Field theoretic cluster expansions and the Brydges-Kennedy forest sum formula

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Dedicated to the memory of Pierre Leroux

Outline:

I _ What is Quantum Field Theory?

II _ Some math

III _ Moebius inversion in partition lattices

IV _ Decoupling or cluster expansion

V _ The 1001 proofs of the Brydges-Kennedy forest sum formula

V.1 _ Proof n°1

V.2 _ Proof n°2

V.3 _ Proof n°3

V.4 _ Proof n°4

V.5 _ Proof n°5

V.6 _ Proof n°6

V.7 _ Proof n°7

⋮

I – What is Quantum Field Theory?

- Several possible answers
- One point of view : measure theory in function spaces

ex: $\phi : \mathbb{R}^d \to \mathbb{R}$ field

$\mathcal{F} = \{ \phi \text{ fields} \}$ function space

$I : \mathcal{F} \to \mathbb{R}$ functional

$d\nu$ probability measure on $\mathcal{F}$

$$\boxed{\int_{\mathcal{F}} d\nu(\phi) \, I(\phi) = \frac{\int_{\mathcal{F}} D\phi \; I(\phi) \, e^{-S(\phi)}}{\int_{\mathcal{F}} D\phi \; e^{-S(\phi)}}}$$

Feynman Path Integral

$$D\phi = \prod_{x \in \mathbb{R}^d} d[\varphi(x)] \qquad \text{Lebesgue measure on } \mathcal{F}$$

$$S(\phi) = \int_{\mathbb{R}^d} dx \left[\underbrace{\frac{1}{2} (\partial\phi)^2(x) + \frac{1}{2}\mu \, \phi(x)^2 + \lambda \, \phi(x)^4}_{\text{local density}} \right]$$

Action functional

Correlation functions: (moments of $d\nu$)

$$S_n(\underbrace{f_1, \dots, f_n}_{\text{test functions}}) = \int d\nu \; \langle \phi, f_1 \rangle \cdots \langle \phi, f_n \rangle$$

$$\langle \phi, f \rangle = \int_{\mathbb{R}^d} dx \; \phi(x) f(x)$$

Connected correlation functions (cumulants)

$$C_n(f_1, \dots, f_n)$$

<u>Def:</u>

$$S_1(f_1) = C_1(f_1)$$

$$S_2(f_1, f_2) = C_2(f_1, f_2) + C_1(f_1)\, C_1(f_2)$$

$$\begin{aligned} S_3(f_1, f_2, f_3) = {} & C_3(f_1, f_2, f_3) \\ & + C_1(f_1)\, C_2(f_2, f_3) \\ & + C_1(f_2)\, C_2(f_1, f_3) \\ & + C_1(f_3)\, C_2(f_1, f_2) \\ & + C_1(f_1)\, C_1(f_2)\, C_1(f_3) \end{aligned}$$

$$\vdots$$

$$S_n(f_1, \dots, f_n) = \sum_{\substack{\pi \text{ partition} \\ \text{of } \{1, \dots, n\}}} \prod_{I \in \pi} C_{|I|}\left((f_i)_{i \in I} \right)$$

$$\Big\downarrow \quad \text{Moebius inversion in partition lattice}$$

$$C_n(f_1, \dots, f_n) = \sum_{\pi} (-1)^{|\pi|-1} (|\pi|-1)! \prod_{I \in \pi} S_{|I|}\left((f_i)_{i \in I} \right)$$

Partition function :

$$Z = \int_{F} D\phi \; e^{-S(\phi)}$$

Pressure :

$$p = \frac{\log Z}{\text{volume of } \mathbb{R}^d}$$

n-point functions :

$$C_n(f_1, \ldots, f_n) = \int_{\mathbb{R}^{nd}} dx_1 \cdots dx_n \, \boxed{C_n(x_1, \ldots, x_n)} \, f_1(x_1) \cdots f_n(x_n)$$

Questions :

