On a Toy Model for Three-Dimensional Conformal Probability

Abdelmalek Abdesselam
Mathematics Department, University of Virginia

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Main references

For big picture:

A.A., "Towards three-dimensional conformal probability", arXiv:1511.03180[math.PR] (27 pages).

For technical details:

A.A., A. Chandra and G. Guadagni, "Rigorous quantum field theory functional integrals over the p-adics I: anomalous dimensions", arXiv:1302.5971 [math.PR] (162 pages).

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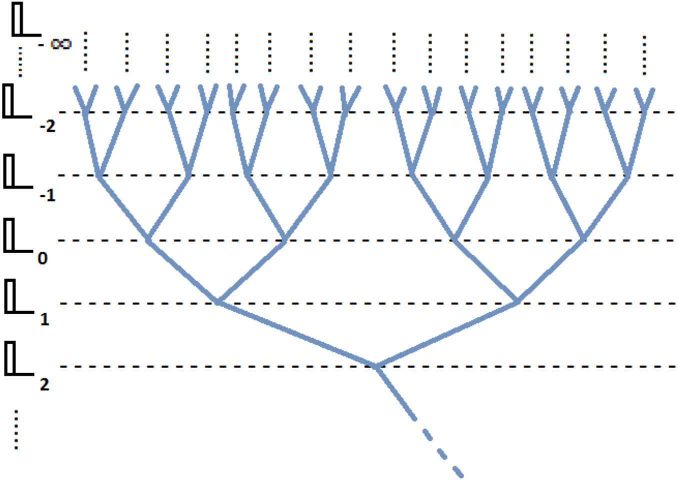
Hence $\mathbb{T} = \cup_{k\in\mathbb{Z}}\mathbb{L}_k$ naturally has the structure of a doubly infinite tree which is organized into layers or generations $\mathbb{L}_k$:



Picture for $d = 1$, $p = 2$

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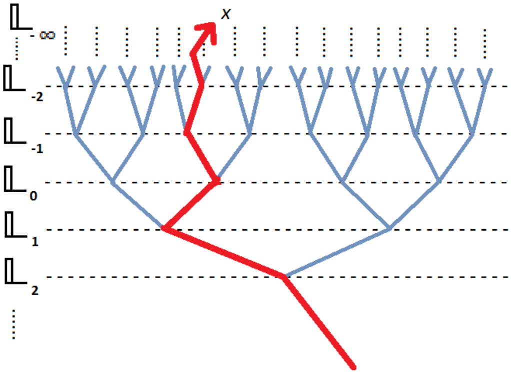
More precisely, these are the infinite bottom-up paths in the tree.



A path representing an element $x \in \mathbb{L}_{-\infty}$

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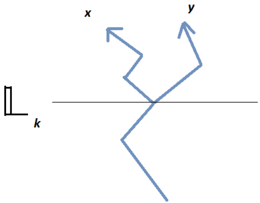

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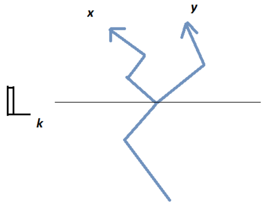

We also define the origin $\mathbf{0} \in \mathbb{L}_0$ as the leftmost path/lattice site. Thus $\#(\bar{B}(\mathbf{0}, p^k)) = p^{dk}$ as for the volume of balls in $\mathbb{R}^d$.

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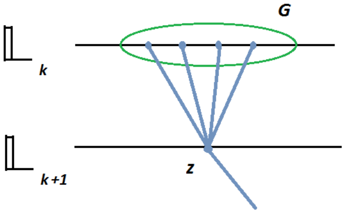

To every litter G of Mama Cat $\mathbf{z} \in \mathbb{L}_{k+1}$ associate a centered Gaussian random vector $(\zeta_{\mathbf{x}})_{\mathbf{x} \in G}$ with $p^d \times p^d$ covariance matrix made of $1 - p^{-d}$'s on the diagonal and $-p^{-d}$'s everywhere else. We impose that Gaussian vectors corresponding to different layers or different litters are independent.

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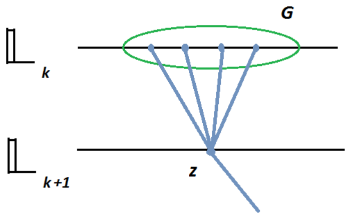

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Define the probability measure μ on $\mathbb{R}^{\mathbb{L}_0}$ as the law of ψ.

We introduce the Wick powers

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From now on: $d = 3$, $[\phi] = \frac{3-\epsilon}{4}$ for $\epsilon > 0$ small. We also fix $L = p^{\ell}$ for some integer $\ell \geq 1$.

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For g in an interval of size $\sim \epsilon^{\frac{3}{2}}$ aroung $\bar{g}_*$ we constructed a critical mass $\mu_c(g)$ for which the infinite volume measure ν is critical, i.e., $\mathbb{E}_\nu \phi_{\mathbf{x}} \phi_{\mathbf{y}} \to 0$ when $d(\mathbf{x}, \mathbf{y}) \to \infty$ and $\sum_{\mathbf{x} \in \mathbb{L}_0} \mathbb{E}_\nu \phi_{\mathbf{0}} \phi_{\mathbf{x}} = \infty$.

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Using a new rigorous renormalization group method, we proved that in an spatially averaged sense

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Remark: For 2D Ising, the scaling dimensions are $[\phi] = \frac{1}{8}$ (spin field) and $[\phi^2] = 1$ (energy field).

More precisely, for $\alpha \in (0,3)$ and $n \to \infty$ one has

$$\sum_{\mathbf{x},\mathbf{y} \in \bar{B}(\mathbf{0},L^n)} \min\left[1, \frac{1}{d(\mathbf{x},\mathbf{y})}\right]^{\alpha}$$

$$= L^{3n}\left[1 + \frac{1-p^{-3}}{1-p^{\alpha-3}}(L^{(3-\alpha)n} - 1)\right] \sim \frac{1-p^{-3}}{1-p^{\alpha-3}} \times L^{(6-\alpha)n} .$$

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What we proved is

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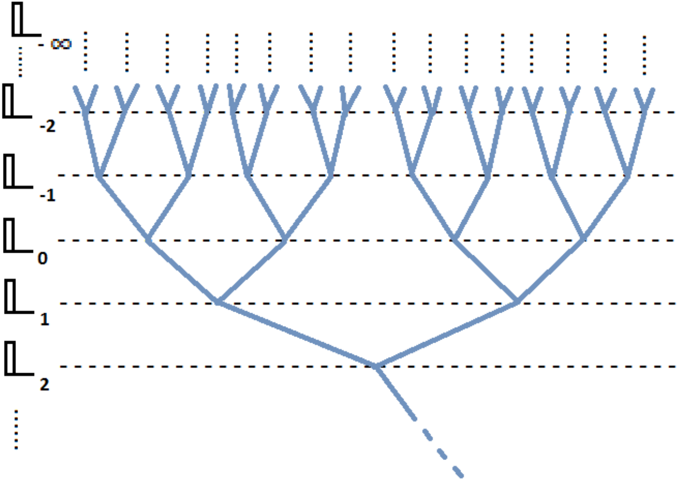
Consistent with predictions by conformal bootsrap for 3D Ising with long-range interactions $J_{\mathbf{xy}} = -(-\Delta_{\mathbb{Z}^3})_{\mathbf{xy}}^{\frac{3+\epsilon}{4}}$. Critical scaling limit believed to be a 3D CFT.



$\sim$ lower half-space model for conformal geometry.

Thank you for your attention.