CRAMER'S RULE AND LOOP ENSEMBLES

A. ABDESSELAM AND D. C. BRYDGES

Abstract. We review a 1986 result of G.X Viennot that expresses a ratio of generating functions for disjoint oriented loops in a finite graph in terms of the generating function of a single path in the graph weighted according to loops in the path, defined by loop erasure. The result is a generalisation of Cramer's formula for the inverse of a matrix. We show that it arises from the Mayer expansion.

1. Introduction

The result reviewed here was noted by G.X. Viennot as an immediate corollary of his theory of heaps and pieces in [Vie86, Proposition 6.3], but perhaps not many people in statistical mechanics have realised that his result is an interesting statement about correlations for loop ensembles such as one encounters in the contour expansion of the Ising model.

It is also a generalisation of Cramer's formula for the inverse of a matrix; indeed we rediscovered it in this context by the methods of statistical mechanics, notably the Mayer expansion. In conversations with combinatorialists we have since learned of another proof based on the involution used in [Str83], which is also nicely explained in [Zei85]. The original proof and the one based on [Str83] are *much* neater than the one we present here, so readers who are not interested in the Mayer expansion will not find anything of interest after this section. In a later draft we hope to add more about the combinatorial proofs and to improve the sometimes sketchy parts of the proof we give here.

For motivation we begin with a combinatorial interpretation of Cramer's formula. Let $A = (A_{xy},\ x, y \in \mathcal{S})$ be a matrix. A *path* ω from a to b is any finite sequence

$$\big(\omega_1, \omega_2, \cdots, \omega_n\big) \in \bigcup_{n\in\mathbb{N}^*} \mathcal{S}^n$$

2000 *Mathematics Subject Classification.* Primary 82B41,05C30,05C50.

Key words and phrases. keywords: species, cramer, polymer, loop ensemble, loop erased random walk, cycle, permutation.

The work of DB was supported in part by NSERC of Canada.

with $\omega_1 = a$ and $\omega_n = b$. The sites $\omega_1, \omega_2, \cdots, \omega_n$ need not be distinct; if they are we say that the path is *self-avoiding*. The set of distinct sites in the sequence ω is called the *support of* ω. If $a = b$ then the only self-avoiding path is (a).

The resolvent expansion in powers of A represents $(I - A)^{-1}$ by the formal power series

$$(I - A)^{-1}_{ab} = \sum_{\omega : a \to b} A^{\omega}$$

where

$$A^{\omega} = \begin{cases} \prod_{i=1}^{n-1} A_{\omega_i \omega_{i+1}} & \text{if } n \geq 2 \\ 1 & \text{if } n = 1. \end{cases}$$

We call $c \subset \mathcal{S} \times \mathcal{S}$ a *self-avoiding loop* if for some $n \in \mathbb{N}^*$,

$$c = \begin{cases} \{(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1)\} & \text{if } n \geq 2 \\ \{(x_1, x_1)\} & \text{if } n = 1, \end{cases}$$

where $x_1, \ldots, x_n$ are *distinct*. The *support* of a loop is the set $\{x_1, \ldots, x_n\} \subset \mathcal{S}$.

By writing $\det(I - A)$ in terms of permutations and decomposing the permutations into cycles, one has

$$\det(I - A) = \sum_{r=0}^{\infty} \sum_{\{c_1, \ldots, c_r\}} (-A^{c_1}) \cdots (-A^{c_r}) \tag{1}$$

where the $r = 0$ term equals 1 by definition and $c_1, \ldots, c_r$ are self-avoiding loops with disjoint supports and

$$A^{c} = \prod_{(x,y) \in c} A_{xy}.$$

There is a similar expansion for the ab cofactor,

$$\det(I - A)^{(b,a)} = \sum_{r=0}^{\infty} \sum_{\bar{\omega}, \{c_1, \ldots, c_r\}} A^{\bar{\omega}} (-A^{c_1}) \cdots (-A^{c_r}), \tag{2}$$

where $\bar{\omega}, c_1, .., c_r$ have disjoint supports, $c_1, .., c_r$ are cycles and $\bar{\omega}$ is a self avoiding path from a to b. If $a = b$ then $\bar{\omega} = (a)$ and $A^{\bar{\omega}} = 1$.

By Cramer's formula

$$\frac{\det(I - A)^{(b,a)}}{\det(I - A)} = (I - A)^{-1}_{ab}, \tag{3}$$

so, as elements in the ring of power series in A with rational coefficients,

$$\frac{\sum_{r=0}^{\infty} \sum_{\bar{\omega}, \{c_1, \ldots, c_r\}} A^{\bar{\omega}} (-A^{c_1}) \cdots (-A^{c_r})}{\sum_{r=0}^{\infty} \sum_{\{c_1, \ldots, c_r\}} (-A^{c_1}) \cdots (-A^{c_r})} = \sum_{\omega : a \to b} A^{\omega}. \tag{4}$$

Our route towards the result of Viennot began when we asked ourselves how the standard theory of the Mayer expansion [Sei82, Bry86], which can be applied to the left hand side, could yield so simple a right hand side.

The result of Viennot generalises this formula to the case where the weights on the self-avoiding loops and the weight on the self-avoiding path are arbitrary. Let $\mathcal{C}$ be the finite set of all self-avoiding loops in $\mathcal{S}$. For each $c \in \mathcal{C}$ we require a formal variable λ_c and let $\lambda = \{\lambda_c : \ c \in \mathcal{C}\}$. Likewise let $\mathcal{C}(a, b)$ be the set of all self-avoiding paths from a to b. For each $\bar{\omega} \in \mathcal{C}(a, b)$ there is a formal variable $\alpha_{\bar{\omega}}$ and $\alpha = \{\alpha_{\bar{\omega}} : \ \bar{\omega} \in \mathcal{C}(a, b)\}$. Let $\mathcal{R}$ be the ring of power series with rational coefficients in λ, α. Define an element of this ring by

$$\langle a, b\rangle = \frac{\sum_{r=0}^{\infty} \sum_{\bar{\omega},\{c_1,\dots,c_r\}} \alpha_{\bar{\omega}} \lambda_{c_1} \cdots \lambda_{c_r}}{\sum_{r=0}^{\infty} \sum_{\{c_1,\dots,c_r\}} \lambda_{c_1} \cdots \lambda_{c_r}}. \tag{5}$$

By traveling along the path ω starting at a and ending at b, recursively erasing self-avoiding loops in the order in which they appear, one obtains a possibly empty list $\mathcal{E}(\omega)$ of erased self-avoiding loops and a self-avoiding path $\bar{\omega}(\omega)$ from a to b.

$$\mathcal{E}(\omega) = \begin{cases} (c_1, c_2, \dots, c_r) & \text{if } r \geq 1 \\ () & \text{if } r = 0 \end{cases}$$

where $c_1, c_2, \dots, c_r$ are self-avoiding loops. In the case where $a = b$, $\bar{\omega}(\omega) = (a)$.

Theorem 1.1. [Vie86]. *For $a, b \in \mathcal{S}$, as an identity in the ring of power series $\mathcal{R}$,*

$$\langle a, b\rangle = \sum_{\omega : a \to b} (-\lambda)^{\mathcal{E}(\omega)} \alpha_{\bar{\omega}(\omega)}$$

where

$$(-\lambda)^{\mathcal{E}(\omega)} = \prod_{i=1}^{r} (-\lambda_{c_i}).$$

Cramer's formula is the particular case where the formal variables are specialised according to

$$\lambda_c = -A^c, \quad \alpha_{\bar{\omega}} = A^{\bar{\omega}}.$$

The rest of the paper is a proof of a generalisation of this theorem. We use the Mayer expansion, especially the versions [Sei82, Bry86] based on trees, to perform the division in (5). Readers familiar with this approach will know that the Mayer expansion expresses $\langle a, b\rangle$ as a sum over a loop ensemble and one self-avoiding path connected by edges of a tree graph. Our proof is an elaboration of the remark that the edges of the tree graph taken in the right order describe how to insert all the loops into the self-avoiding path to obtain a single path. Loop erasure is the inverse operation, which, when applied to a path, recreates the ensemble of loops and the self-avoiding path and the tree.

It is hard to convert these words into a careful argument without combinatoric ambiguities, which is why the proof that follows is lengthy. We have found that the theory of combinatorial species [Joy81, BLL98, Abd04] is helpful to this end and use it throughout our proof.

2. Preliminaries

We will need some standard facts and notations from elementary set theory and graph theory. Let E be a finite set. We denote by $\mathcal{P}(E)$ the power set of E, or the set of subsets of E. The cardinality of a finite set A is denoted by $\#(A)$. For any $k \in \mathbb{N}$, we let

$$\mathcal{P}_k(E) \stackrel{\mathrm{def}}{=} \{A \in \mathcal{P}(E) | \#(A) = k\} \tag{6}$$

An *unoriented graph* on E is any subset of $\mathcal{P}_2(E)$. An element of this graph is called an *edge*. An *oriented graph* is any subset $\mathcal{G}$ of the Cartesian product $E \times E$. An element $(a, b) \in \mathcal{G}$, also called an edge, is said to go from a to b. Given an oriented graph $\mathcal{G}$ on E one associates to it an unoriented graph $\mathcal{G}^u$ by

$$\mathcal{G}^u \stackrel{\mathrm{def}}{=} \{A \in \mathcal{P}_2(E) | \exists (a, b) \in \mathcal{G}, A = \{a, b\}\} \tag{7}$$

Note that such a and b are then necessarily distinct. To an unoriented graph G corresponds a set theoretic partition $\Pi(G)$ of E, the set of connected components, i.e., maximal subsets A of E such that for any $a, b \in A$ there exists a sequence $e_0 = a, e_1, \ldots, e_k = b$ in E with $k \geq 0$ and $\{e_i, e_{i+1}\} \in G$ for any i, $0 \leq i < k$. Given two elements a, b of E one can define their distance $d(a, b)$ by letting it equal ∞ if there is no such sequence $(e_0, \ldots, e_k)$, and letting $d(a, b) = k$ if such a sequence exists and k is minimal for this property.