- QFT dv nontrivial ? $\Leftrightarrow$ non Gaussian

$$(\exists\, n \geqslant 3 \,,\quad C_n \not\equiv 0)$$

- Decay of $C_2(x_1, x_2)$ when $|x_1 - x_2| \to \infty$

$$\sim e^{-\underset{\text{mass}}{m}|x_1 - x_2|} \qquad \text{QFT massive}$$

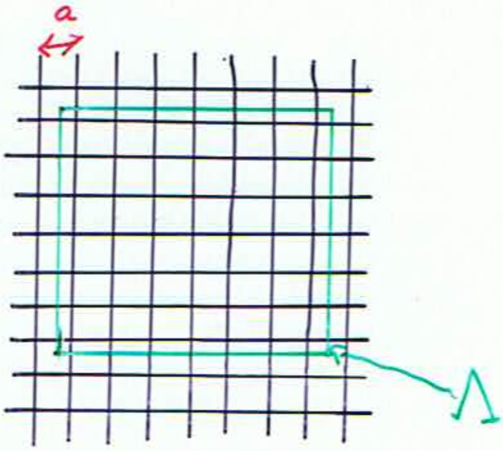
$$\sim |x_1 - x_2|^{-p} \qquad \text{QFT massless critical}$$

⋮

II Some math:

UV Regularization: $\mathbb{R}^d \longrightarrow (a\mathbb{Z})^d$

Volume cut-off Λ



$$D\phi \to \prod_{x\in(a\mathbb{Z})^d\cap\Lambda} d\phi(x)$$

$$\int_{\mathbb{R}^d} dx \to a^d \sum_{x\in(a\mathbb{Z})^d\cap\Lambda}$$

$$\partial_i\phi(x) = \frac{\partial\phi}{\partial x_i}(x) \longrightarrow \frac{1}{a}\left[\varphi(x+a\vec{e}_i)-\varphi(x)\right]$$

$d\nu_{a,\Lambda}$ well defined probability measure on $F_{a,\Lambda} = \mathbb{R}^{(a\mathbb{Z})^d\cap\Lambda}$

limits $a\to 0$, $\Lambda \nearrow \mathbb{R}^d$ Thermodynamic limit

need to let μ, λ vary with a

Renormalization, Renormalization Group

$$Z(\Lambda) = \int_{\mathbb{R}^\Lambda} \prod_{x\in\Lambda} d\phi(x) \underbrace{\exp\left[-\sum_{x\in\Lambda}\left\{\frac{1}{2}(\partial\phi)^2(x) + \frac{\mu}{2}\phi(x)^2 + \lambda\phi(x)^4\right\}\right]}$$

a=1 subset of $\mathbb{Z}^d$

Gaussian measure $d\mu_C(\phi)$

Covariance

$$C = \text{"}(-\Delta + \mu \,\mathrm{Id})^{-1}\text{"}$$

$$= (C_{xy})_{x,y\in\Lambda} \quad \text{matrix}$$

$$|C_{x,y}| \le K e^{-m|x-y|}$$

free propagator decay

pressure

$$\tilde{p} = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{\log \tilde{Z}(\Lambda)}{|\Lambda|}$$

(at small λ)

$$\tilde{Z}(\Lambda) = \int d\mu_C(\phi) \exp\left[-\lambda \sum_{x\in\Lambda} \phi(x)^4\right]$$

$\Downarrow$ Cluster Expansion

$$= \sum_{\pi \text{ partition of } \Lambda} \prod_{Y\in\pi} A_0(Y)$$

polymer

[Brydges-Kennedy formula → decouples covariance C

If $Y = \{x\}$ singleton & translation invariance (e.g. if periodic b.c.)

$$\mathcal{A}_0(Y) = c_0 \approx 1$$

$$c_0^{-|\Lambda|}\,\tilde{Z}(\Lambda) = \underbrace{\boxed{\sum_{N\geq 0}\frac{1}{N!}\sum_{\substack{Y_1,\dots,Y_N \\ \text{disjoint} \\ \text{polymers in } \Lambda}} \mathcal{A}_1(Y_1)\cdots\mathcal{A}_1(Y_N)}}_{\tilde{\tilde{Z}}(\Lambda)}$$

$$\mathcal{A}_1(Y) = c_0^{-|Y|}\,\mathbb{1}\{|Y|\geq 2\}\,\mathcal{A}_0(Y)$$

$\Downarrow$ Mayer expansion

$$\log \tilde{\tilde{Z}}(\Lambda) = \sum_{N\geq 1}\frac{1}{N!}\sum_{Y_1,\dots,Y_N} \underbrace{\varphi(Y_1,\dots,Y_N)}_{\substack{\text{Mayer} \\ \text{coefficient} \\ \text{Moebius function}}}\,\mathcal{A}_1(Y_1)\cdots\mathcal{A}_1(Y_N)$$

