Lecture Notes for MATH 7320

Theory of Distributions

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Notes taken by Math 7320 class

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Dedicated to Smurfs everywhere.

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- ??

0

Motivation and Preview

0.1 Some Motivating Examples

In most texts, there are two types of courses after measure theory

1. Applications $\longrightarrow$ PDE's

   Text: *A Guide to Distribution Theory and Fourier Transforms* by Robert Strichartz

2. Functional Analysis (nuts and bolts) $\longrightarrow$ topological vector spaces and underlying theory

The point of this course is to do both. Time permitting, we will discuss probability theory on spaces of distributions, which does not appear in texts. We will combine the two approaches into one economical approach by choosing bases.

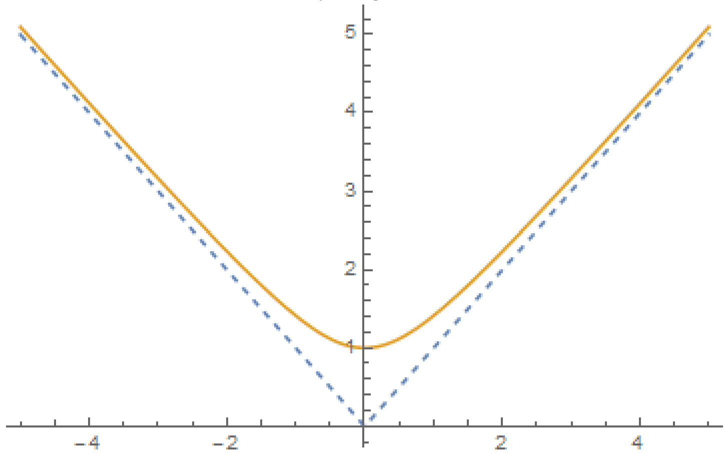
0.1.1 Notation

$\mathbb{N}_0 = \{0, 1, ...\}$
$\mathbb{N} = \{1, 2, ...\}$
$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$
$[n] = \{1, 2, ..., n\}$
If I is a finite set, $|I|$ = Cardinality of I.
For any sets A, B the set of functions from A to B is denoted $\mathscr{F}(A, B)$.
$x = (x_1, ..., x_d) \in \mathbb{R}^d$
$d^d x$ = Lebesgue measure[1] on $\mathbb{R}^d$

[1] We will only consider Borel sets and Borel functions.

$|x| = \sqrt{x_1^2 + ... + x_d^2}$ Euclidean nom
$\langle x \rangle = \sqrt{1 + |x|^2}$

Note that $\langle x \rangle$ is smooth and does not vanish at the origin. This will be used as a basic unit for analysing behavior of a function at infinity.



Multi-index Notation
$\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$
$|\alpha| = \sum_1^d \alpha_i$ is the length of α
$\alpha! = \prod_i^d \alpha_i!$
$x^\alpha = (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})$
$\partial^\alpha f = \dfrac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}$
$\partial_i f = \frac{\partial f}{\partial x_i}$

Notation reference
The set of smooth functions in $\mathscr{F}(U, \mathbb{K})$ with compact support is denoted $\mathscr{D}(U)$.

0.1.2 Idea of Distribution

Suppose $f \in \mathscr{F}(\mathbb{R}^d, \mathbb{R})$. We can evaluate f at a point $x \in \mathbb{R}^d$ to get a value $f(x) \in \mathbb{R}$. The idea behind distributions is that such an evaluation does not exist in the real world.

Example 0.1.1. Suppose $f \in \mathscr{F}(\mathbb{R}^3, \mathbb{R})$ gives the temperature at a point x in space. Unfortunately, a themometer cannot have zero thickness, i.e. it does not have infinite resolution. Instead, it takes some weighted average

$$\int_{\mathbb{R}^3} \varphi(y) f(y) \; d^3y$$

where $\varphi(y)$ is a weight, preferably with mass centered at x.

In this example,

$$\text{Knowing } f \quad \Longleftrightarrow \quad \text{Knowing } \int_{\mathbb{R}^3} \varphi(y) f(y)\, d^3y$$
$$\text{for all "test functions" } \varphi$$

Definition 0.1.2. We define the **space of test functions** to be

$$\mathcal{D}(\mathbb{R}^d, \mathbb{K}) = \{\varphi \in \mathcal{F}(\mathbb{R}^d, \mathbb{K}) \mid \varphi \in C^\infty,\ \mathrm{supp}(\varphi) \text{ is compact}\}.$$

Observe that the map $\mathcal{D} \to \mathbb{K}$ sending $\varphi \mapsto \int_{\mathbb{R}^3} \varphi f$ is a linear form (functional). This brings us to a definition of a distribution.

Definition 0.1.3. A continuous linear form $f : \mathcal{D} \to \mathbb{K}$ is called a **distribution** (or generalized function).

The space of distributions is the topological dual of $\mathcal{D}$, which we will denote

$$\mathcal{D}'(\mathbb{R}^d, \mathbb{K}).$$

To guarantee continuity, we need a topology on $\mathcal{D}$, which will make $\mathcal{D}$ into a TVS. This topology will be given later, as will the topology on its dual. Unfortunately, this topology will be complicated. However, if we relax our requirements a little, we can get a space with a much nicer topology.

Definition 0.1.4. We define the **Schwartz Space** to be

$$\mathcal{S} = \{f \in \mathcal{F}(\mathbb{R}^d, \mathbb{K}) \mid f \in C^\infty \text{ and } \|\langle x\rangle^k \partial^\alpha f(x)\|_{L^\infty} < \infty\ \forall\ \alpha \in \mathbb{N}_0, k \in \mathbb{N}_0\}.$$

Observe that $\mathcal{D} \subseteq \mathcal{S}$, and therefore, once $\mathcal{S}$ has a topology, its topological dual $\mathcal{S}'$ will contain $\mathcal{D}'$. We call $\mathcal{S}'$ the **space of (tempered) temperate distributions**.

Example 0.1.5 (Charge Distributions from Calc III/ Electrostatics)**.** Suppose we have a charge q at a point x with force felt $\vec{F} = q\vec{E}(x)$ where $\vec{E}$ is the electric field at position x. Then we have

$$\vec{E} = -\vec{\nabla}\varphi$$

where $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is the electric potential. Now suppose ρ is a charge distribution. Then $\rho(x)d^3x$ gives a charge in an infinitesimal neighborhood. We want to find the electric charge potential φ generated by q, i.e. we want to solve the Poisson Equation

$$\Delta\varphi = -\rho$$

where Δ is the Laplacian, and

$$\Delta\varphi = \sum_1^3 \frac{\partial}{\partial x_i}\left(\frac{\partial \varphi}{\partial x_i}\right) = \sum_1^3 \partial_i^2 \varphi.$$

Recall that Gauss' Law gives us

$$\vec{\nabla} \cdot \vec{E} = \rho$$

where $\vec{\nabla} \cdot \vec{E} = \operatorname{div}(\vec{E}) = \sum_1^3 \frac{\partial E_i}{\partial x_i}$.