An unoriented graph T is called a *spanning tree* on E iff $\Pi(T) = \{E\}$ and there is no sequence $(e_1, \ldots, e_k)$ of distinct elements in E, with $k \geq 3$, and all of $\{e_1, e_2\}$, $\{e_2, e_3\}, \ldots$, $\{e_{k-1}, e_k\}$, $\{e_k, e_1\}$ in T. If e_* is a privileged element in E, called the *root*, one can canonically associate to T an oriented graph T^o by letting $(a, b) \in T^o$ iff $\{a, b\} \in T$ and $d(e_*, a) < d(e_*, b)$, i.e., the chosen orientation is *away* from the root.

An order relation on E is viewed as a subset of $E \times E$. Given another finite set F, the set of maps from E to F is denoted by $\mathrm{Maps(E, F)}$.

We now proceed with the definitions that are specific to our subject matter; these are the notions of *canvas* and *precanvas* on a finite set E, and are examples of *combinatorial species* in the sense of Joyal [Joy81, BLL98, Abd04]. Throughout we let $\mathcal{S}$, called the *state space* be a fixed finite set one can think of as a subset of a lattice $\mathbb{Z}^d$. Given a finite set E, a precanvas on E is any element $P = (e_{\mathrm{in}}, e_{\mathrm{out}}, \mathcal{G}, \phi)$ of $E \times E \times \mathcal{P}(E \times E) \times \mathrm{Maps}(E, \mathcal{S})$, satisfying the following axiom.

(PC) :

- $\forall e \in E$, $(e, e_{\mathrm{in}}) \notin \mathcal{G}$ and $(e_{\mathrm{out}}, e) \notin \mathcal{G}$.
- $\forall e \in E \backslash \{e_{\mathrm{in}}\}$, $\exists ! e' \in E$, $(e', e) \in \mathcal{G}$.
- $\forall e \in E \backslash \{e_{\mathrm{out}}\}$, $\exists ! e' \in E$, $(e, e') \in \mathcal{G}$.

The set of precanvases on E is denoted by $Precanvas(E)$. The correspondence $E \to Precanvas(E)$ is a covariant endofunctor of the groupoid category of finite sets with morphisms given by bijections. Indeed if $\sigma : E \to F$ is a bijection and $P = (e_{\mathrm{in}}, e_{\mathrm{out}}, \mathcal{G}, \phi)$ is a precanvas on E, one can transport the latter via σ in order to obtain a precanvas on F denoted by $Precanvas(\sigma)(P)$ and equal to $(\sigma(e_{\mathrm{in}}), \sigma(e_{\mathrm{out}}), \mathcal{G}^{\sigma}, \phi \circ \sigma^{-1})$ where

$$\mathcal{G}^{\sigma} \stackrel{\mathrm{def}}{=} \left\{(\sigma(a), \sigma(b)) \in F^2 | (a, b) \in \mathcal{G}\right\} \tag{8}$$

One has the following trivial lemma.

Lemma 2.1. *There is a unique component $E_{\mathrm{ch}} \in \Pi(\mathcal{G}^u)$ containing both e_{in} and e_{out}.*

The component E_{ch} is called the *chain* of P and the other components are called the *loops* of P.

We can move on to the notion of canvas on E. This is any element $C = (P, \mathcal{M}, \mathcal{O})$ of

$$Precanvas(E) \times \mathcal{P}\left(\mathcal{P}_2(E)\right) \times \mathcal{P}\left(\mathcal{P}_2(E) \times \mathcal{P}_2(E)\right)$$

satisfying axioms **C1**-**C4** to be stated below. The *Mayer tree* $\mathcal{M}$ is required to satisfy

(C1) :

- $\forall A \in \Pi(\mathcal{G}^u)$, $\forall a, b \in A$, $\{a, b\} \notin \mathcal{M}$.
- $\forall A_1, A_2 \in \Pi(\mathcal{G}^u)$, with $A_1 \neq A_2$, there is at most one $l \in \mathcal{M}$ with $l \cap A_1 \neq \emptyset$ and $l \cap A_2 \neq \emptyset$.

Define the graph $\bar{\mathcal{M}}$ *induced* by $\mathcal{M}$ on $\Pi(\mathcal{G}^u)$ by letting $\{A_1, A_2\} \in \bar{\mathcal{M}}$ iff A_1, A_2 are distinct elements of $\Pi(\mathcal{G}^u)$ for which there exists $a_1 \in A_1$ and $a_2 \in A_2$ such that $\{a_1, a_2\} \in \mathcal{M}$.

We also require the axiom

(C2) :

- $\bar{\mathcal{M}}$ is a spanning tree on $\Pi(\mathcal{G}^u)$.

We let the chain $E_{\rm ch}$ be the root of the tree $\bar{\mathcal{M}}$; and once again orient the edges away from it. This allows to also orient the underlying graph $\mathcal{M}$, i.e., one defines $\mathcal{M}^o$ to be the set of pairs $(a,b) \in E \times E$ such that $\{a,b\} \in \mathcal{M}$ and $(A,B) \in (\bar{\mathcal{M}})^o$ where A is the connected component of a and B is that of b with respect to the graph $\mathcal{G}^u$. We now require the next axiom.

(C3) :

- One cannot have two edges $l_1 = \{a,b\}$ and $l_2 = \{b,c\}$ in $\mathcal{M}$ with l_1 oriented from a to b and l_2 oriented from b to c.

Now one defines a binary relation denoted by $\sim$ on the Mayer tree $\mathcal{M}$. Two edges l_1, l_2 in $\mathcal{M}$ are said to be *adjacent*, i.e., satisfy $l_1 \sim l_2$, iff $l_1 \cap l_2 \neq \emptyset$. It is easy to deduce from (**C2**) and (**C3**) the following.

Lemma 2.2. *The adjacency relation $\sim$ is an equivalence relation on $\mathcal{M}$.*

Denote the set of equivalence classes for $\sim$ by $\mathrm{Ad}(\mathcal{M})$. The remaining item $\mathcal{O}$ is a *partial order* on $\mathcal{M}$ denoted by $\preceq_{\mathcal{O}}$. The last axiom needed in the definition of a canvas is the following.

(C4) :

- If two edges $l_1, l_2 \in \mathcal{M}$ belong to different equivalence classes in $\mathrm{Ad}(\mathcal{M})$, l_1 and l_2 are not comparable with respect to the partial order $\mathcal{O}$.
- Within each class in $\mathrm{Ad}(\mathcal{M})$, $\mathcal{O}$ is a *total* order.

Given a finite set E, the set of canvases on E which by definition satisfy **C1–C4**, is denoted by $Canvas(E)$. Again this produces an endofunctor $E \to Canvas(E)$ for the category of finite sets with bijections. The definition of the canvas $Canvas(\sigma)(C)$ on F obtained from a canvas C on E via the bijection $\sigma : E \to F$ is straightforward and left to the reader. As customary in the theory of combinatorial species, given two pairs (E,P) and (E',P') made of a finite set and a precanvas on the latter, one says that such pairs are equivalent iff there exists a bijection $\sigma : E \to E'$ for which $P' = Precanvas(\sigma)(P)$. Besides, one has the notion of automorphism group of a pair (E,P). It is the group $Aut(E,P)$ of bijections $\sigma : E \to E$ such that $Precanvas(\sigma)(P) = P$. One also has the analogous definitions for canvases.

At this point it is appropriate to use a planar representation for canvases. An edge $(a,b) \in \mathcal{G}$ is represented by a solid straight arrow joining the nodes corresponding to the elements a and b of E. An edge $\{a,b\} \in \mathcal{M}$ is represented by a squiggly line

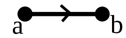

FIGURE 1

FIGURE 2

on which one can put a directional arrow

FIGURE 3

pointing away from the root $E_{\rm ch}$ in $\Pi(\mathcal{G}^u)$. Now axiom **C3** means that a situation such as

FIGURE 4

is *forbidden.* The only way some squiggly lines can share a common node is if they are all oriented away from it. The collection of such lines pointing away from some node corresponds to an equivalence class in $\mathrm{Ad}(\mathcal{M})$. The total ordering within this class, due to $\mathcal{O}$, is indicated by a clockwise rotating arrow

An example of canvas on the finite set

$$E = \{a, b, c, d, e, f, g, h, 1, 2, 3, 4, 5, 6, 7, 8, 9, \spadesuit, \heartsuit, \clubsuit, \Delta, \nabla\}$$

with $\#(E) = 22$, is (apart from the map ϕ which needs to be specified separately) faithfully described by the next picture.