III Moebius inversion in partition lattices:

- E finite set
- $(\Pi_E, \leq)$ partition lattice

$$\pi_1 \leq \pi_2 \iff \pi_1 \text{ finer than } \pi_2$$

$f : \Pi_E \to \mathbb{R}$ function

<u>Def.</u>: $g : \Pi_E \to \mathbb{R}$ Moebius inverse of f

$$\forall \pi \in \Pi_E, \qquad f(\pi) = \sum_{\tau \in \Pi_E} \mathbb{1}\{\tau \leq \pi\} \, g(\tau)$$

$$\to \quad g(\pi) = \sum_{\tau \leq \pi} \mu(\tau, \pi) \, f(\tau)$$

$$\mu(\tau, \pi) = \prod_{X \text{ block of } \pi} \left[(-1)^{m_X - 1} (m_X - 1)! \right]$$

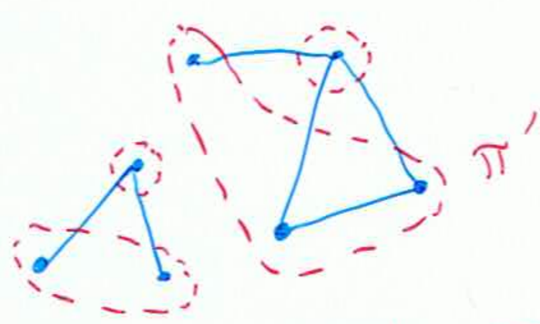
($\to$ formula for C_n)

m_X = # blocks of τ in X

<u>Refinement</u>:

- G graph on the vertex set E (sorry!)
- $\Pi_{E,G}$ set of partitions π such that $\forall X \in \pi$ the induced subgraph G_X connects X

π in $\Pi_{E,G}$



π' not in $\Pi_{E,G}$

- $\preccurlyeq$ same notion on $\Pi_{E,G} \subset \Pi_E$

- $Y_1, \ldots, Y_N$ polymers in Λ

  $E = \{1, 2, \ldots, N\}$

  G : $\quad i \bullet\!\!-\!\!-\!\!\bullet j \iff Y_i \cap Y_j \neq \emptyset$

$$\Psi(Y_1, \ldots, Y_N) = 0 \quad \text{if} \quad G \text{ does not connect } E.$$

otherwise

$$\Psi(Y_1, \ldots, Y_N) = \mu_{\Pi_{E,G}}(0, 1)$$

all singletons $\qquad$ one block

<u>Aside:</u> hyperplane arrangements A_{n-1}

$H_{ij} = \{ x_i - x_j = 0 \} \subset \mathbb{C}^n$

subvector space V obtained by intersections of H_{ij}'s

$\to$ lattice (for inclusion)

$\to$ Moebius functions

$\to$ combinatorial identities with sums over forests.

Important property :

$$\Psi(y_1,\dots,y_N) = (-1)^{N-1} \sum_{\substack{T \\ \text{some} \\ \text{spanning trees} \\ \text{of } G}} 1$$

Rota 64

O. Penrose 66

⋮

IV Decoupling or cluster expansion :

$$\tilde{Z}(\Lambda) = \int d\mu_C(\phi) \exp\left[-\lambda \sum_{x\in\Lambda} \phi(x)^4\right]$$

$$= \exp\left[\frac{1}{2}\sum_{x,y\in\Lambda} \frac{\partial}{\partial\phi(x)} \mathbb{1} C_{x,y} \frac{\partial}{\partial\phi(y)}\right] \exp\left[-\lambda\sum_{x\in\Lambda}\phi(x)^4\right]\Bigg|_{\phi}$$

$\mathbb{1} = t_{\{x,y\}}$ if $x \neq y$

$\mathbb{1} = 1$ if $x = y$

$$= f(\vec{t})\Big|_{\vec{t}=\vec{1}}$$

compare with $\vec{t} = \vec{0}$.