For simplicity, we assume the charge q is concentrated near the origin. Moreover, we assume

- $\operatorname{supp}(\rho) \sim \{0\}$,
- ρ is rotation invariant (**radial**), i.e, ρ is a function of $|x|$ only.

We reasonably guess that ϕ is also radial. That is,

$$\varphi(x) = f(|x|).$$

Can we find f?

Since $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, it follows from repeated applications of the chain rule that

$$\begin{aligned}
\partial_i |x| &= \partial_i (x_1^2 + x_2^2 + x_3^2)^{1/2} \\
&= \frac{1}{2(x_1^2 + x_2^2 + x_3^2)^{1/2}} \partial_i (x_1^2 + x_2^2 + x_3^2) \\
&= \frac{1}{2|x|} 2x_i \\
&= \frac{x_i}{|x|}
\end{aligned}$$

which is a useful little fact to know, and then applying it to our specific situation, we have

$$\begin{aligned}
\partial_i \varphi &= \partial_i f(|x|) \\
&= f'(|x|) \partial_i |x| \\
&= f'(|x|) \frac{x_i}{|x|}
\end{aligned}$$

where "x has to avoid 0" because of the $|x|$ in the denominator.

Let u be the unit vector pointing in the same direction as x:

$$u := \frac{x}{|x|}$$

and let R be the magnitude of x (the "radius"):

$$R := |x|$$

Then we have

$$\begin{aligned}
\nabla\varphi &= (\partial_i \varphi)_{i=1}^3 \\
&= \left(f'(|x|)\frac{x_i}{|x|}\right)_{i=1}^3 \\
&= f'(|x|)\frac{(x_i)_{i=1}^3}{|x|} \\
&= f'(|x|)\frac{x}{|x|} \\
&= f'(R)u
\end{aligned}$$

Point charge Q

We will now calculate the flux of $(E)_{k=1}^3$ through a sphere of radius R centered at Q in two different ways in order to discover an expression for Q:

1. First calculation:

$$\begin{aligned}
\iint_S \nabla\varphi \bullet dS &= \iint_S f'(R)u \bullet dS \\
&= f'(R) \iint_S u \bullet dS \\
&= f'(R) \times \text{ surface area of a sphere} \\
&= f'(R)4\pi R^2
\end{aligned}$$

2. In this second calculation, we apply the divergence theorem to get a 3-dimensional integral:

$$\begin{aligned}
f'(R)4\pi R^2 &= \iint_S \nabla\varphi \bullet dS \\
&= \iiint_B (\nabla \bullet \nabla\varphi)\, d^3x \\
&= \iiint_B \Delta\varphi\, d^3x \\
&= \iiint_B -\rho\, d^3x \\
&= -Q
\end{aligned}$$

and so we conclude that $Q = -f'(R)4\pi R^2$. Rearranging, we have

$$f'(R) = -\frac{Q}{4\pi R^2}.$$

By integrating both sides with respect to R, we obtain

$$\varphi(x) = \frac{Q}{4\pi|x|}$$

General charge distribution ρ

We can express the electric potential $\varphi(x)$ by integrating $\frac{\rho(y)d^3y}{|x-y|}$ over $\mathbb{R}^3 \setminus \{x\}$:

$$\varphi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus \{x\}} \frac{\rho(y)}{|x-y|}\, d^3y \qquad (\star)$$

Recall $-\Delta\varphi = \rho$ is a linear nonhomogeneous equation.

Point charge case:

Let δ be the 1-dimensional Dirac delta function given by

$$\delta(t) \text{``}=\text{''} \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

As a probability density distribution, it satisfies the property

$$\int_{\mathbb{R}} \delta(t)\, dt = 1.$$

Define the 3-dimensional Dirac delta function by

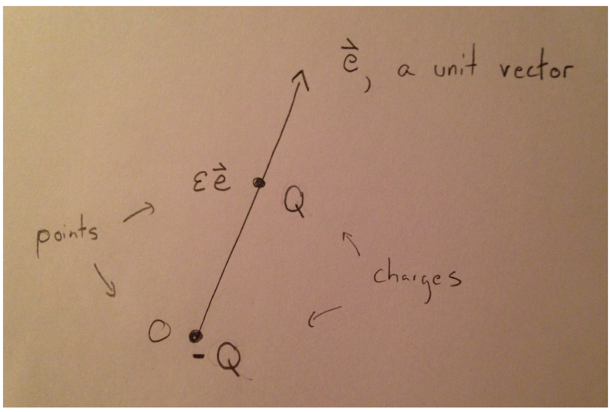
$$\delta^3(x) := \delta(x_1)\delta(x_2)\delta(x_3)$$

Then

$$-\Delta\varphi = Q\delta^3(x)$$

dipoles

A dipole is two point charges: one with charge Q and the other with opposite charge $-Q$:



$$\rho(y) = Q\delta^3(y - \varepsilon e) - Q\delta^3(y)$$

$$\rho(y) = \lim_{\varepsilon \to 0} \frac{Q\delta^3(y - \varepsilon e) - Q\delta^3(y)}{\varepsilon}$$

This is a nontrivial charge distribution called a dipole.

$$Q|e\nabla|\delta^3$$

Is ($\star$) well defined? Yes it is, but we must prove it! We now go about proving this, which takes quite a while, and won't even be finished this lecture...

$$\varphi(x) = \int_{\mathbb{R}^3} K(x,y)\rho(y)\, d^3y$$

where

$$K(x,y) := \begin{cases} \frac{1}{4\pi|x-y|} & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

integral operator $\rho \mapsto \varphi$ forall K
well defined if e.g.
ρ Borel measure

$$|\rho(x)| \leq C\langle x\rangle^{-\alpha}$$

works for $\alpha > 2$. But why? A quick digression((

$$\int_{\mathbb{R}^3 \setminus \{x\}} \frac{\rho(y)}{|x-y|}\, d^3y$$

has a problem at "y near x" and a problem at "y near ∞"

1. First, the problem at infinity...

   Change to polar coords.

   $$= \int_1^\infty \frac{1}{r^{1+\alpha}} 4\pi r^{3-1}\, dr$$

   $$= \int_{|y|\geq 1} \frac{1}{|y|^{1+\alpha}}\, d^3y$$

   So that $\alpha > 2$ is necessary.

2. Finally, the problem at x...

   The $\rho(y)$ is bounded or something near x, so it doesn't matter. So the expression, as far as convergence goes, behaves the same as

$$\int_{|x-y|\leq 1} \frac{1}{|x-y|}\, d^3y < \infty$$

   which in polar becomes

$$\int_0^1 \frac{r^2}{r^3}\, dr$$

   and therefore converges with no extra restrictions on α.

end of digression.