The conventions of planar representation we use are such that the chain $E_{\rm ch}$ is drawn at the bottom going from left to right; the loops are drawn above it and oriented clockwise; and the squiggly lines are oriented upwards away from the root $E_{\rm ch}$. In this example, $e_{\rm in} = c$, $e_{\rm out} = \spadesuit$, the oriented graph is given by

$$\mathcal{G} = \{(c, b), (b, 5), (5, a), (a, 3), (3, \spadesuit), (\heartsuit, \heartsuit),$$
$$(d, 1), (1, d), (2, e), (e, 4), (4, \clubsuit), (\clubsuit, 2), (9, 9),$$
$$(\nabla, \nabla), (6, g), (g, \Delta), (\Delta, 7), (7, 6), (f, 8), (8, h), (h, f)\}$$

The Mayer tree is

$$\mathcal{M} = \{\{5, \heartsuit\}, \{d, 3\}, \{6, \spadesuit\}, \{\spadesuit, f\}, \{1, 2\}, \{9, 4\}, \{\nabla, 4\}\}$$

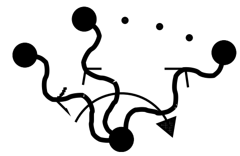

FIGURE 5

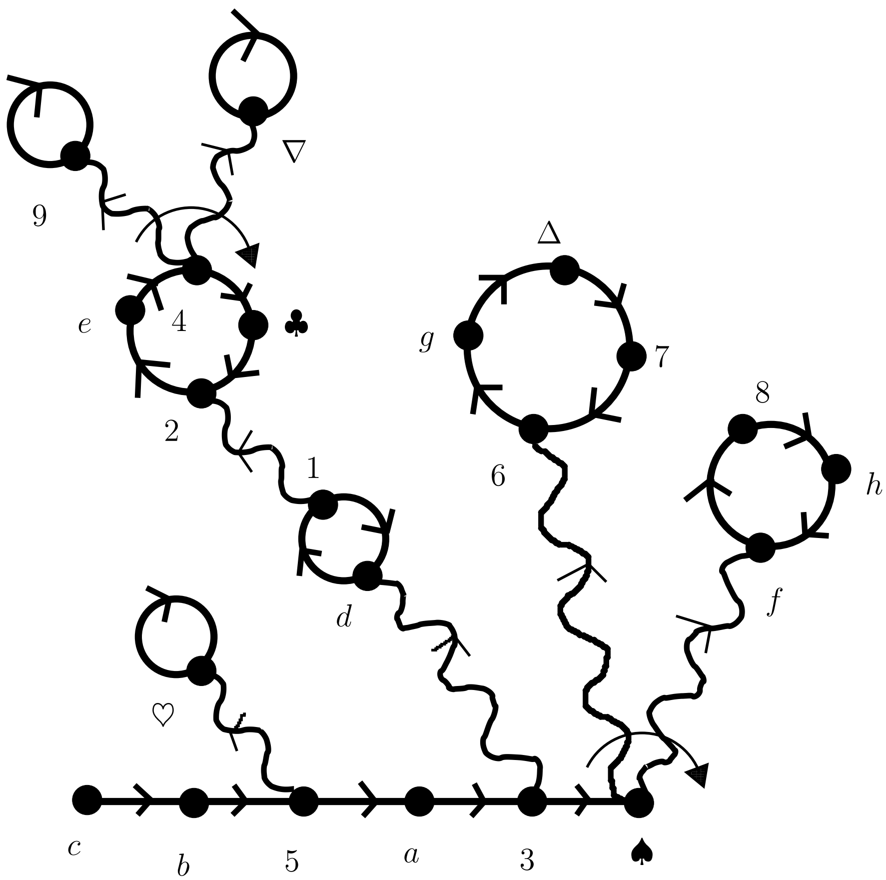

FIGURE 6

which has three trivial adjacency classes in $\mathrm{Ad}(\mathcal{M})$ and two nontrivial ones:

$$A_1 = \{\{6, \spadesuit\}, \{\spadesuit, f\}\} \qquad \text{and} \qquad A_2 = \{\{9, 4\}, \{\nabla, 4\}\}$$

The ordering $\mathcal{O}$ corresponds to the total ordering of A_1 by $\{6, \spadesuit\} \prec_{\mathcal{O}} \{\spadesuit, f\}$, and of A_2 by $\{9, 4\} \prec_{\mathcal{O}} \{\nabla, 4\}$.

We can now define a *canonical total order* on E which is associated to a canvas $C \in Canvas(E)$. One starts with e_{in}, as a smallest element, and turns around the tree and counts vertices $x \in E$ as encountered in this order provided one reaches them by following a straight arrow of $\mathcal{G}$ rather than a squiggly line of $\mathcal{M}$. On the previous example the canonical total order corresponds to the succession:

$$c, b, 5, \heartsuit, a, 3, 1, e, 4, 9, \nabla, \clubsuit, 2, d, \spadesuit, g, \Delta, 7, 6, 8, h, f$$

Remark 2.3. *One always starts with* $e_{\rm in}$*; but does not necessarily finish with* $e_{\rm out}$*. This depends on wherther or not squiggly lines are attached to* $e_{\rm out}$*.*

3. Loop-erasure Combinatorial Coefficients

The previous considerations did not involve the map ϕ from the finite set E, also called the *label set*, to the state space $\mathcal{S}$. This map will now play an important role in the definition of some combinatorial coefficients associated to canvases and precanvases. Let E be a finite set of cardinality N and C be a canvas on E. We will use the notations of the previous section. We will firstly classify the $\frac{N(N-1)}{2}$ unordered pairs $l \in \mathcal{P}_2(E)$ into four disjoint categories.

(1) **Intralinks :** These are the pairs $l \in \mathcal{P}_2(E)$ for which there exists $A \in \Pi(\mathcal{G}^u)$ such that $l \subset A$.
(2) **Mayer interlinks :** These are the elements of $\mathcal{M}$.
(3) **Hard interlinks :** These are the pairs $l \in \mathcal{P}_2(E)\backslash\mathcal{M}$, which can be written $l = \{a, b\}$, with a *preceding* b in the canonical total order of E given by the canvas C and satisfying the following property. Let $A, B \in \Pi(\mathcal{G}^u)$ be the components of a and b respectively. We impose that $A \neq B$ (so as to rule out intralinks) and that B be a *descendant* of A in the tree $\bar{\mathcal{M}}$ rooted at $E_{\rm ch}$.
(4) **Void interlinks :** These are the pairs $l \in \mathcal{P}_2(E)$ which do not fall in any of the three previous categories.

As an illustration of these definitions, let us consider the example of Figure 6, for which the following hold.

- The pairs $\{c, a\}$, $\{e, \clubsuit\}$, $\{6, g\}$ are examples of intralinks.
- The pairs $\{9, 4\}$, $\{\nabla, 4\}$ are examples of Mayer interlinks (the squiggly lines).
- The pairs $\{e, 9\}$, $\{c, \nabla\}$, $\{5, g\}$, $\{1, \nabla\}$ are examples of hard interlinks.
- The pairs $\{9, \clubsuit\}$, $\{\heartsuit, a\}$, $\{1, g\}$, $\{\nabla, f\}$ are examples of void interlinks.

Now let $\delta : \mathcal{S} \times \mathcal{S} \to \mathbb{N}$ be the delta function on the state space defined by

$$\delta(s, s') \stackrel{\text{def}}{=} \begin{cases} 1 \text{ if } s = s' \\ 0 \text{ otherwise} \end{cases}$$

For any $l \in \mathcal{P}_2(E)$ we define the quantity $\omega(C, l)$ by the following rule. Let $l = \{a, b\}$ for some distinct elements a, b of E. We impose

$$\omega(C, l) \stackrel{\text{def}}{=} \begin{cases} 1 - \delta(\phi(a), \phi(b)) & \text{if} \quad l \text{ is an intralink} \\ -\delta(\phi(a), \phi(b)) & \text{if} \quad l \text{ is a Mayer interlink} \\ 1 - \delta(\phi(a), \phi(b)) & \text{if} \quad l \text{ is a hard interlink} \\ 1 & \text{if} \quad l \text{ is a void interlink} \end{cases} \tag{9}$$

Now we definte the *loop-erasure coefficient* of a canvas C on a finite set E by

$$\tilde{\psi}_{\mathrm{LE}}(E, C) \stackrel{\text{def}}{=} \prod_{l \in \mathcal{P}_2(E)} \omega(C, l) \tag{10}$$

This trivially only depends on the equivalence class of the pair (E, C). Now given a precanvas P on E we also define its *loop-erasure coefficient* by

$$\psi_{\mathrm{LE}}(E, P) \stackrel{\text{def}}{=} \sum_{C \text{ over } P} \tilde{\psi}_{\mathrm{LE}}(E, C) \tag{11}$$

where the sum is over canvases $C = (P, \mathcal{M}, \mathcal{O})$ where the precanvas P is the given one. Therefore the summation is only on the Mayer tree $\mathcal{M}$ and the ordering $\mathcal{O}$. Once again $\psi_{\mathrm{LE}}(E, P)$ only depends on the equivalence class of the pair with respect to transport of structure.

4. Mayer combinatorial coefficients

Let P be a precanvas on a finite set E; this defines a partition $\Pi(\mathcal{G}^u) = \{E_{\mathrm{ch}}, L_1, \ldots, L_r\}$ of E. We now define the *Mayer coefficient* $\psi_{\mathrm{M}}(E, P)$ of the precanvas P as follows.

If there is two elements $a \neq b$ in the same component of $\Pi(\mathcal{G}^u)$ with $\phi(a) = \phi(b)$, we let $\psi_{\mathrm{M}}(E, P) \stackrel{\text{def}}{=} 0$.