Decoupling expansion

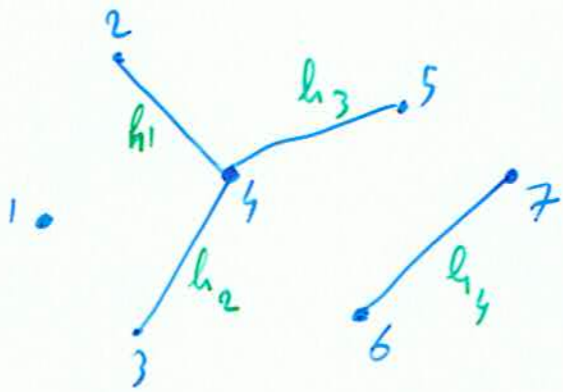
- E finite (vertex) set
- P_E = set of unordered pairs $\ell = \{i,j\}$, $i \neq j$ in E
- $\vec{t} = (t_\ell)_{\ell \in P_E} \in \mathbb{R}^{P_E} = \mathbb{R}^{\binom{|E|}{2}}$
- $f : \mathbb{R}^{P_E} \to \mathbb{R}$, $\quad \vec{t} \mapsto f(\vec{t})$ $\qquad \vec{1} = (1,1,\dots)$, $\quad \vec{0} = (0,0,\dots)$

<u>The Brydges-Kennedy forest formula:</u> (& A.A.-Rivasseau)

$$\boxed{f(\vec{1}) = \sum_{\substack{\mathcal{F} \\ \text{forest on } E}} \left(\prod_{\ell \in \mathcal{F}} \int_0^1 dh_\ell\right) \left(\prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial t_\ell}\right) f(\vec{t}) \Big|_{\vec{t} = \vec{w}^{AR}(\mathcal{F}, \vec{h})}}$$

$$w^{AR}_{\{i,j\}}(\mathcal{F}, \vec{h}) = \begin{cases} 0 & \text{if } i,j \text{ not connected by } \mathcal{F} \\ \text{else} & \\ \text{minimum of } h_\ell \text{ for } \ell \text{ in path } i \leftrightarrow j \text{ in } \mathcal{F} & \end{cases}$$

ex:



$w_{\{1,3\}} = 0$

$w_{\{4,5\}} = h_3$

$w_{\{3,5\}} = \min(h_2, h_3)$

$=\Lambda)$ Application to first step, cluster expansion

$$\tilde{Z}(\Lambda) = \sum_{\substack{F \\ \text{forest on } \Lambda}} \prod_{l\in F}\int_0^1 dh_l \int d\mu_{\underbrace{C(F,h)}_{C_{xy}\, w^{AR}_{\{x,y\}}(F,h)}}(\phi) \overbrace{\left(\prod_{l\in F} \Delta_{C,l}\right)}^{\text{acts on}} e^{-\lambda \sum_{x\in\Lambda} \phi(x)}$$

factorizes over connected components

$$\Delta_{C,\{x,y\}} = C_{xy} \frac{\partial}{\partial\phi(x)} \frac{\partial}{\partial\phi(y)}$$

$$= \sum_{\pi \text{ partition of } \Lambda} \prod_{Y\in\pi} A_0(Y)$$

formula is same, but using spanning trees for Y instead of forests.

$$\lambda \text{ small} \Rightarrow \sum_{Y \ni y_0} |A_0(Y)|\, cst^{|Y|} < +\infty$$

K-P condition

$A_0(Y)$ decays if Y bigger or more spread out.

Proof:

$$\Pi\Delta = \underbrace{C_{xz}\,C_{yz}\,C_{uz}}_{\lesssim e^{-m(|x-z|+|z-y|+|z-u|)}}\ \frac{\partial}{\partial\phi(x)}\ \frac{\partial}{\partial\phi(y)}\left(\frac{\partial}{\partial\phi(z)}\right)^3\frac{\partial}{\partial\phi(u)}$$

acting on

$$\exp\left[-\lambda\left(\phi(x)^4+\phi(y)^4+\phi(z)^4+\phi(u)^4\right)\right]$$