Differentiating φ

It is now our goal to differentiate $\varphi(x)$. That is, we want to calculate the partial derivative $\partial_i \varphi(x)$. Recall equation $(\star)$:

$$\varphi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus \{x\}} \frac{\rho(y)}{|x-y|}\, d^3y$$

We might try differentiating under the integrand, but that won't work. So messy. Let's use convolution...

$$K(x,y) = G(x-y)$$

where

$$G(x) := \begin{cases} \frac{1}{4\pi|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

that is, K is a convolution kernel.

$$\varphi(x) = \int_{\mathbb{R}^3} G(x-y)\rho(y)\, d^3y$$

(note: something about $G * \rho$ and x fixed)

We will perform a change of variables using $z = x - y$. This uses the global change of variables formula (or "abstract" change or variables formula) from MATH 7310. If we let $z = f(y) = x - y$, and then compute the Jacobian matrix J, we get $J = -I_3$. The multiplying factor is the absolute value of the Jacobian, which is $|\det(J)| = |-1| = 1$, so the change of variables results in:

$$\varphi(x) = \int_{\mathbb{R}^3} G(z)\rho(x-z)\, d^3z$$

Now we still want to differentiate under the integral sign! But in order to do so, we must satisfy two hypotheses:

1. The integrand must be continuously differentiable (it is).
2. We need a dominating function. Like some decay function where $|\delta_1\rho(x)| \leq C\langle x\rangle^{-\alpha}$.

Let's take $\rho \in S(\mathbb{R}^3; \mathbb{R})$

....fill in the blanks here...by repeatedly applying differentiation under the integrand....

$$\varphi is C^\infty$$

$$(\delta\varphi)(x) = \int_{\mathbb{R}^3} G(z)(\delta\rho)(x-z)\, d^3z$$

$$\frac{1}{4\pi}\int_{\mathbb{R}^3\setminus\{0\}} \frac{1}{|z|}(\delta\rho)(x-z)\, d^3z \tag{1}$$

Back to a point charge $Q = 1$

$$-\delta\left(\frac{1}{4\pi|z|}\right) = \delta^3(z)$$

let's check it

$$\frac{\delta}{\delta z_i}\frac{1}{|z|} = -\frac{z_i}{|z|^3}$$

By differentiating again, we obtain

$$\delta_i^2\frac{1}{|z|} = -\frac{1}{|z|^3} - z_i(-3)|z|^{-4}\frac{z_i}{|z|}$$

$$= -|z|^{-3} + 3\frac{z_i^2}{|z|^5}$$

We wish to apply Integration By Parts to $(*)$, but we can't do it because there is a problem at 0. So we have to do another step first. This will be explained next lecture!

Now, we continue to seek ρ where $-\Delta\varphi = \rho \in S(\mathbb{R}^3, \mathbb{R})$. Recall that

$$\varphi(x) := \frac{1}{4\pi}\int_{\mathbb{R}^3-\{0\}} \frac{1}{|x-y|}\rho(y) d^3y$$

where φ is C^∞. A change of variables gives us

$$\varphi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3-\{0\}} \frac{1}{|y|}\rho(x-y)d^3y.$$

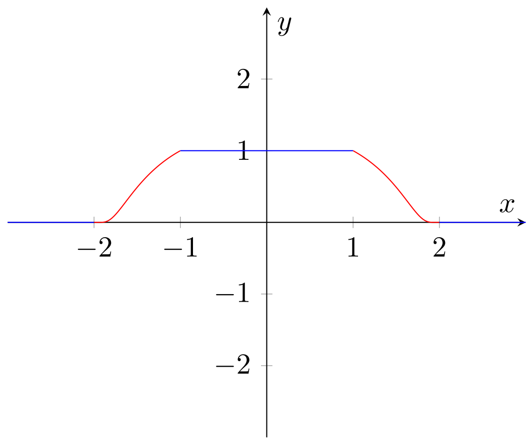
We thus have

$$|\Delta\varphi|(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3-\{0\}} \frac{1}{|y|}\Delta\rho(x-y)d^3y.$$

We want to integrate by parts, keeping in mind that $\Delta\frac{1}{|y|} = 0$ for $y \neq 0$; first, we propose a lemma.

Lemma. *There exists an even C^∞ function $\chi : \mathbb{R} \to \mathbb{R}$ such that $0 \leq \chi \leq 1$, $\chi = 1$ on $[-1, 1]$, $\chi = 0$ on $\mathbb{R} - (-2, 2)$.*

Graph of $\chi(x)$:

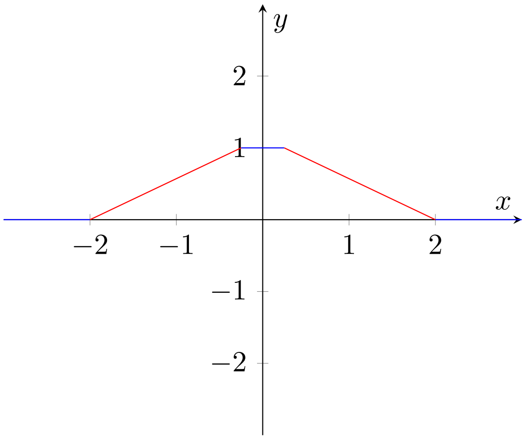


A proof of this lemma appears in the toolkit section of our course. For now, we take the lemma for granted and consider χ as described in the statement. Define $\psi : \mathbb{R}^3 \to \mathbb{R}$ by $\psi(x) = \chi(|x|)$. By definition, this function ψ is radial. Note that ψ is also C^∞ because it is the composition of C^∞ functions: $|x|$ is C^∞ except at 0, and in a neighborhood of 0, χ is a constant function equal to 1.

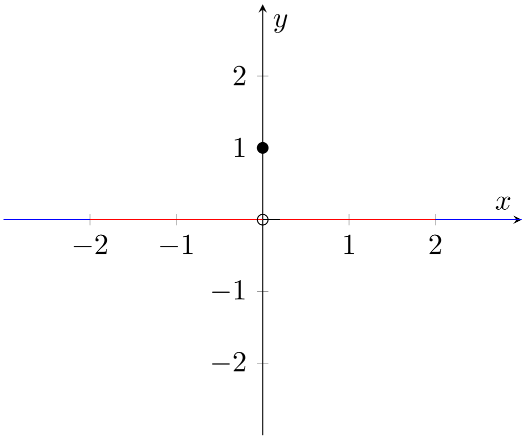
Using the dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3-\{0\}} \frac{1-\psi(nz)}{|z|}\Delta\rho(x-z)d^3z = (\Delta\varphi)(x).$$

The graph of $1-\psi(nz)$ looks like the graph below. Here, the red lines no longer end at -1 and 1; they end at $-1/n$ and $1/n$.



Allowing n to go to infinity, the graph of $1-\psi(nz)$ tends to the graph below.