Otherwise, the map ϕ restricts to an injection into $\mathcal{S}$, within each component of $\Pi(\mathcal{G}^u)$. In this case consider the images by ϕ of these components $Y_0 = \phi(E_{\mathrm{ch}})$, $Y_1 = \phi(L_1)$,$\ldots Y_r = \phi(L_r)$, which we call *polymers* in the state space $\mathcal{S}$. We then let

$$\psi_{\mathrm{M}}(E, P) \stackrel{\text{def}}{=} \psi(Y_0, Y_1, \ldots, Y_r) \tag{12}$$

the standard Mayer expansion coefficient of the collection of polymers $(Y_0, Y_1, \ldots, Y_r)$ (see e.g. [Bry86, Riv91, Abd97]). One has the well-known expression

$$\psi(Y_0, Y_1, \ldots, Y_r) = \sum_G \prod_{\{i,j\} \in G} \left(-\mathbb{1}_{\{Y_i \cap Y_j \neq \emptyset\}}\right) \tag{13}$$

Here the sum is over all unoriented graphs G on the finite set of polymer indices $\{0, 1, \ldots, r\}$ which entirely connect the latter, i.e., such that $\Pi(G) = \{\{0, 1, \ldots, r\}\}$. The product is over all unordered pairs $\{i, j\}$ in G, and $\mathbb{1}_{\{\cdots\}}$ denotes the characteristic function of the condition between braces. One can readily rewrite the Mayer coefficient as a sum of unoriented graphs on the underlying set E. Indeed, if one writes $A_0 = E_{\mathrm{ch}}$, $A_1 = L_1, \ldots, A_r = L_r$, so that $Y_i = \phi(A_i)$ for all i, $1 \leq i \leq r$; one has

$$\begin{aligned} \mathbb{1}_{\{Y_i \cap Y_j \neq \emptyset\}} &= 1 - \mathbb{1}_{\{Y_i \cap Y_j = \emptyset\}} & (14) \\ &= 1 - \prod_{a \in Y_i} \prod_{b \in Y_j} (1 - \delta(\phi(a), \phi(b))) & (15) \end{aligned}$$

By replacing the characteristic functions in (13) by the last expression and expanding all the products one easily obtains

$$\psi_{\mathrm{M}}(E, P) = \sum_H \prod_{\{a,b\} \in H} (-\delta(\phi(a), \phi(b))) \tag{16}$$

where the sum is over all unoriented graphs H on E, only made of *interlinks* (i.e., no edge of H is contained in a block of $\Pi(\mathcal{G}^u)$), and such that H and $\mathcal{G}^u$ together connect E, i.e., $\Pi(H \cup \mathcal{G}^u) = \{E\}$.

In order to have a formula which is valid in all cases, including the previously considered one when there are elements $a \neq b$ in the same component of $\Pi(\mathcal{G}^u)$ with $\phi(a) = \phi(b)$, we rewrite (16) as

$$\psi_{\mathrm{M}}(E, P) = \prod_{\substack{\{a,b\} \\ \text{intralink}}} (1 - \delta(\phi(a), \phi(b))) \times \sum_H \prod_{\{a,b\} \in H} (-\delta(\phi(a), \phi(b))) \tag{17}$$

where the nomenclature of intralink is the same as in Section 3.

5. The key theorem and the loop-insertion recursion

We can now state the key theorem of this article

Theorem 5.1. *For any precanvas P on a finite set E*

$$\psi_{\mathrm{LE}}(E, P) = \psi_{\mathrm{M}}(E, P) \tag{18}$$

The proof is by induction on the number of loops and proceeds by showing that both coefficients satisfy the same loop-insertion recursion. The start of the induction, or the equality when no loops are present is trivial. Indeed, in this case $E = E_{\rm ch}$ and the only possible canvas is such that $\mathcal{M} = \emptyset$ and $\mathcal{O} = \emptyset$. All $l \in \mathcal{P}_2(E)$ are the intralinks and

$$\psi_{\rm LE}(E, P) = \prod_{\{a,b\}\in\mathcal{P}_2(E)} (1 - \delta(\phi(a), \phi(b)))$$

Likewise,

$$\psi_{\rm M}(E, P) = \prod_{\{a,b\}\in\mathcal{P}_2(E)} (1 - \delta(\phi(a), \phi(b)))$$

since the graph H has to be empty. As a result one has the following.

Lemma 5.2. *If there are no loops in the precanvas P on E*

$$\psi_{\rm LE}(E, P) = \psi_{\rm M}(E, P) \tag{19}$$

Now let us return to the general case and let us consider a precanvas P on a finite set E. Given a subset A of E we let

$$\rho(A) \stackrel{\rm def}{=} \prod_{\{a,a'\}\in\mathcal{P}_2(A)} (1 - \delta(\phi(a), \phi(a')))$$

Given another subset B such that $A \cap B = \emptyset$ we denote

$$\rho(A, B) \stackrel{\rm def}{=} \prod_{a\in A}\prod_{b\in B} (1 - \delta(\phi(a), \phi(b)))$$

Now let $e_1, e_2, \ldots, e_p$, with $p \geq 1$, be a numbering of the chain $E_{\rm ch}$ in the precanvas P in such a way that $e_1 = e_{\rm in}$, $e_p = e_{\rm out}$ and $(e_i, e_{i+1}) \in \mathcal{G}$ for any i, $1 \leq i < p$.

If x is an element of $E_{\rm ch}$; then $x = e_i$ for a unique i, $1 \leq i \leq p$, and we can therefore define the subsets

$$E_{\rm ch}^{\leq x} \stackrel{\rm def}{=} \{e_1, e_2, \ldots, e_i\}$$

and

$$E_{\rm ch}^{>x} \stackrel{\rm def}{=} \{e_{i+1}, e_{i+2}, \ldots, e_p\}$$

If y is an element of $E\backslash E_{\rm ch}$; then there is a unique loop $L \in \Pi(\mathcal{G}^u)$ containing y. The elements of this loop can be numbered as $f_1, f_2, \ldots, f_q$, with $q \geq 1$, in such a way that $f_1 = y$, and (f_1, f_2), (f_2, f_3),$\ldots$,(f_{q-1}, f_q), (f_q, f_1) belong to the graph $\mathcal{G}$ (the list reduces to (f_1, f_1) if $q = 1$).

The points x and y allow one to canonically define a new finite set $\tilde{E}_{x,y}$ and a new precanvas $\tilde{P}_{x,y}$ on it by a loop-insertion procedure. We let

$$\tilde{E}_{x,y} \stackrel{\rm def}{=} E\backslash\left(\{y\} \cup E_{\rm ch}^{>x}\right) \tag{20}$$

and $\tilde{P}_{x,y} \stackrel{\mathrm{def}}{=} (\tilde{e}_{\mathrm{in}}, \tilde{e}_{\mathrm{out}}, \tilde{\mathcal{G}}, \tilde{\phi})$ where $\tilde{e}_{\mathrm{in}} \stackrel{\mathrm{def}}{=} e_{\mathrm{in}}$,

$$\tilde{e}_{\mathrm{out}} \stackrel{\mathrm{def}}{=} \begin{cases} x & \text{if} \quad q = 1 \\ f_q & \text{if} \quad q > 1 \end{cases} \tag{21}$$

and

$$\tilde{\mathcal{G}} \stackrel{\mathrm{def}}{=} \begin{cases} \mathcal{G} \cap (\tilde{E}_{x,y} \times \tilde{E}_{x,y}) & \text{if} \quad q = 1 \\ \{(x, f_2)\} \cup \Big(\mathcal{G} \cap (\tilde{E}_{x,y} \times \tilde{E}_{x,y}) \Big) & \text{if} \quad q > 1 \end{cases} \tag{22}$$

and $\tilde{\phi} \stackrel{\mathrm{def}}{=} \phi|_{\tilde{E}_{x,y}}$. It is visually obvious, although a bit tedious to formally check in both cases $q = 1$ and $q > 1$, that **PC** holds for $\tilde{P}_{x,y}$; so it is indeed a precanvas on $\tilde{E}_{x,y}$.

Suppose $\psi(E, P)$ is a quantity associated to finite sets equiped with a precanvas. The focus of the next sections is the following identity.
Loop-insertion recursion :

$$\psi(E,P) = \sum_{x \in E_{\mathrm{ch}}} \sum_{y \in E \backslash E_{\mathrm{ch}}} [-\delta(\phi(x), \phi(y))]\, \rho(E_{\mathrm{ch}}^{>x}) \rho(E_{\mathrm{ch}}^{\le x}, E_{\mathrm{ch}}^{>x}) \psi(\tilde{E}_{x,y}, \tilde{P}_{x,y}) \tag{23}$$

We will show in Section 6 that the loop-erasure coefficients ψ_{LE} satisfy this recursion; and we will do the same for the Mayer coefficient ψ_{M} in Section 7. Since the precanvas $\tilde{P}_{x,y}$ clearly has one loop less than P; an easy induction on the number of loops, together with Lemma 5.2 will finally establish Theorem 5.1.