$\Rightarrow$ $\boxed{\text{small factor } \lambda \text{ per site}}$

vertex degrees $d_x = d_y = d_u = 1,\ d_z = 3$

$$\left(\frac{\partial}{\partial\phi}\right)^d e^{-\lambda\phi^4}\ ,\quad \#\text{ways of performing derivatives} \lesssim d!$$

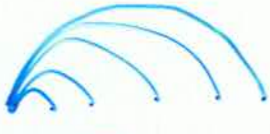
$\boxed{\text{Local factorial bound}}$

$$\left|\int d\mu_C(\phi)\ \prod_{x\in Y}\phi(x)^{m_x}\right| \lesssim \prod (m_x!)^{1/2}$$

Glimm-Jaffe-Spencer 73

Eckmann-Magnen-Sénéor 75

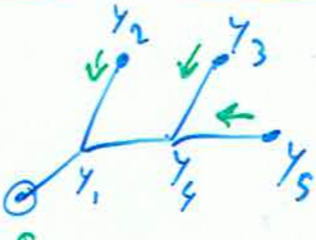
$\boxed{\text{Volume effect}}$



$$\varepsilon^{\frac{d(d+1)}{2}} \quad \text{beats} \quad d\log d$$

$\boxed{\text{Tree summation}}$ (pin & sum)

$$\sum_{\substack{Y\\ \text{set}}}\ \sum_{\substack{T\\ \text{tree on } Y}} \longrightarrow \sum_n \frac{1}{n!}\ \underbrace{\sum_{\substack{T \text{ on}\\ \{1,\dots,n\}}}}_{n^{n-2}}\ \underbrace{\sum_{y_1,\dots,y_n}}_{\le cst^n}$$



fixed

V The 1001 proofs of the Brydges-Kennedy forest sum formula:

. Moebius again

- E set
- $\mathcal{P}_E$ pairs
- $(\Pi_E, \leq)$ partition lattice
- $(\mu_\ell)_{\ell \in \mathcal{P}_E}$ variables

$$f(\pi) = \prod_{X \in \pi} F(X)$$

$$F(X) = \exp\Big(\sum_{\substack{\ell \in \mathcal{P}_E \\ \ell \subset X}} \mu_\ell\Big)$$

$$\xRightarrow{\text{Moebius inverse}} \quad g(\pi) = \prod_{X \in \pi} G(X)$$

Naïve yet fundamental formula:

$$\boxed{G(X) = \sum_{\substack{\text{graph } H \subset \mathcal{P}_X \\ H \text{ connects } X}} \prod_{\ell \in H} \left(e^{\mu_\ell} - 1\right)}$$

too many graphs ⇝ resummation in terms of trees

- <u>Tree sum formulas:</u>

?... , Groeneveld 62, Rota 64, Penrose 66, Glimm-Jaffe-Spencer 73/74, Brydges-Federbush 76/78, Malyshev-Minlos 79?, Seiler 82, Battle-Federbush 82/8[illegible] Brydges-Kennedy J. Stat. Phys 87, Brydges-Wright 88, A.A.-Rivasseau 94/97, A.A. 97, A.A.-Magnen-Rivasseau 0[illegible] Brydges-Imbrie 03, Rivasseau 07, Magnen-Rivasseau 07, ...

- <u>Differential equation with quadratic nonlinearity:</u>

(Brydges-Kennedy 87)

$$G(X) = \sum_{H \text{ connects } X} \prod_{\ell \in H} \left(e^{u_\ell} - 1\right)$$

$$G(t,X) = \sum_{H \text{ connects } X} \prod_{\ell \in H} \left(e^{t u_\ell} - 1\right)$$

- ? diff. eq. for infinitesimal variation of $\{G(t,X)\}_{\substack{X \subset E \\ X \neq \emptyset}}$ with t