Therefore

$$\Delta\varphi = \lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{1-\psi(nz)}{|z|} \Delta\rho(x-z) d^3z,$$

where $\frac{1-\psi(nz)}{|z|}$ is C^∞ and we can apply integration by parts. Let

$$dv = \Delta_z \rho(x-z) \Rightarrow v = \rho(x-z)$$

and

$$u = \frac{1-\psi(nz)}{|z|} \Rightarrow du = \Delta_z \frac{1-\psi(nz)}{|z|}$$

so that

$$\begin{aligned}(\Delta\varphi)(x) &= \lim_{n\to\infty} \Big(\frac{1-\psi(nz)}{|z|} \rho(x-z)\Big|_{\mathbb{R}^3} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \Delta_z \Big(\frac{1-\psi(nz)}{|z|} \Big) \rho(x-z) d^3z \Big) \\ &= \lim_{n\to\infty} \frac{1}{4\pi} \int_{\mathbb{R}^3} \Delta_z \Big(\frac{1-\psi(nz)}{|z|} \Big) \rho(x-z) d^3z.\end{aligned}$$

Now we note that on $|z| \leq 1/n$, $\Delta_z(\frac{1-\psi(nz)}{|z|}) = 0$ so that we need only determine the behavior of $\Delta_z(\frac{1-\psi(nz)}{|z|})$ for $|z| > 1/n$. We have

$$\Delta_z\Big(\frac{1-\psi(nz)}{|z|}\Big) = \sum_{i=1}^{3} \frac{\partial^2}{\partial z_i^2}\Big(\frac{1-\psi(nz)}{|z|}\Big)$$

and we apply the Leibniz rule so that

$$\Delta_z\left(\frac{1-\psi(nz)}{|z|}\right) = \sum_{i=1}^{3}\left[\frac{\partial^2}{\partial z_i^2}\left(\frac{1-\psi(nz)}{|z|}\right) + 2\frac{\partial(1-\psi(nz))}{\partial z_i}\frac{\partial(1/|z|)}{\partial z_i} + \frac{\partial^2}{\partial z_i^2}\left(\frac{1}{|z|}\right)\right].$$

We have $\frac{\partial^2}{\partial z_i^2}\left(\frac{1}{|z|}\right) = 0$, so

$$\Delta_z\left(\frac{1-\psi(nz)}{|z|}\right) = \frac{-n^2(\Delta\psi)(nz)}{|z|} - 2n(\nabla\psi)(nz)\cdot\nabla\left(\frac{1}{|z|}\right)$$

which equals zero when $|z| \leq 2/n$.

Substituting back, we have

$$(\Delta\varphi)(x) = \frac{1}{4\pi}\lim_{n\to\infty}\int_{\{\frac{1}{n}\leq|z|\leq\frac{2}{n}\}}\left(\frac{-n^2}{|z|}(\Delta\varphi)(nz) + 2n\left(\nabla\psi(nz)\cdot\frac{z}{|z|^3}\right)\right)\rho(x-z)d^3z$$

where the limit is guaranteed to exist due to an application of the dominated convergence theorem. Note that as $n \to \infty$, we have $\{\frac{1}{n} \leq |z| \leq \frac{2}{n}\}$ shrinks to the origin, so $\rho(x-z) \sim \rho(x)$. With that in mind, we consider the following:

$$\begin{aligned}&\frac{1}{4\pi}\int_{\{\frac{1}{n}\leq|z|\leq\frac{2}{n}\}}\Big(\frac{-n^2}{|z|}(\Delta\varphi)(nz) + 2n\Big(\nabla\psi(nz)\cdot\frac{z}{|z|^3}\Big)\Big)\rho(x-z)d^3z\\ &= E + \frac{1}{4\pi}\int_{\{\frac{1}{n}\leq|z|\leq\frac{2}{n}\}}\Big(\frac{-n^2}{|z|}(\Delta\varphi)(nz) + 2n\Big(\nabla\psi(nz)\cdot\frac{z}{|z|^3}\Big)\Big)\rho(x)d^3z\end{aligned}$$

where E is an error term that we will attempt to bound. We note that that

$$|E| \leq \frac{1}{4\pi}\int_{\{\frac{1}{n}\leq|z|\leq\frac{2}{n}\}}\Big(\frac{n^2}{|z|}|\Delta\psi(nz)| + 2n|\nabla\psi(nz)|\frac{1}{|z|^2}\Big)|\rho(x-z)-\rho(x)|d^3z$$

where $|\Delta\psi(nz)|$ and $|\nabla\psi(nz)|$ are both bounded by some appropriate constants. We have

$$|E| \leq \sum_{i=1}^{3}||\partial_i\rho||_{L^\infty}|z|$$

i.e. $\rho(x-z) = f(t)|_{t=1}$ where $f(t) = \rho(x-(-z))$. We have

$$f(1) - f(0) = \int f'(t)dt = \int_0^1(\sum_{i=1}^{3}(-z_i)\partial_i\rho(x-tz))dt < ||\partial_i\rho||_{L^\infty}.$$

Therefore, for some constant C we have

$$|E| \leq C \int_{\{\frac{1}{n} \leq |z| \leq \frac{2}{n}\}} \left(\frac{n^2}{|z|} + \frac{2n}{|z|} \right) d^3 z$$

so that the u-substitution $\mu = nz$ grants us

$$|E| \leq \int_{\{1 \leq |\mu| \leq 2\}} \left(n^2 + \frac{2n}{|\mu|/n} \right) \frac{1}{n^3} d^3 \mu \to 0$$

so we've shown that $(\Delta\varphi)(x) = \rho(x)C$. All that remains is to show that $C = -1$. Recall that

$$C = \frac{1}{4\pi} \lim_{n \to \infty} \int_{\{\frac{1}{n} \leq |z| \leq \frac{2}{n}\}} \left(\frac{-n^2}{|z|} \Delta\psi(nz) + 2n(\nabla\psi)(nz) \frac{z}{|z|^3} \right) d^3 z$$

where we can drop the limit because C is a constant. We replace the z here with a new variable z/n and obtain

$$C = \int_{\{1 \leq |z| \leq 2\}} \left(\frac{-1}{|z|} (\Delta\psi)(z) + 2(\nabla\psi)(z) \frac{z}{|z|^3} \right) d^3 z$$

where $2(\nabla\psi)(z)\frac{z}{|z|^3} = \chi'(|z|)\frac{1}{|z|^2}$. Note $\psi(z) = \chi(|z|)$ and $\partial_i\psi(z) = \chi'(|z|)\frac{z_i}{|z|}$ so

$$\partial_i^2 \psi(z) = \chi'(|z|) \frac{z_i}{|z|} \frac{z_i}{|z|} + \frac{\chi'(|z|)}{|z|} - \frac{\chi'(|z|) z_i}{|z|^2} \frac{z_i}{|z|}.$$

Therefore

$$\begin{aligned} (\Delta\psi)(z) &= \chi'(|z|) + 3\frac{\chi'(|z|)}{|z|} - \frac{\chi'(|z|)}{|z|} = \chi''(|z|) + \frac{2\chi'(|z|)}{|z|} \\ \implies C &= \frac{1}{4\pi} \int_{\{1 \leq |z| \leq 2\}} \left(\frac{-\chi'(|z|)}{|z|} - \frac{2\chi'(|z|)}{|z|^2} + \frac{2\chi'(|z|)}{|z|^2} \right) d^3 z. \end{aligned}$$

We use spherical coordinates with $r = |z|$ to rewrite

$$\begin{aligned} C = \frac{1}{4\pi} \int_1^2 \frac{\chi'(r)}{r} 4\pi r^2 dr &= -\int_1^2 \chi''(r) r dr \\ &= -\chi'(r)\Big|_1^2 + \int_1^2 \chi'(r) dr \\ &= \chi(r) - \chi(1) \\ &= -1. \end{aligned}$$

Therefore, we have established that $-\Delta\varphi = \rho$. We remark here that the proof of this result was rather complicated because of our use of "classical" functions and notions of derivatives. We shall later employ the language of distributions, which will reduce tedious computations considerably.