6. The loop-insertion recursion for the loop-erasure coefficients

Let C be a canvas with a nonzero number of loops then one can uniquely define an element $x(C)$ of E_{ch} and an element $y(C)$ of $E \backslash E_{\mathrm{ch}}$ in the following manner. Let again $e_1, e_2, \ldots, e_p$, with $p \ge 1$, be a numbering of the chain E_{ch} in the precanvas P in such a way that $e_1 = e_{\mathrm{in}}$, $e_p = e_{\mathrm{out}}$ and $(e_i, e_{i+1}) \in \mathcal{G}$ for any i, $1 \le i < p$. By axiom **C2**, there must exist an i, $1 \le i \le p$, such that e_i belongs to some edge $l \in \mathcal{M}$. If $i_{\max}$ is the largest index with this property we let $x(C) \stackrel{\mathrm{def}}{=} e_{i_{\max}}$. Now let $A_{\max} \in \mathrm{Ad}(\mathcal{M})$ be the adjacency class of edges in $\mathcal{M}$ emanating from $x(C)$; and let $l_{\max}$ be the greatest element in this class for the ordering $\mathcal{O}$. By definition, $y(C)$ is the element of $l_{\max}$ other than $x(C)$.

We will organize the sum in (11) according to the values of $x(C)$ and $y(C)$:

$$\psi_{\mathrm{LE}}(E,P) = \sum_{x\in E_{\mathrm{ch}}} \sum_{y\in E\backslash E_{\mathrm{ch}}} \sum_{C\in\mathcal{C}_{x,y}} \tilde{\psi}_{\mathrm{LE}}(E,C) \tag{24}$$

where $\mathcal{C}_{x,y}$ is the set of canvases C on E, built on the fixed precanvas P, and such that $x(C)=x$ and $y(C)=y$. We need to carefully analyse the product $\tilde{\psi}_{\mathrm{LE}}(E,C)$ in order to factor it into different pieces, one of which should be an analogous product for a canvas $\tilde{C}_{x,y}$ on $\tilde{E}_{x,y}$ built on $\tilde{P}_{x,y}$. By definition, $\tilde{C}_{x,y} = (\tilde{P}_{x,y}, \tilde{\mathcal{M}}, \tilde{\mathcal{O}})$ where

$$\tilde{\mathcal{M}} \stackrel{\mathrm{def}}{=} \mathcal{M}\backslash\{\{x,y\}\} \tag{25}$$

and $\tilde{\mathcal{O}}$ is the partial order

$$\tilde{\mathcal{O}} \stackrel{\mathrm{def}}{=} \mathcal{O}\cap(\mathcal{M}\times\mathcal{M}) \tag{26}$$

Although a bit tedious, checking the axioms **C1-C4** for $\tilde{C}_{x,y}$ offers no difficulty. It might be worth mentioning that the requirement $\mathcal{M}\in\mathcal{P}(\mathcal{P}_2(\tilde{E}_{x,y}))$ rests on the definition of $x=x(C)$ which implies that no edge of $\mathcal{M}$ touches $E_{\mathrm{ch}}^{>x}$, and on the axiom **C3** for the canvas C. Likewise, the connectedness requirement in **C2** for $\tilde{C}_{x,y}$ needs the statement **C3** for C. The latter ensures that if $L(y)$ is the loop of C containing y; no other loop is attached to $L(y)$ by a Mayer link hooked precisely at y. In the end, one gets $\tilde{C}_{x,y}\in Canvas(\tilde{E}_{x,y})$. We now have the following.

Lemma 6.1. *For every $C\in\mathcal{C}_{x,y}$ one has the identity*

$$\tilde{\psi}_{\mathrm{LE}}(E,C) = \left[-\delta(\phi(x),\phi(y))\right]\rho(E_{\mathrm{ch}}^{>x})\rho(E_{\mathrm{ch}}^{\le x},E_{\mathrm{ch}}^{>x})\tilde{\psi}_{\mathrm{LE}}(\tilde{E}_{x,y},\tilde{P}_{x,y}) \tag{27}$$

Proof. We need to consider many cases for the pairs $l\in\mathcal{P}_2(E)$. In order to adequately transform some of the factors involved we will also need the trivial identities

$$\delta(\phi(a),\phi(b))\,(1-\delta(\phi(a),\phi(c))) = \delta(\phi(a),\phi(b))\,(1-\delta(\phi(b),\phi(c))) \tag{28}$$

and

$$(1-\delta(\phi(a),\phi(b)))^2 = (1-\delta(\phi(a),\phi(b))) \tag{29}$$

We need some more notation. Recall that the loop of C (or rather P) containing y is denoted by $L(y)$. We let $U \stackrel{\mathrm{def}}{=} L(y)\backslash\{y\}$. We let V be the union of loops, in C, which branch off $L(y)$. Note that by axiom **C3**, $V=\emptyset$ if $q=1$, i.e. if $U=\emptyset$. Finally we let W be the union of loops, in C, which branch off $E_{\mathrm{ch}}^{\le x}$. Note that E is the disjoint union of $E_{\mathrm{ch}}^{<x}$, $\{x\}$, $E_{\mathrm{ch}}^{>x}$, $\{y\}$, U, V, W. The cases to be considered are as follows.

(1) $\underline{l = \{x, y\}}$: By construction l is a Mayer interlink of C; therefore

$$\omega(C, l) = -\delta(\phi(x), \phi(y))$$

which is the first factor in the right hand side of (27). Note that this factor play an important role, somewhat as a catalyst, in order to transform other factors $\omega(C, l)$ via (28).

(2) $\underline{l \subset E_{\text{ch}}^{>x}}$: Then l is an intralink of C. If $l = \{a, b\}$ then

$$\omega(C, l) = 1 - \delta(\phi(a), \phi(b))$$

The collection of such factors make up $\rho(E_{\text{ch}}^{>x})$ in the right hand side of (27).

(3) $\underline{l \text{ between } E_{\text{ch}}^{\le x} \text{ and } E_{\text{ch}}^{>x}}$: Then l is an intralink of C, and if $l = \{a, b\}$,

$$\omega(C, l) = 1 - \delta(\phi(a), \phi(b))$$

The collection of such factors make up $\rho(E_{\text{ch}}^{\le x}, E_{\text{ch}}^{>x})$ in the right hand side of (27).

(4) $\underline{l \text{ between } \tilde{E}_{x,y} \backslash E_{\text{ch}}^{\le x} \text{ and } E_{\text{ch}}^{>x}}$: It is easy to see that such l's are void interlinks of C; because the elements of $E_{\text{ch}}^{>x}$ come *after* those of $\tilde{E}_{x,y}$ in the canonical total order of E associated to C. Besides, any element of $\tilde{E}_{x,y} \backslash E_{\text{ch}}^{\le x}$ belongs to a loop which automatically is a descendent of the root E_{ch} which contains $E_{\text{ch}}^{>x}$. The $\omega(C, l)$'s in this case are all equal to 1.

(5) $\underline{l \text{ between } y \text{ and } E_{\text{ch}}^{>x}}$: These are all void interlinks of C, for the same reasons as for the previous case. They contribute a factor of 1.

(6) $\underline{l \text{ is between } y \text{ and } V \cup W}$: Let $l = \{y, a\}$, with $a \in V \cup W$. If $a \in W$ then neither of the components of a and y descends from the other, and therefore l cannot be a hard intralink of C. It cannot be an intralink because $L(y) \cap W = \emptyset$ and neither a Mayer link because of the absence of descendence relation between the relevant components. As a result l is a void interlink of C. Now if $a \in V$ then the component of a descends from that of y (i.e., the loop $L(y)$); but y *follows* a in the canonical order so l cannot be a hard interlink of C. It cannot be an intralink because $L(y) \cap V = \emptyset$ and neither a Mayer link because of axiom **C3** for C. As a result l is a void interlink of C. In sum the factors in this case are all equal to 1.

(7) $\underline{l \text{ between } y \text{ and } U}$: If $l = \{y, a\}$ with $a \in L(y) \backslash \{y\}$; l is an intralink of C and

$$\omega(C, l) = 1 - \delta(\phi(y), \phi(a))$$

This is rewritten thanks to the factor $\delta(\phi(x),\phi(y))$ from case 1), using identity (28), as

$$1-\delta(\phi(x),\phi(a))$$

to be later absorbed by identity (29) in case 9) below.

(8) $\underline{l \text{ between } y \text{ and } E_{\rm ch}^{<x}}$: If $l=\{y,a\}$ with $a\in E_{\rm ch}^{<x}$ then clearly a precedes y in the canonical order, and the component $L(y)$ of y descends from the component $E_{\rm ch}$ of a. Therefore l is a hard interlink of C. It contributes a factor

$$\omega(C,l)=1-\delta(\phi(y),\phi(a))$$

which is rewritten as $1-\delta(\phi(x),\phi(a))$ via (28) before absorption via (29) in case 9) below.

(9) $\underline{l\subset \tilde{E}_{x,y} \text{ and intralink for } \tilde{C}_{x,y}}$: Then $l=\{a,b\}$ is also an intralink for C except when l is between $E_{\rm ch}^{\le x}$ and U in which case l is a hard interlink of C. In both situations

$$\omega(C,l)=\omega(\tilde{C}_{x,y},l)=1-\delta(\phi(a),\phi(b))$$

If l is between x and $a\in U$ one gets the factor $1-\delta(\phi(x),\phi(a))$ *twice*. Once is from the present case, and once is from case 7); both are combined into one such factor equal to $\omega(\tilde{C}_{x,y},l)$ using (29). If l is between x and $a\in E_{\rm ch}^{<x}$ one does the same to combine the present factor $1-\delta(\phi(x),\phi(a))$ with the one produced by case 8) in order to end up with a single factor $\omega(\tilde{C}_{x,y},l)$.

(10) $\underline{l\subset \tilde{E}_{x,y} \text{ and Mayer interlink for } \tilde{C}_{x,y}}$: Then l is also a Mayer interlink for C and $\omega(C,l)=\omega(\tilde{C}_{x,y},l)$.