→ 1- integral form
2- iterate
3- trees

- Wilson 74, Polchinski 84 RG

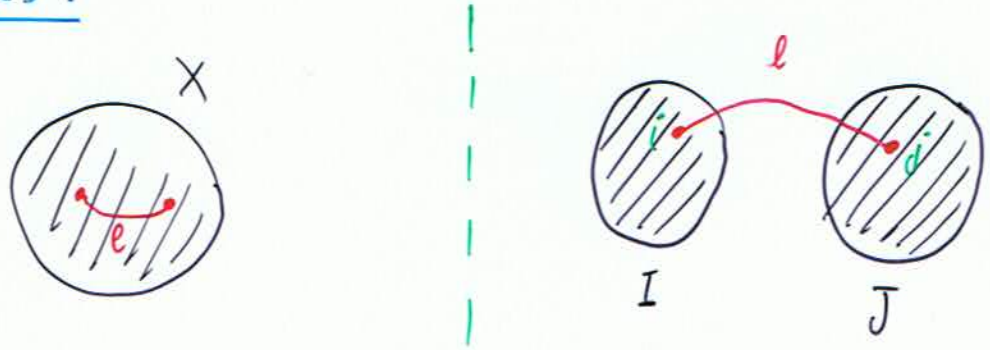
⇝ Gallavotti-Nicoló trees 85

(Feldman-Hurd-Rosen-Wright 88
Hurd 89)

$$G(t,X) = \sum_{\substack{H \text{ simple graph} \\ \text{connects } X}} \prod_{\ell \in H} \left(e^{t\mu_\ell} - 1\right) = \sum_{\substack{M \text{ multigraph} \\ \text{connects } X}} \frac{(t\mu)^M}{M!}$$

$$\frac{d}{dt} G(t,X) = \sum_{\substack{M,\ell \\ M \text{ connects } X \\ \ell \in M}} \mu_\ell \cdot \frac{(t\mu)^{M-\ell}}{(M-\ell)!}$$

two cases :



$$\boxed{\frac{d}{dt} G(t,X) = \left(\sum_{\ell \subset X} \mu_\ell\right) G(t,X) + \sum_{\substack{I,J,i,j \\ X = I \uplus J \\ i \in I,\, j \in J}} \frac{1}{2} \mu_{\{ij\}} \, G(t,I)\, G(t,J)}$$

solve :

$$G(t,X) = \sum_{T \text{ span } X} \left(\prod_{\ell \in T} \int_0^1 dh_\ell\right)\left(\prod_{\ell \in T} t\mu_\ell\right) e^{\sum_{\ell \in P_X} w_\ell^{BK}(T,h)\, t\mu_\ell}$$

$$w^{BK}_{\{ij\}} = 1 - \max(h\text{'s in path } i \leftrightarrow j)$$

(cut-join ?)

• Partial fraction decomposition:

(A.A. - Rivasseau 94)

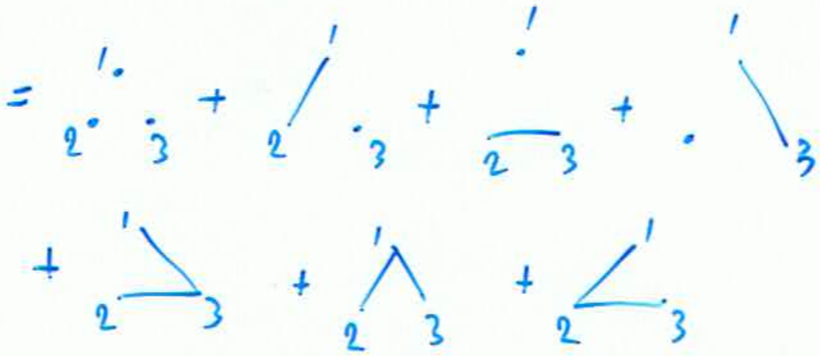
$$f(\vec{1}) = e^{\sum_{\ell \in E} \mu_\ell}$$

$$= \sum_{\substack{F \subset P_E \\ \text{forest}}} \prod_{\ell \in F} \int_0^1 dh_\ell \prod_{\ell \in F} \mu_\ell \; e^{\sum_{\ell \in P_E} w_\ell^{AR}(F,h)\, \mu_\ell}$$

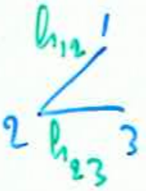
ex: $E = \{1, 2, 3\}$

$\mu_{12} \quad \mu_{23} \quad \mu_{13}$

$f(\vec{1}) = e^{\mu_{12}+\mu_{23}+\mu_{31}}$



 $= 1$,  $= \int_0^1 dh_{12}\, \mu_{12}\, e^{h_{12}\mu_{12} + 0\mu_{13} + 0\mu_{23}} = e^{\mu_{12}} - 1$