0.2 Multilinear and Tensor Algebra

If R is a commutative ring with identity element 1 and M, N are R-modules, then one can construct the R-module $M \otimes_R N$, called the **tensor product** of M and N, in the following way. We first define the set of "almost finite" functions on $M \times N$:

$$\begin{aligned}\mathscr{F}_{af}(M \times N, R) &:= \{f \in \mathscr{F}(M \times N, R) \mid f(m,n) = 0 \text{ for all but finitely many } (m,n) \in M \times N\} \\ &\cong \bigoplus_{M \times N} R\end{aligned}$$

$\mathscr{F}_{af}$ is generated by "simple tensors" $m \otimes n$ $(m \in M, n \in N)$ defined by

$$(m \otimes n)(m', n') = \begin{cases} 1, & (m,n) = (m', n') \\ 0, & \text{else} \end{cases}$$

Let J be the submodule of $\mathscr{F}_{af}$ generated by the set of all elements of the form

- $(m + m') \otimes n - m \otimes n - m' \otimes n$
- $m \otimes (n + n') - m \otimes n - m \otimes n'$
- $(\lambda m) \otimes n - \lambda(m \otimes n)$
- $m \otimes (\lambda n) - \lambda(m \otimes n)$

ranging over all $m, m' \in M, n, n' \in N$, and all $\lambda \in R$. The tensor product is then defined to be the quotient module

$$M \otimes_R N = \mathscr{F}_{af}(M \times N, R)/J$$

For our purposes, $R = \mathbb{K}$ will always be a field, hence M, N are $\mathbb{K}$-vector spaces. Recall that if V is a finite dimensional vector space and V' denotes the (algebraic) dual of V, then there is a canonical isomoprhism $(V')' \cong V$. That is to say, V is **reflexive**. For infinite dimensional vector spaces V, we cannot hope for an isomorphism between V and its double algebraic dual, so by *reflexivity* we mean that V is isomorphic to its double topological dual.

If $V_1, V_2, \ldots, V_n, W$ are vector spaces, we denote by $\mathscr{L}_n(V_1, \ldots, V_n; W)$ the space of n-multilinear maps $V_1 \times \cdots \times V_n \to W$. Later, we will demand that these maps are continuous. If V, W are finite-dimensional $\mathbb{K}$-vector spaces, the universal property of the tensor product induces a canonical isomorphism

$$\begin{aligned} V' \otimes W' &\longrightarrow \mathscr{L}_2(V, W; \mathbb{K}) \\ \phi \otimes \psi &\longmapsto [(v,w) \mapsto \phi(v)\psi(w)] \end{aligned}$$

When V, W are finite dimensional, this gives us a cheap construction of $V \otimes_{\mathbb{K}} W$:

$$V \otimes_{\mathbb{K}} W = \mathscr{L}_2(V', W'; \mathbb{K}) \tag{NT1}$$

Notice that a vector $x = (x_1, \ldots, x_d) \in \mathbb{K}^d$ can be reinterpreted as a function $[d] \to \mathbb{K}$ given by $i \mapsto x_i$. We will make use of this later.

Categories

Category	FinSet	$\text{FinVect}_{\mathbb{K}}$
Objects	finite sets	finite dimensional $\mathbb{K}$-vector spaces
Morphisms	maps of sets	$\mathbb{K}$-linear maps

Consider the contravariant functor $\mathscr{B} = \mathscr{F}(-, \mathbb{K})$ from the category FinSet of finite sets to the category $\text{FinVect}_{\mathbb{K}}$ of finite dimensional $\mathbb{K}$-vector spaces which associates to each finite set I the $\mathbb{K}$-vector space $\mathscr{F}(I, \mathbb{K})$ of dimension $|I|$ and to each map of finite sets $\tau : I \to J$ the $\mathbb{K}$-linear map $\mathscr{B}(\tau) : \mathscr{B}(J) \to \mathscr{B}(I)$ given by $\mathscr{B}(\tau)(f) = f \circ \tau$.

Because a vector space is uniquely determined up to isomorphism by the cardinality of any basis, the functor $\mathscr{B}$ is **essentially surjective**, i.e., for all $V \in \text{FinVect}_{\mathbb{K}}$, there exists $I \in \text{FinSet}$ such that V is isomorphic to $\mathscr{B}(I)$ as $\mathbb{K}$-vector spaces. The upshot of this fact is that any finite-dimensional vector space (with a fixed basis) can be concretely realized as a space of functions.

Let I, J be finite sets, and put $V = \mathscr{F}(I, \mathbb{K}), W = \mathscr{F}(J, \mathbb{K})$. Fix a basis $\{e_i \mid i \in I\}$ of V and $\{f_j \mid j \in J\}$ of W. Then there is an isomorphism

$$\begin{aligned} V \otimes_{\mathbb{K}} W &\longrightarrow \mathscr{F}(I \times J, \mathbb{K}) \\ x \otimes y &\longmapsto [(i,j) \mapsto x_i y_j] \end{aligned}$$

which associates simple tensors in $V \otimes_{\mathbb{K}} W$ with functions $I \times J \to \mathbb{K}$ which factor into the form fg where $f \in \mathscr{F}(I, \mathbb{K})$ and $g \in \mathscr{F}(J, \mathbb{K})$. Another nice property of $\text{FinVect}_{\mathbb{K}}$ is that there is a canonical isomorphism

$$\begin{aligned} V' \otimes_{\mathbb{K}} W &\longrightarrow \text{Hom}(V, W) \\ \phi \otimes y &\longmapsto [\phi(-)y : V \to W, x \mapsto \phi(x)y] \end{aligned} \tag{NT2}$$

In summary, for finite dimensional $\mathbb{K}$-vector spaces V, W, there are several equivalent ways to define $V \otimes_{\mathbb{K}} W$:

(a) algebraically, as a quotient of $\mathscr{F}_{af}(V \times W, \mathbb{K})$
(b) $\text{Hom}(W', V)$
(c) $\text{Hom}(V', W)$

(d) $\mathscr{L}_2(V', W'; \mathbb{K})$
(e) $\mathscr{F}(I \times J, \mathbb{K})$, where I is a basis of V and J is a basis of W.

What about infinite dimensional spaces (e.g. Hilbert, Banach, normed)? For such spaces V, W we cannot hope for the spaces (a)–(e) listed above to each be equivalent to $V \otimes_{\mathbb{K}} W$; even by taking topological duals, $V \otimes_{\mathbb{K}} W$ is "too small". In addition to taking topological duals, we can endow $V \otimes_{\mathbb{K}} W$ with an appropriate topology. The completion $V \widehat{\otimes}_{\mathbb{K}} W$ of $V \otimes_{\mathbb{K}} W$ with respect to this topology, called the **topological tensor product**, will take the place of $V \otimes_{\mathbb{K}} W$.

