\le p<n} \left(\prod_{p<j<n} f'(\bar{g}_j) \right) \left[-L^{2\epsilon} a\, \delta g_p^2 \;+ \xi_g(\bar{g}_p + \delta g_p, \mu_p, R_p) \right]$$

$\forall n < 0,$

$$\delta g_n' = -\sum_{n\le p<0} \left(\prod_{n\le j\le p} \frac{1}{f'(\bar{g}_j)} \right) \left[-L^{2\epsilon} a\, \delta g_p^2 \;+ \xi_g(\bar{g}_p + \delta g_p, \mu_p, R_p) \right]$$

$\forall n \in \mathbb{Z},$

$$\mu_n' = -\sum_{p\ge n} L^{-\left(\frac{3+\epsilon}{2}\right)(p-n+1)}\; \xi_\mu(\bar{g}_p + \delta g_p, \mu_p, R_p)$$

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- Use contraction mapping argument in a big Banach space of double-sided sequences

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- The fixed point (in the space of sequences) is a trajectory

6. Functional Analysis, Norms:

Fields:

- Δ a closed cube. Sobolev imbedding $W^{4,2}(\mathring{\Delta}) \hookrightarrow C^2(\Delta)$
- $\phi \in \mathrm{Fld}(X) = \bigoplus_{\Delta \subset X} W^{4,2}(\mathring{\Delta})$ plus C^2 gluing conditions
- $||\phi||_{\mathrm{Fld}(X)} = \left(\sum_{\Delta \subset X} \sum_{|\nu| \leq 4} ||\partial^\nu \phi_\Delta||^2_{L^2(\mathring{\Delta})} \right)^{\frac{1}{2}}$
- Fluctuation measure $d\mu_\Gamma$ realized in Hilbert spaces $\mathrm{Fld}(X)$
- Also need $||\phi||_{C^2(X)} = \sup_{x \in X} \max_{|\nu| \leq 2} |\partial^\nu \phi(x)|$
- ϕ's are real-valued

Functionals:

Functionals:

$$||K|| = \sup_{\Delta_0} \sum_{X \supset \Delta_0} L^{5|X|} \sup_{\phi \in \mathrm{Fld}(X)} \left\{ e^{-\kappa \sum_{\Delta \subset X} \sum_{1 \le |\nu| \le 4} ||\partial^\nu \phi||^2_{L^2(\mathring{\Delta})}} \right.$$

$$\left. \times \sum_{0 \le n \le 9} \frac{(c g^{-\frac{1}{4}})^n}{n!} \sup_{\phi_1, \ldots, \phi_n \in \mathrm{Fld}(X) \setminus \{0\}} \frac{|D^n K(X, \phi; \phi_1, \ldots, \phi_n)|}{||\phi_1||_{C^2(\Delta)} \cdots ||\phi_n||_{C^2(\Delta)}} \right\}$$

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- K's allowed to be complex-valued
- Fibered norm problem: norms depend on the dynamical variable g
- use fake solution to calibrate

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- Study more refined dynamical systems features: construct full invariant curve, regularity properties, smoothness at $g = 0$ (asked by K. Gawędzki), first on HM with Ph. D. student Ajay Chandra

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