(11) $\underline{l\subset \tilde{E}_{x,y} \text{ and hard interlink for } \tilde{C}_{x,y}}$: Then l is also a hard interlink of C and $\omega(C,l)=\omega(\tilde{C}_{x,y},l)$. This follows from two easily checked facts. The canonical total order on $\tilde{E}_{x,y}$ associated to $\tilde{C}_{x,y}$ is the restriction of that on E associated to C. Descendence is preserved when going from the Mayer tree of C to that of $\tilde{C}_{x,y}$.

(12) $\underline{l\subset \tilde{E}_{x,y} \text{ and void interlink for } \tilde{C}_{x,y}}$: Then l is also a void interlink of C for the same reasons as in the previous case.

In the end, the collected factors over all $l\in\mathcal{P}_2(E)$ reproduce the right hand side of (27). $\square$

Now one has

Lemma 6.2.

$$\sum_{C\in\mathcal{C}_{x,y}} \tilde{\psi}_{\mathrm{LE}}(\tilde{E}_{x,y}, \tilde{C}_{x,y}) = \psi_{\mathrm{LE}}(\tilde{E}_{x,y}, \tilde{P}_{x,y}) \tag{30}$$

PROOF. This is because the map $C \mapsto \tilde{C}_{x,y}$ is a bijection from $\mathcal{C}_{x,y}$ onto the set of canvases on $\tilde{E}_{x,y}$ built over $\tilde{P}_{x,y}$. The injectivity follows easily from the definitions of $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{O}}$. For the surjectivity, given a canvas $\tilde{C} = (\tilde{P}_{x,y}, \tilde{\mathcal{M}}, \tilde{\mathcal{O}})$ one lets

$$\mathcal{M} \stackrel{\mathrm{def}}{=} \tilde{\mathcal{M}} \cup \{\{x, y\}\}$$

and

$$\mathcal{O} \stackrel{\mathrm{def}}{=} \tilde{\mathcal{O}} \cup (\cup_l \{(l, \{x, y\})\})$$

where the union is over l's in the adjacency class of Mayer links emanating from x in the canvas $\tilde{C}$. The verification of the canvas axioms for $C = (P, \mathcal{M}, \mathcal{O})$, as well as $x(C) = x$, and $y(C) = y$, although tedious offers no difficulty. □

We have finally proved

Proposition 6.3. *The loop-erasure coefficients ψ_{LE} satisfies the loop-insertion recursion.*

7. The loop-insertion recursion for the Mayer coefficients

(Sketch) One organizes the sum in (17), as follows. First release the condition that H is made of interlinks; which is made possible by the prefactor

$$\prod_{\substack{\{a,b\}\\ \mathrm{intralink}}} (1 - \delta(\phi(a), \phi(b)))$$

When loops are present, there must be links $l \in H$ attached to a point e_i of the chain E_{ch}. We take $x(H) \stackrel{\mathrm{def}}{=} e_{i_{\max}}$ the point with the highest index i, for this property. Then one considers the connected component $K(H) \stackrel{\mathrm{def}}{=} \{x, y_1, \ldots, y_s\}$ of the point x in $\Pi(H)$, i.e., one only takes into account the edges of H, not those of $\mathcal{G}^u$. Then H is the disjoint union of H_K and H_{K^c} where H_K is the set of edges of H contained in K and $H_{K^c} \stackrel{\mathrm{def}}{=} H \backslash H_K$. One then rewrites (17) as

$$\psi_{\mathrm{M}} = \sum_{x\in E_{\mathrm{ch}}} \sum_K \sum_H \prod_{\substack{\{a,b\}\\ \mathrm{intralink}}} (1 - \delta(\phi(a), \phi(b)))$$

$$\times \prod_{\{a,b\}\in H_K} (-\delta(\phi(a), \phi(b))) \prod_{\{a,b\}\in H_{K^c}} (-\delta(\phi(a), \phi(b)))$$

where the sum over K is over subsets of $\mathcal{P}_2(E)$ containing x such that $K\backslash\{x\}$ is nonempty and contained in $E\backslash E_{\rm ch}$; the sum over H is over unoriented graphs on E such that $H\cup\mathcal{G}^u$ connects E and $x(H)=x$ and $K(H)=K$. One can rewrite the last sum as

$$\sum_{x\in E_{\rm ch}}\sum_K\sum_{H_{K^c}}\prod_{\substack{\{a,b\}\\ \text{intralink}}}(1-\delta(\phi(a),\phi(b)))$$
$$\times\prod_{\{a,b\}\in H_{K^c}}(-\delta(\phi(a),\phi(b)))\sum_{H_K}\prod_{\{a,b\}\in H_K}(-\delta(\phi(a),\phi(b)))$$

where the last sum is over unoriented graphs H_K contained in K and connecting it; and where the sum over H_{K^c} is over graphs on E with edges not entirely in K and which together with the tree, made of the links from x to the remaining elements of K, connect E. Now

$$\begin{aligned}\Gamma(K)&\stackrel{\rm def}{=}\sum_{H_K}\prod_{\{a,b\}\in H_K}(-\delta(\phi(a),\phi(b)))\\ &=(-1)^s s!\,\mathbb{1}\{\text{all } \phi(a) \text{ with } a\in K \text{ are equal}\}\end{aligned}\tag{31}$$

by a classical Mayer coefficient calculation (the expansion of $\log(1+t)$) This can be written by interpreting the s factor in (31) as a sum over $K\backslash\{x\}=\{y_1,\ldots,y_s\}$, as

$$\sum_{H_K}\prod_{\{a,b\}\in H_K}(-\delta(\phi(a),\phi(b)))=\sum_{y\in K\backslash\{x\}}[-\delta(\phi(x),\phi(y))]\Gamma(K\backslash\{y\})$$

then reexpand $\Gamma(K\backslash\{y\})$ (some details missing here).

8. The single-loop-erasure morphism of functors

Let $C=(P,\mathcal{M},\mathcal{O})$ be a canvas on a finite set E built on the precanvas $P=(e_{\rm in},e_{\rm out},\mathcal{G},\phi)$. One can canonically associate to it a new canvas $\rho_E(C)\stackrel{\rm def}{=}(P',\mathcal{M}',\mathcal{O}')$ on the same set E with $P'\stackrel{\rm def}{=}(e'_{\rm in},e'_{\rm out},\mathcal{G}',\phi')$, as follows. As in Section 5 let $e_1=e_{\rm in}$, $e_2,\ldots,e_p=e_{\rm out}$ be the numbering of the chain $E_{\rm ch}$ following the arrows of $\mathcal{G}$. If $\phi(e_1),\ldots,\phi(e_p)$ are all distinct we let $\rho_E(C)\stackrel{\rm def}{=}C$. Else, one considers j the minimal index $1<j\le p$ for which there exists an index i, $1\le i<j$ (necessarily unique) such that $\phi(e_i)=\phi(e_j)$. We then let $e'_{\rm in}\stackrel{\rm def}{=}e_{\rm in}$. We define $e'_{\rm out}$ as equal to $e_{\rm out}$ if $j<p$ and to e_i if $j=p$. Then if $j=p$ we let

$$\mathcal{G}'\stackrel{\rm def}{=}(\mathcal{G}\backslash\{(e_i,e_{i+1})\})\cup\{(e_j,e_{i+1})\}\tag{32}$$

whereas if $j<p$ we let

$$\mathcal{G}'\stackrel{\rm def}{=}(\mathcal{G}\backslash\{(e_i,e_{i+1}),(e_j,e_{j+1})\})\cup\{(e_i,e_{j+1}),(e_j,e_{i+1})\}\tag{33}$$

Next, one always lets $\phi' \stackrel{\text{def}}{=} \phi$. Furthermore, if no edge of $\mathcal{M}$ contains e_j we let

$$\mathcal{M}' \stackrel{\text{def}}{=} \mathcal{M} \cup \{\{e_i, e_j\}\} \tag{34}$$

and define the ordering $\mathcal{O}'$ by putting the extra link $\{e_i, e_j\}$ *after* those of the adjacency class of links leaving e_i in $\mathcal{M}$. Otherwise, if the adjacency class of Mayer links emanating from e_j is nonempty and given by

$$\{e_j, a_1\} \prec_{\mathcal{O}} \ldots \prec_{\mathcal{O}} \{e_j, a_r\}$$

one lets

$$\mathcal{M}' \stackrel{\text{def}}{=} (\mathcal{M} \backslash \{\{e_j, a_1\}, \ldots, \{e_j, a_r\}\}) \cup \{\{e_i, e_j\}\} \cup \{\{e_i, a_1\}, \ldots, \{e_i, a_r\}\} \tag{35}$$

and defines $\mathcal{O}'$ by specifying the ordering of the adjacency class of links emanating from e_i in $\mathcal{M}'$ as follows:

- First, we put the links that were already present in $\mathcal{M}$ with their $\mathcal{O}$ order.
- Then we put $\{e_i, e_j\}$.
- Last, we put the links $\{e_i, a_1\}, \ldots, \{e_i, a_r\}$ in this order.

The point of this construction is that it is *functorial*. Indeed if $\sigma : E \to F$ is a bijection of finite sets one has the commutation

$$\rho_F \circ Canvas(\sigma) = Canvas(\sigma) \circ \rho_E \tag{36}$$

i.e. the *single-loop-erasure* ρ is morphism of functors from the $Canvas$ functor to itself.