We shall use the term **nuclear** to refer to spaces which are "effectively finite dimensional". Nuclear spaces V, W have the property that the definitions (a)–(e) above equivalently define $V \widehat{\otimes}_{\mathbb{K}} W$. The spaces $\mathscr{D}, \mathcal{S}, \mathscr{D}', \mathcal{S}'$ with which we will work are nuclear and reflexive. In other words, these are "good" spaces.

Duality Pairings

There are techniques besides taking duals and tensor products which will be useful for us. Here, we describe the method of "duality pairing." This is a technique for constructing new vector spaces out of existing vector spaces defined as finite tensor products of V and V', for V a finite-dimensional vector space.

Let V be a finite-dimensional $\mathbb{K}$-vector space with basis $(e_i)_{1 \le i \le d}$. The dual space V' has a basis $(e'_i)_{1 \le i \le d}$, where $e'_i(e_j) := \delta_{ij}$ for $i, j \in [d]$. The following is an example of a duality pairing. For any $A \in V \otimes V' \otimes V \otimes V'$ and $B \in V' \otimes V \otimes V$, we associate a pairing $A \circ B \in V \otimes V' \otimes V$. The idea is to pair the adjacent factors $V \otimes V'$ and $V' \otimes V$ which are colored here.

Here, we express the vector $A \circ B$ explicitly in terms of a basis. The vector A can be expressed uniquely as a finite sum

$$A = \sum_{(i,j,k,\ell) \in [d]^4} A_{i,j,k,\ell} \cdot e_i \otimes e'_j \otimes e_k \otimes e'_\ell,$$

where each $A_{i,j,k,\ell} \in \mathbb{K}$. And B can be expressed uniquely as

$$B = \sum_{r,s,t \in [d]^3} B_{r,s,t} \cdot e'_r \otimes e_s \otimes e_t.$$

We then define

$$A \circ B := \sum_{(i,j,t) \in [d]^3} C_{i,j,t} \cdot e_i \otimes e'_j \otimes e_t,$$

where

$$C_{i,j,t} := \sum_{(k,\ell)\in[d]^2} A_{i,j,k,\ell} B_{k,\ell,t}.$$

To show this, we observe that the map

$$\widetilde{\widetilde{S}} : ((V \otimes V') \times V \times V') \times (V' \times V \times V) \to \mathbb{K}$$

given by

$$\widetilde{\widetilde{S}} : (x, v_1, f_1, f_2, v_2, z) \mapsto f_1(v_1) f_2(v_2) x \otimes z$$

is multilinear. [Here, $x \in V \otimes V'$, $v_i \in V$ and $f_i \in V'$ for $i = 1, 2$, and z is a member of the final factor V.] Hence, there is a unique linear map

$$\widetilde{S} : ((V \otimes V') \otimes V \otimes V') \otimes (V' \otimes V \otimes V) \to \mathbb{K}$$

such that

$$\widetilde{S} : (x \otimes v_1 \otimes f_1 \otimes f_2 \otimes v_2 \otimes z) \mapsto f_1(v_1) f_2(v_2) x \otimes z.$$

Finally, we let $S := \widetilde{S}\iota$, where ι is the canonical map

$$\begin{aligned}\iota : &((V \otimes V') \otimes V \otimes V') \times (V' \otimes V \otimes V)) \\ &\to ((V \otimes V') \otimes V \otimes V') \otimes (V' \otimes V \otimes V)).\end{aligned}$$

The map S is a well-defined linear map whose definition is independent of any choice of basis on V. Therefore, we aim to show that $S = T$, where

$$T : ((V \otimes V') \otimes V \otimes V') \times (V' \otimes V \otimes V) \to \mathbb{K}$$

is such that

$$T : (A, B) \mapsto A \circ B.$$

T is linear, and so it suffices to show S, T agree on basis vectors

$$e = (e_i \otimes e'_j \otimes e_k \otimes e'_\ell, e'_r \otimes e_s \otimes e_t),$$

for $i, j, k, \ell, r, s, t \in [d]$. This vector e is mapped to $e_i \otimes e'_j \otimes e_t$ under both S and T, and so the two maps agree.

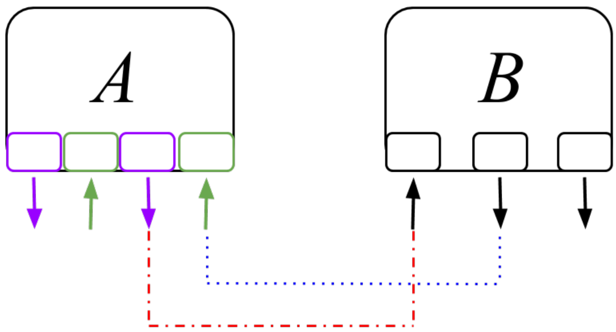
The key element of this construction is the natural *duality pairing* between V' and V given by evaluation. The map $V' \times V \to \mathbb{K}$ given by $(\phi, v) \mapsto \phi(v)$ is bilinear. Hence, there is unique linear map $V' \otimes V \to \mathbb{K}$ such that $\phi \otimes v \mapsto \phi(v)$.

Note that in the explicit definition of $A \circ B$, we sum over the repeated indices k, ℓ. It is often convenient to use Einstein's convention that repeated indices are automatically summed.

$$A_{i,j,k,\ell} B_{k,\ell,t} \quad \textit{denotes} \quad \sum_{(k,\ell)\in[d]} A_{i,j,k,\ell} B_{k,\ell,t}.$$

Example 0.2.1. Many familiar operations such as trace and matrix products can be cast as duality pairings. For instance, if $V := \mathbb{K}^d$ with canonical basis $(e_i)_{1\le i\le d}$ and dual basis $(e'_i)_{1\le i\le d}$, we have an isomorphism between $\mathrm{Hom}(V, V)$ and $V' \otimes V$ which identifies $e'_i \otimes e_j$ with the linear map $f_{i,j}$ such that $f_{i,j}(e_k) = \mathbb{1}\{k = i\} e_j$. The duality pairing sends $e'_i \otimes e_j$ to $e'_i(e_j) = \delta_{ij}$, and the trace of each $f_{i,j}$ is δ_{ij}. Hence, the trace is a duality pairing.

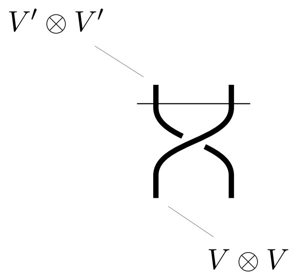
To illustrate the duality pairing construction construction, it is helpful to illustrate the pairing as a pairing of centipedes' legs. The diagram below illustrates this particular pairing of A and B. Downward arrows, colored purple in the "A" diagram, represent tensor factors V and upward arrows, colored green in the "A" diagram, represent tensor factors V' in the underlying vector spaces. The dotted lines represent pairings. Because three arrows are now bound, we see from the diagram that $A \circ B$ is a member of $V \otimes V' \otimes V$.