 $= \int_0^1\int_0^1 dh_{12}\, dh_{23}\; \mu_{12}\mu_{23}\; e^{h_{12}\mu_{12} + h_{23}\mu_{23} + \min(h_{12},h_{23})\mu_{13}}$

$$= \frac{\mu_{12}\mu_{23}}{(\mu_{23}+\mu_{13})} \left[\frac{e^{\mu_{12}+\mu_{23}+\mu_{13}} - 1}{\mu_{12}+\mu_{23}+\mu_{13}} - \frac{e^{\mu_{12}} - 1}{\mu_{12}} \right]$$

$$+ \left(\mu_{12} \rightleftarrows \mu_{23} \right)$$

1) $\mathcal{F} = \{\ell_1, \dots, \ell_k\}$ nonordered forest

$\rightarrow \quad (\ell_1, \dots, \ell_k)$ ordered forest

$$\prod_{\ell \in \mathcal{F}} \int_0^1 dh_\ell \rightarrow \int_{1 > h_1 > \dots > h_k > 0} dh_1 \dots dh_k$$

$$\boxed{\begin{array}{l} \text{B-K with ordered forests :} \\ f(\vec{1}) = \displaystyle\sum_{\mathcal{F} = (\ell_1, \dots, \ell_p)} \int_{1 > h_1 > \dots > h_p > 0} dh_1 \dots dh_p \;\; \partial_{\ell_1} \dots \partial_{\ell_p} \; e^{\sum_\ell w_\ell^{ord} \partial_\ell} \end{array}}$$

$$w^{ord}_{\{ij\}} = \begin{cases} * \; 0 \text{ if } i, j \text{ not connected by } \mathcal{F} = (\ell_1, \dots, \ell_p) \\ * \; h_\nu \text{ if } i, j \text{ connected by } (\ell_1, \dots, \ell_\nu) \\ \quad \text{but not by } (\ell_1, \dots, \ell_{\nu-1}) \end{cases}$$

2) change of var $\quad h_i = t_i + h_{i+1}$

$$\int_{\substack{t_0 \geq 0, \dots, t_p \geq 0 \\ \sum t_i = 1}} dt_0 \dots dt_p \;\; e^{t_0 a_0 + \dots + t_p a_p} = \sum_i \frac{e^{a_i}}{\prod_{j \neq i} (a_i - a_j)}$$

new variables $\quad v_\ell = e^{\partial_\ell}$

$\rightarrow$ Algebra

Identity:

$$\prod_{\ell \in P_E} v_\ell = \sum_{\substack{F=(\ell_1,\dots,\ell_r) \\ \text{ordered forest}}} \mu_{\ell_1}\cdots\mu_{\ell_r} \left(\sum_{\nu=0}^{r} \frac{b_\nu^F}{\prod_{\substack{\mu=0 \\ \mu\neq\nu}}^{r} (a_\nu^F - a_\mu^F)} \right)$$

where

$$a_\nu^F = \sum_{\substack{\{ij\} \text{ connected} \\ \text{by } (\ell_1,\dots,\ell_\nu)}} \mu_{\{ij\}} \quad , \quad b_\nu^F = \prod_{\substack{\{ij\} \text{ connected} \\ \text{by } (\ell_1,\dots,\ell_\nu)}} v_{\{ij\}}$$

Proof: coefficient by coefficient % v's

ex: constant monomial

$$\sum_{F=(\ell_1,\dots,\ell_r)} (-1)^r \frac{\mu_{\ell_1}\cdots\mu_{\ell_r}}{a_1^F \cdots a_r^F} = \begin{cases} 1 & \text{if } |E|=1 \\ 0 & \text{if } |E|\geq 2 \end{cases}$$

proof in case $|E| \geq 2$: $F = (\ell_1,\dots,\ell_r)$

$$\mathcal{A}_F = (-1)^r \frac{\mu_{\ell_1}\cdots\mu_{\ell_r}}{a_1^F \cdots a_r^F} \quad , \quad R_F = \mathcal{A}_F \cdot \frac{a_r^F}{a_{total}}$$

$$a_{total} = \sum_{\ell \in P_E} \mu_\ell$$

$$= a_r^F + \sum_{\substack{F'=(\ell_1,\dots,\ell_r,\ell_{r+1}) \\ \text{1-extension} \\ \text{of } F}} \mu_{\ell_{r+1}}$$

$$a_r^F = a_{total} + \sum_{F'} \left(- \frac{\mu_{e_{r+1}}}{a_{r+1}^{F'}} \right) a_{r+1}^{F'}$$