A precanvas, or a canvas, is said to be *linear* iff $E = E_{\text{ch}}$.

9. The main theorem

We define a few more combinatorial species. The state or site space $\mathcal{S}$ is again fixed.

We define the specie of self-avoiding loops emmbedded in $\mathcal{S}$ given by a functor $E \to SAL(E)$ where the elements of the set $SAL(E)$ are all pairs $L = (\mathcal{G}, \phi)$ made of an oriented graph $\mathcal{G}$ on E which is the graph of a cyclic permutation of E, and an injective map $\phi : E \to \mathcal{S}$. If E is empty one lets $SAL(E) = \emptyset$ as well. Transport of structure is defined in the obvious manner. The equivalence class of a pair (E, L) where $L \in SAL(E)$ is denoted by $[E, L]$. The set of such classes is finite since $\mathcal{S}$ is finite by assumption. We suppose that a formal variable $\lambda_{[E,L]}$ is associated to each class $[E, L]$.

We now let s_{in} and s_{out} be two elements of $\mathcal{S}$ chosen once and for all. These elements are not necessarily distinct. We now define the specie of self-avoiding walks from s_{in} to s_{out} in $\mathcal{S}$ as a functor $E \to SAW(E)$.

The set $SAW(E)$ is the subset of $Precanvas(E)$ made of fourtuples $W = (e_{\rm in}, e_{\rm out}, \mathcal{G}, \phi)$ such that $\mathcal{G}$ is linear (i.e. $E = E_{\rm ch}$), $\phi(e_{\rm in}) = s_{\rm in}$, $\phi(e_{\rm out}) = s_{\rm out}$, and $\phi : E \rightarrow \mathcal{S}$ is injective. Note that if $s_{\rm in} = s_{\rm out}$ then by the injectivity of ϕ one also has $e_{\rm in} = e_{\rm out}$; and by the linearity of the graph $\mathcal{G}$ the set E has to be a singleton. We now suppose that a formal variable $\alpha_{[E,W]}$ is associated to each equivalence class $[E, W]$ for this new specie.

The main theorem will be an identity in the ring $\mathcal{R}$ of formal power series with rational coefficents in the variables $\alpha_{[E,W]}$ and $\lambda_{[E,L]}$.

Given a precanvas $P = (e_{\rm in}, e_{\rm out}, \mathcal{G}, \phi)$ on a set E, we define the amplitude $\mathcal{B}_{-}(E, P)$ in the ring $\mathcal{R}$ as follows. Let $\Pi = \{E_{\rm ch}, E_1, \ldots, E_r\}$ be the partition of E corresponding to the connected components of the unoriented graph $\mathcal{G}^u$; with $E_{\rm ch}$ corresponding to the chain and $E_1, \ldots, E_r$ to the loops. We let $\mathcal{B}_{-}(E, P) \stackrel{\rm def}{=} 0$ unless $\phi(e_{\rm in}) = s_{\rm in}$, $\phi(e_{\rm out}) = s_{\rm out}$ and the restrictions of ϕ to the blocks of the partition Π are *injective*. If this condition is fulfilled then

$$W \stackrel{\rm def}{=} (e_{\rm in}, e_{\rm out}, \mathcal{G} \cap (E_{\rm ch} \times E_{\rm ch}), \phi|_{E_{\rm ch}}) \in SAW(E_{\rm ch}) \qquad (37)$$

and likewise for any i, $1 \leq i \leq r$,

$$L_i \stackrel{\rm def}{=} (\mathcal{G} \cap (E_i \times E_i), \phi|_{E_i}) \in SAL(E_i)\ . \qquad (38)$$

We then let

$$\mathcal{B}_{-}(E, P) \stackrel{\rm def}{=} \alpha_{[E_{\rm ch}, W]} \prod_{i=1}^{r} \left(-\lambda_{[E_i, L_i]}\right)\ . \qquad (39)$$

We also define the analogous expression $\mathcal{B}_{+}(E, P)$ without the minus signs.

If C is a canvas on E we denote by $P(C)$ the underlying precanvas. Let ρ^∞ correspond to the infinite iteration of the single-loop-erasure morphism of functors ρ^∞. Note that the result stabilizes after a finite number of iterations depending on the canvas C on which these are applied.

The main identity concerns the quantity

$$\Delta \stackrel{\rm def}{=} \sum_{[E,C]} \frac{\mathcal{A}(E, C)}{\#Aut(E, C)} \qquad (40)$$

where the sum is over equivalence classes of pairs made of a finite set E of labels and a canvas C on E. The amplitude $\mathcal{A}(E, C)$ is the product

$$\mathbb{1}\{\phi(e_{\rm in}) = s_{\rm in}\}\mathbb{1}\{\phi(e_{\rm out}) = s_{\rm out}\}\mathbb{1}\{C \text{ linear}\} \times \mathcal{B}_{-}\left(\rm E, P(\rho^\infty_E(C))\right)$$

One now has the following lemmata.

Lemma 9.1. *If C is a canvas on a set E then $\#Aut(E, C) = 1$.*

Lemma 9.2. *A canvas C on E is fixed by ρ_E if and only if the restriction of ϕ to $E_{\rm ch}$ is injective.*

Lemma 9.3. *Let C be a linear canvas on E with $\phi(e_{\rm in}) = s_{\rm in}$ and $\phi(e_{\rm out}) = s_{\rm out}$. Denote by C' the canvas $\rho_E^\infty(C)$ and by P' the corresponding precanvas $P(C')$. Let $\{E'_{\rm ch}, E'_1, \ldots, E'_r\}$ be the partition of E given by the chain and the loops of P'. Then the embedding map ϕ of P' (which is the same as that of C) has injective restrictions to $E'_{\rm ch}$ and the E'_i. Therefore*

$$\mathcal{B}_-\left(E, P(\rho_E^\infty(C))\right) \neq 0$$

in the ring $\mathcal{R}$.

We now apply Theorem 8 from [Abd04] for the natural transformation ρ^∞ in order to get

$$\Delta = \sum_{[E,C']} \frac{1}{\#Aut(E,C')} \sum_{\substack{C\in Canvas(E)\\ \rho_E^\infty(C)=C'}} \mathcal{A}(E,C) \tag{41}$$

We now have

Lemma 9.4. *A canvas C' on E is the image by ρ_E^∞ of a linear canvas C with $\phi(e_{\rm in}) = s_{\rm in}$ and $\phi(e_{\rm out}) = s_{\rm out}$ if and only if*

$$\mathbb{1}\{\phi(e_{\rm in}) = s_{\rm in}\}\mathbb{1}\{\phi(e_{\rm out}) = s_{\rm out}\}\tilde{\psi}_{\rm LE}(E,C') \neq 0$$

Besides, if this condition is satisfied, then one can uniquely recover the linear canvas C from C'.

Therefore

$$\Delta = \sum_{[E,C']} \frac{1}{\#Aut(E,C')} \sum_{\substack{C\in Canvas(E)\\ \rho_E^\infty(C)=C'}}$$

$$\mathbb{1}\{\phi(e_{\rm in}) = s_{\rm in}\}\mathbb{1}\{\phi(e_{\rm out}) = s_{\rm out}\}\mathbb{1}\{C \text{ linear}\}\mathcal{B}_-\left(\mathrm{E}, \mathrm{P(C')}\right) \tag{42}$$

$$= \sum_{[E,C']} \frac{\mathbb{1}\{\phi(e_{\rm in}) = s_{\rm in}\}\mathbb{1}\{\phi(e_{\rm out}) = s_{\rm out}\}\tilde{\psi}_{\rm LE}(E,C')\mathcal{B}_+\left(E, P(C')\right)}{\#Aut(E,C')} \tag{43}$$

Note that the signs in $\mathcal{B}_-$ have been absorbed in the Mayer interlinks of $\tilde{\psi}_{\rm LE}(E,C')$.

We now again apply Theorem 8 of [Abd04] to the last expression using the morphism of functors $C \mapsto P(C)$. The result is, after changing the dummy variable C' to C,

$$\Delta = \sum_{[E,P]} \frac{1}{\#Aut(E,P)} \sum_{\substack{C\in Canvas(E)\\ P(C)=P}}$$

$$\mathbb{1}\{\phi(e_{\rm in})=s_{\rm in}\}\mathbb{1}\{\phi(e_{\rm out})=s_{\rm out}\}\tilde{\psi}_{\rm LE}(E,C)\mathcal{B}_+\left(E,P(C)\right) \tag{44}$$

where the last sum is over precanvas classes. Now

$$\Delta=\sum_{[E,P]}\frac{\mathbb{1}\{\phi(e_{\rm in})=s_{\rm in}\}\mathbb{1}\{\phi(e_{\rm out})=s_{\rm out}\}\mathcal{B}_+\left(E,P\right)}{\#Aut(E,P)}$$

$$\times\sum_{\substack{C\in Canvas(E)\\ P(C)=P}}\tilde{\psi}_{\rm LE}(E,C) \tag{45}$$

By definition of the $\psi_{\rm LE}$ coefficient and Theorem 5.1 this becomes

$$\Delta=\sum_{[E,P]}\frac{\mathcal{D}(E,P)}{\#Aut(E,P)} \tag{46}$$

where

$$\mathcal{D}(E,P)\stackrel{\rm def}{=}\mathbb{1}\{\phi(e_{\rm in})=s_{\rm in}\}\mathbb{1}\{\phi(e_{\rm out})=s_{\rm out}\}\psi_M(E,P)\mathcal{B}_+(E,P) \tag{47}$$