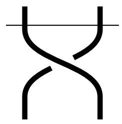
In centipede diagrams, it is convent to represent incoming factors on top of and outgoing factors on the bottom of a centipede instead of putting both on the centipede's bottom. This allows for compact depictions of duality pairings via braid diagrams; paired factors are stacked vertically. For example, if $V = \mathbb{C}^2$ with canonical basis (e_1, e_2), let

$$R \in \mathrm{Hom}(V \otimes V, V \otimes V) \cong V \otimes V \otimes V' \otimes V'$$

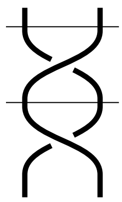
be the map which sends $a \otimes b$ to $b \otimes a$. From the isomorphism, we depict R as a centipede with two incoming factors and two outgoing factors. The strands depict the interchanging of factors. Viewed as a braid, this centipede diagram is a standard building block for braid diagrams:



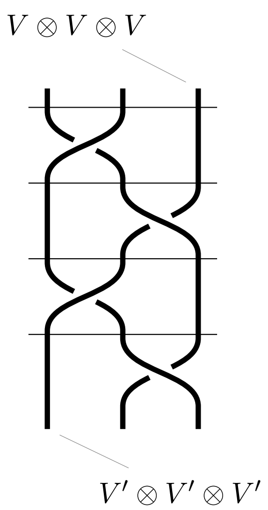
R^{-1} is visualized similarly:



and by composing the two we obtain the identity by pulling the top strand over the bottom. This is a *Reidemeister Type II move*, one of three transformations on braid diagrams which generate all well-defined transformations on braid representations of knots.



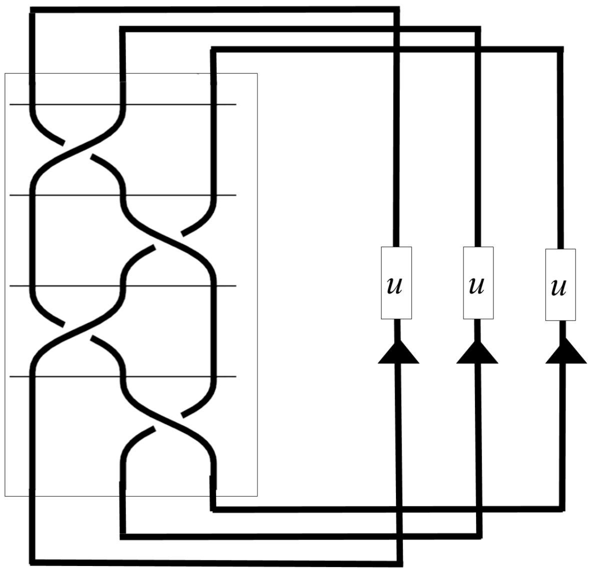
We can extend this construction to arbitrary finite tensor products of V and V':



In the above braid diagram, the following factors are stacked from top to bottom: $R \otimes I_V$, $I_V \otimes R^{-1}$, $R \otimes I_V$, and $I_V \otimes R^{-1}$, where I_V denotes the identity map on V. Hence, the above braid is a centipede diagram for the composite

$$B := (I_V \otimes R^{-1}) \circ (R \otimes I_V) \circ (I_V \otimes R^{-1}) \circ (R \otimes I_V).$$

Now, let u denote the pairing between the corresponding incoming factors V' and outgoing factors V belonging to B. This produces a knot, a braid with no loose strands, as shown below. This entire diagram represents the composition of $u \otimes u \otimes u$ with B.



Note that $u \circ B$ is a member of the base field $\mathbb{K}$, since tensoring with $u \otimes u \otimes u$ evaluates all dual pairs (V', V).

The *Jones Polynomial*, a knot invariant, can be defined as a centipede-braid diagram such as B (i.e., a member of a finite tensor product with an equal number of factors V and V'). These polynomials are evaluated by taking traces. To take the trace of such a finite tensor product is to compose the braid B with the the factor $u \otimes \ldots \otimes u$ which pairs each corresponding incoming and outgoing arrow. Theorems of knot theory show that every knot can be unwound to be a closed braid as in the above diagram and that braids can be generated by simple factors R and R^{-1}. Hence, we obtain a definition of the Jones Polynomial for arbitrary knots.

1

A toolbox

1.1 Combinatorial Analysis: The Arbogast Faà di-Bruno Formula

Theorem 1.1.1. *Let I and J be non-empty intervals in $\mathbb{R}$. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be C^k-functions. Suppose $f(I) \subseteq J$. Then $g \circ f : I \to \mathbb{R}$ is also in $C^k(I)$ and for every $x \in I$:*

$$(g \circ f)^{(k)}(x) = k! \sum_{n \geq 0} \left(\sum_{k_1, \ldots, k_n \geq 1} \mathbb{1} \left\{ \sum_{i=1}^{n} k_i = k \right\} \frac{g^{(n)}(f(x)) f^{(k_1)}(x) \cdots f^{(k_n)}(x)}{n! k_1! \cdots k_n!} \right)$$

Notation: Recall that given a property P:

$$\mathbb{1}\{P\} = \begin{cases} 1 \text{ if P is true;} \\ 0 \text{ if P is false.} \end{cases}$$

Remark. The sum in Theorem 1.1.1 is finite (i.e., the sum terminates). In fact note that:

$$\mathbb{1} \left\{ \sum_{i=1}^{n} k_i = k \right\} \neq 0 \Rightarrow n \leq \sum_{i=1}^{n} k_1 = k$$

since $k_1 \geq 1$ for $1 \leq i \leq n$ and $k_i \leq k$. This implies the sum must be finite.

How to remember it. Use the special case $f(0) = 0$, and f, g analytic functions. Then:

$$\frac{(g \circ f)^{(k)}(0)}{k!} = \text{ coefficient of } z^k \text{ in } f(g(z))$$

Proof. Proceed by induction on k. Suppose the result is true for k, i.e.,:

$$(g \circ f)^{(k)}(x) = k! \sum_{n \geq 0} \left(\sum_{\alpha_1, \ldots, \alpha_n \geq 1} \mathbb{1} \left\{ \sum_{i=1}^{n} \alpha_i = k \right\} \frac{g^{(n)}(f(x)) f^{(\alpha_1)}(x) \cdots f^{(\alpha_n)}(x)}{n! k_1! \cdots k_n!} \right)$$

$$= k! \sum_{n \geq 0} \sum_{\alpha \in \mathbb{N}_0^n} \mathbb{1} \left\{ \begin{array}{c} |\alpha| = k \\ \forall i, \alpha_i \geq 1 \end{array} \right\} \frac{g^{(n)}(f(x))}{n! \alpha!} f^{(\alpha)}(x)$$

Now we differentiate with respect to x:

$$(g \circ f)^{(k+1)}(x) = k! \sum_{n \geq 0} (A_n + B_{n,1} + \cdots + B_{n,n})$$

where A_n is defined as follows:

$$A_n = \sum_{\alpha \in \mathbb{N}_0^n} \mathbb{1} \left\{ \begin{array}{c} |\alpha| = k \\ \forall i, \alpha_i \geq 1 \end{array} \right\} \frac{g^{(n+1)}(f(x))}{n! \alpha!} f^{(\alpha)}(x) f'(x)$$