$\downarrow \times \frac{A_F}{a_{total}}$

$$\boxed{R_F = A_F + \sum_{\substack{F' \\ 1\text{-ext} \\ \text{of } F}} R_{F'}}$$

"R antiderivative of A"

iterate

$$\Rightarrow \sum_F A_F = \sum_{\substack{F \\ \text{ext of } \emptyset}} A_F = R_\emptyset = \underbrace{A_\emptyset}_{=1} \frac{\overbrace{a_0^\emptyset}^{=0}}{a_{total}} = 0$$

"integral of A"

$\boxtimes$

Monomials in v all of the form

$$\prod_{\substack{e \text{ in a} \\ \text{block of } \pi}} v_e \qquad \text{where } \pi \text{ partition of } E$$

case $|\pi| \geq 2$: same as before with π instead of E

case $|\pi| = 1$: special treatment induction <u>contract e_1</u>

- proof of algebraic BK using ⊗ copies of matrices of stirling numbers A.A. 94
- proof using minimal decompositions of permutations into transpositions V. Lafforgue 94
- Taylor expansion with stopping rule, iterated integrals:

Rk: (A-R 94)

$$e^{\sum_{\ell \in E} u_\ell} = \sum_F \prod_{\ell \in F} \int_0^1 dh_\ell \prod_{\ell \in F} u_\ell \quad e^{\sum_{\ell \in P_E} w_\ell^{AR}(F,h) u_\ell}$$

$\Updownarrow$

$$f(\vec{1}) = \sum_F \prod_{\ell \in F} \int_0^1 dh_\ell \left(\prod_{\ell \in F} \frac{\partial}{\partial t_\ell} \right) f\left(\vec{w}^{AR}(F,h)\right)$$

$\Uparrow$ trivial : $f(\vec{t}) = e^{\sum_{\ell \in P_E} t_\ell u_\ell}$

$\Downarrow$ $u_\ell = \dfrac{\partial}{\partial t_\ell}$

Taylor BK with ordered forests:

$$f(t_\ell = 1, \forall \ell) = \sum_{F=(\ell_1,\dots,\ell_n)} \int_{1>h_1>\dots>h_n>0} dh_1 \dots dh_n \frac{\partial}{\partial t_{\ell_1}} \dots \frac{\partial}{\partial t_{\ell_n}} f\left(\vec{w}^{ord}(F,h)\right)$$

$w_{ij}^{ord} = 0$ if never connected, h_r if i,j connected at time r

Proof: (A.R. 97 based on idea of Knörrer, Magnen, Rivasseau 94)

$$f(1,1,\dots) = f(h_1,h_1,\dots)\Big|_{h_1=1}$$

$$= \underbrace{f(0,0,\dots)}_{F=\emptyset} + \int_0^1 dh_1 \frac{d}{dh_1} f(h_1,h_1,\dots)$$

$$\sum_{\ell_1} \int_0^1 dh_1 \, \frac{\partial f}{\partial t_{\ell_1}}(h_1,h_1,\dots)$$

$$\frac{\partial f}{\partial t_{\ell_1}}(h_1,h_1,\dots) = \frac{\partial f}{\partial t_{\ell_1}}(h_2,\dots,h_2,\underbrace{h_1}_{\substack{t_{\ell_1}\text{ entry} \\ \text{frozen at} \\ \text{value } h_1}},h_2,\dots)\Big|_{h_2=h_1}$$

$$= \dots\Big|_{h_2=0} + \int_0^{h_1} dh_2 \frac{d}{dh_2}\dots$$

etc. etc.

A.R, Poirot 96 generalizes → hypergraphs
→ p-particle irreducible
(= (p+1)-edge connected)
→ ...
→ ...

$\boxed{\text{The end}}$