Now, we once more define a new specie of *enriched precanvases* $E\to EnP(E)$. An element $Q\in EnP(E)$ consists of a precanvas P together with a total ordering or numbering $E_1,\ldots,E_r$ of the loops which appear in the partition $\{E_{\rm ch},E_1,\ldots,E_r\}$ of E into connected components for P. We define the amplitude $\mathcal{E}(E,Q)\stackrel{\rm def}{=}\frac{1}{r!}\mathcal{D}(E,P)$ where P is the underlying precanvas of Q and r is the number of loops. By Theorem 8 of [Abd04] again one has

$$\sum_{[E,Q]}\frac{\mathcal{E}(E,Q)}{\#Aut(E,Q)}=\sum_{[E,P]}\frac{1}{\#Aut(E,P)}\sum_{\substack{Q\in EnP(E)\\ Q\ \text{over}\ P}}\frac{\mathcal{D}(E,P)}{r!} \tag{48}$$

$$=\sum_{[E,P]}\frac{\mathcal{D}(E,P)}{\#Aut(E,P)}=\Delta \tag{49}$$

Now one can introduce a specific *model* for the classes $[E,Q]$. Let $r\ge 0$, $k\ge 1$, $l_1,\ldots,l_r\ge 1$ be some integers. Let $E=\{1,2,\ldots,N\}$ with $N=k+l_1+\cdots+l_r$. Let

- $E_{\rm ch}=\{1,2,\ldots,k\}$,
- $E_1=\{k+1,\ldots,k+l_1\}$,
- $E_2=\{k+l_1+1,\ldots,k+l_1+l_2\}$,

  $\vdots$
- $E_r=\{k+l_1+\cdots+l_{r-1}+1,\ldots,k+l_1+\cdots+l_r\}$,

Let $e_{\rm in}=1$ and $e_{\rm out}=k$ in $E_{\rm ch}$. We define the oriented graph $\mathcal{G}$ by listing its edges:

- $(1,2),\ldots,(k-1,k)$ in $E_{\rm ch}$,

- $(k+1,k+2),\ldots,(k+l_1-1,k+l_1),(k+l_1,k+1)$ in E_1,

  $\vdots$
- $(k+l_1+\cdots+l_{r-1}+1,k+l_1+\cdots+l_{r-1}+2),\ldots,$
  $(k+l_1+\cdots+l_r-1,k+l_1+\cdots+l_r),$
  $(k+l_1+\cdots+l_r,k+l_1+\cdots+l_{r-1}+1)$ in E_r.

The maps $\phi: E \to \mathcal{S}$ is allowed to be arbitrary. Finally the numbering of the loops is the one given by the ordering $E_1,\ldots,E_r$.

It is easy to see that Δ can now be rewritten

$$\Delta = \sum_{r\geq 0}\sum_{k\geq 1}\sum_{l_1,\ldots,l_r\geq 1}\sum_{\phi}\frac{1}{\#Aut(E,Q)}\frac{1}{r!}$$

$$\mathbb{1}\{\phi(e_{\rm in})=s_{\rm in}\}\mathbb{1}\{\phi(e_{\rm out})=s_{\rm out}\}\psi_M(E,P)\mathcal{B}_+(E,P) \tag{50}$$

where the pair (E,Q) is the one contructed by the previous process, and P is the underlying precanvas. Note that if $\mathcal{B}_+(E,P)\neq 0$ the map ϕ has to be an injection in each component. This together with the ordering on the loops forces $\#Aut(E,Q)=1$. Let, as in Section 4, $Y_0=\phi(E_{\rm ch})$, $Y_1=\phi(E_1)$,...,$Y_r=\phi(E_r)$. Recalling the definition of Mayer coefficients in Section 4 and summing independently over the restrictions of ϕ within each component one readily obtains

$$\Delta = \sum_{r\geq 0}\sum_{(Y_0,\ldots,Y_r)}\frac{1}{r!}\psi(Y_0,\ldots,Y_r)B(Y_0)A(Y_1)\ldots A(Y_r) \tag{51}$$

where $(Y_0,\ldots,Y_r)$ is a sequence of *polymers* or subsets of the site space $\mathcal{S}$, and where the polymer amplitudes A and B are defined as follows.

One lets

$$A(Y)\stackrel{\rm def}{=}\sum_{\substack{[E,L]\\ \phi(E)=Y}}\lambda_{[E,L]} \tag{52}$$

where the sum is over classes of pairs (E,L) with $L\in SAL(E)$, and ϕ is the embedding map coming with L. One also lets

$$B(Y)\stackrel{\rm def}{=}\sum_{\substack{[E,W]\\ \phi(E)=Y}}\alpha_{[E,W]} \tag{53}$$

where the sum is over classes of pairs (E,W) with $W\in SAW(E)$, and ϕ is the corresponding embedding map. The conditions $\phi(e_{\rm in})=s_{\rm in}$, $\phi(e_{\rm out})=s_{\rm out}$ are implicitly assumed. Note that because of the forced injectivity of the corresponding maps ϕ the automorphism groups of these classes are trivial. Therefore one does not need to normalize by their cardinality.

Now by the fundamental theorem of Mayer expansion theory one can rewrite (51) as

$$\Delta = \frac{\sum_{r\ge 0}\sum_{\substack{Y_0,\{Y_1,\ldots,Y_r\}\\ \text{disjoint}}} B(Y_0)A(Y_1)\ldots A(Y_r)}{\sum_{r\ge 0}\sum_{\substack{\{Y_1,\ldots,Y_r\}\\ \text{disjoint}}} A(Y_1)\ldots A(Y_r)} \tag{54}$$

i.e. Δ is the correlation function $< s_{\text{in}}s_{\text{out}} >$ of a loop ensemble. In these sums the collection $\{Y_1,\ldots,Y_r\}$ is *unordered*. The polymers $(Y_0), Y_1,\ldots,Y_r$ are all assumed to be disjoint subsets of $\mathcal{S}$.

We have proven the following, which is the main result of this article.

Theorem 9.5. *In the power series ring $\mathcal{R}$, one has the identity*

$$\sum_{[E,C]} \mathbb{1}\{\phi(e_{\text{in}}) = s_{\text{in}}\}\mathbb{1}\{\phi(e_{\text{out}}) = s_{\text{out}}\}\mathbb{1}\{C \text{ linear}\}\mathcal{B}_-\left(\mathrm{E}, \mathrm{P}(\rho_{\mathrm{E}}^{\infty}(\mathrm{C}))\right)$$

$$= \frac{\sum_{r\ge 0}\sum_{\substack{Y_0,\{Y_1,\ldots,Y_r\}\\ \text{disjoint}}} B(Y_0)A(Y_1)\ldots A(Y_r)}{\sum_{r\ge 0}\sum_{\substack{\{Y_1,\ldots,Y_r\}\\ \text{disjoint}}} A(Y_1)\ldots A(Y_r)} \tag{55}$$

10. Cramer's rule

Let $A = (A_{xy})_{x,y\in\mathcal{S}}$ be a matrix of formal variables. For a and b some fixed points in $\mathcal{S}$, which are not necessarily distinct, we consider Cramer's rule for the matrix $I - A$ i.e.

$$\frac{\det[(I-A)^{(b,a)}]}{\det[I-A]} = (I-A)^{-1}_{ab} \tag{56}$$

The determinants are with respect to an arbitrary ordering of $\mathcal{S}$ which is the same for lines and columns, and therefore need not be specified. The minor determinant is that of a matrix of the same size as $I - A$ using the trick of defining the matrix $(I - A)^{(b,a)}$ on $\mathcal{S}$ with entries

$$(I-A)^{(b,a)}_{xy} \stackrel{\text{def}}{=} \mathbb{1}\{x \neq b\}(I-A)_{xy}\mathbb{1}\{y \neq a\} + \mathbb{1}\{x = b\}\mathbb{1}\{y = a\} \tag{57}$$

If one takes $s_{\text{in}} = a$, $s_{\text{out}} = b$, in the previous section and specializes the variables α and λ to:

- $\alpha_{[E,W]}$ is the product of the A entries along the steps of the self-avoiding walk W. If $a = b$ this walk makes zero step and the corresponding α is equal to 1.
- $\lambda_{[E,L]}$ is *minus* the product of the A entries along the steps of the self-avoiding loop L. Note that $E = \emptyset$ is forbidden since $SAL(\emptyset) = \emptyset$ by definition, and $L \in SAL(E)$ is assumed to

exist. If E is a singleton $\{e\}$, then the oriented graph has to be $\mathcal{G} = \{(e,e)\}$. If x is the image by ϕ of e then

$$\lambda_{[E,L]} = -A_{x,x} \tag{58}$$

Although it is a void one, the loop actually makes a step!

With these definitions, and regardless whether a is equal to b or not, the statement of Theorem 9.5 is exactly Cramer's rule (56). Note that with the definition of the functor ρ, in case $a = b$ the chain that remains is always trivial. Also in this situation the loops in the numerator are simply forbidden to touch the site $a = b$ in $\mathcal{S}$.

Acknowledgment. The work of DB was supported in part by NSERC of Canada. We gratefully acknowledge very helpful communications with Pierre Leroux and Christian Krattenthaler.

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LAGA, Institut Galilée, CNRS UMR 7539, Université Paris XIII, Avenue J.B. Clément, F93430, Villetaneuse, France

Department of Mathematics, University of British Columbia, Vancouver, B.C. V6T 1Z2