Set $\beta = (\alpha, 1)$, where $|\alpha| = k$. Then:

$$A_n = \sum_{\beta \in \mathbb{N}_0^{n+1}} \mathbb{1} \left\{ \begin{array}{c} |\beta| = k+1 \\ \forall i, \beta_i \geq 1 \\ \beta_{n+1} = 1 \end{array} \right\} \frac{g^{(n+1)}(f(x))}{n! \beta!} f^{(\beta)}(x) f'(x)$$

$$= \sum_{\beta \in \mathbb{N}_0^{n+1}} \mathbb{1} \left\{ \begin{array}{c} |\beta| = k+1 \\ \forall i, \beta_i \geq 1 \\ \beta_j = 1 \end{array} \right\} \frac{g^{(n+1)}(f(x))}{n! \beta!} f^{(\beta)}(x) f'(x)$$

(Note that for every $j \in [n+1]$ there is $\sigma \in S_{n+1}$ such that $\sigma(n+1) = j$).

$$= \sum_{\beta \in \mathbb{N}_0^{n+1}} \mathbb{1} \left\{ \begin{array}{c} |\beta| = k+1 \\ \forall i, \beta_i \geq 1 \end{array} \right\} \mathbb{1} \left\{ \begin{array}{c} |\beta| = k+1 \\ \beta_j = 1 \end{array} \right\} \frac{g^{(n+1)}(f(x))}{n! \beta!} f^{(\beta)}(x) f'(x)$$

Note that $k + 1 = \beta_1 \cdots + \beta_j + \cdots + \beta_{n+1}$ and $j \in [n+1]$, and so:

$$\mathbb{1} \left\{ \begin{array}{c} |\beta| = k+1 \\ \beta_j = 1 \end{array} \right\} = \frac{1}{n+1} \sum_{j=1}^{n+1} \mathbb{1} \left\{ \ \beta_j = 1 \ \right\}$$

Substituting we see that:

$$A_n = \sum_{\beta \in \mathbb{N}_0^{n+1}} \mathbb{1} \left\{ \begin{array}{c} |\beta| = k+1 \\ \forall i, \beta_i \geq 1 \end{array} \right\} \left(\frac{1}{n+1} \sum_{j=1}^{n+1} \mathbb{1}\{\beta_j = 1\} \right) \frac{g^{(n+1)}(f(x))}{n! \beta!} f^{(\beta)}(x) f'(x)$$

so substituting back in the formula:

$$= k! \sum_{n \geq 0} \sum_{\alpha \in \mathbb{N}_0^n} \mathbb{1} \left\{ \begin{array}{c} |\beta| = k+1 \\ \forall i, \beta_i \geq 1 \end{array} \right\} \left(\sum_{j=1}^{n+1} \mathbb{1}\{\beta_j = 1\} \right) \frac{g^{(n+1)}(f(x))}{(n+1)! \beta!} f^{(\beta)}(x)$$

$$= k! \sum_{n \geq 1} \sum_{\alpha \in \mathbb{N}_0^{n+1}} \mathbb{1} \left\{ \begin{array}{c} |\alpha| = k+1 \\ \forall i, \alpha_i \geq 1 \end{array} \right\} \left(\sum_{j=1}^{n} \mathbb{1}\{\alpha_j = 1\} \right) \frac{g^{(n)}(f(x))}{n! \alpha!} f^{(\alpha)}(x) = \heartsuit$$

On the other hand:

$$B_{n,j} = \sum_{\alpha \in \mathbb{N}_0^n} \mathbb{1}\left\{ \begin{array}{c} |\alpha| = k \\ \forall i, \alpha_i \geq 1 \end{array} \right\} \frac{g^{(n)}(f(x))}{n!\alpha!} f^{(\alpha + e_{n,j})}(x)$$

Set $\beta = \alpha + e_{n,j}$. Then:

$$\begin{aligned} B_{n,j} &= \sum_{\alpha \in \mathbb{N}_0^n} \mathbb{1}\left\{ \begin{array}{c} |\beta| = k+1 \\ \forall i, \beta_i \geq 1 \\ \beta_j \geq 2 \end{array} \right\} \frac{g^{(n)}(f(x))}{n!(\beta - e_{n,j})!} f^{(\beta)}(x) \\ &= \sum_{\alpha \in \mathbb{N}_0^n} \mathbb{1}\left\{ \begin{array}{c} |\beta| = k+1 \\ \forall i, \beta_i \geq 1 \\ \beta_j \geq 2 \end{array} \right\} \frac{g^{(n)}(f(x))}{n!} \frac{\beta_j}{\beta!} f^{(\beta)}(x) \end{aligned}$$

and so we see that:

$$\sum_{j=1}^{n} B_{n,j} = \sum_{\alpha \in \mathbb{N}_0^n} \mathbb{1}\left\{ \begin{array}{c} |\alpha| = k+1 \\ \forall i, \alpha_i \geq 1 \end{array} \right\} (\mathbb{1}\{\alpha_i \geq 2\}) \frac{g^{(n)}(f(x)) f^{(\alpha)}(x)}{n!\alpha!} = \clubsuit$$

And since $(g \circ f)^{(k+1)}(x) = \heartsuit + \clubsuit$ we see that:

$$(g \circ f)^{(k+1)}(x) = k! \sum_{n \geq 0} \sum_{\alpha \in \mathbb{N}_0^n} \mathbb{1}\left\{ \begin{array}{c} |\alpha| = k \\ \forall i, \alpha_i \geq 1 \end{array} \right\} \frac{g^{(n)}(f(x)) f^{(\alpha)}(x)}{n!\alpha!} C_{n,\alpha}$$

where $C_{n,\alpha}$ is as follows:

$$C_{n,\alpha} = \mathbb{1}_{\{n \geq 1\}} \sum_{j=1}^{n} \mathbb{1}\{\alpha_j = 1\} + \sum_{j=1}^{n} \mathbb{1}\{\alpha_j \geq 2\}\alpha_j = \sum_{j=1}^{n} \mathbb{1}\{\alpha_j = 1\}\alpha_j + \sum_{j=1}^{n} \mathbb{1}\{\alpha_j \geq 2\}\alpha_j$$

$$\sum_{j=1}^{n} \alpha_j (\mathbb{1}\{\alpha_j = 1\} + \mathbb{1}\{\alpha_j \geq 2\}) = \sum_{j=1}^{n} \alpha_j \mathbb{1}\{\alpha_j \geq 1\} = \sum_{j=1}^{n} \alpha_j = |\alpha| = k+1$$

□

We now take a look at the Leibniz Product Rule in $\mathbb{R}^d$.

Proposition 1.1.2. *Let $f : U \to \mathbb{K}, g \to \mathbb{K}$ be C^k functions. Then on U, for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, we have*

$$\partial^\alpha (fg) = \sum_{\beta, \gamma \in \mathbb{N}_0^d} \mathbb{1}\{\beta + \gamma = \alpha\} \frac{\alpha!}{\beta! \, \gamma!} (\partial^\beta f)(\partial^\gamma g)$$

Proof. We prove this using induction on $|\alpha|$.

• $|\alpha| = 1$: Suppose w.l.g. that $\alpha_1 = 1, \alpha_i = 0 \ \forall \ i = 2, 3, ..., d$. $\beta + \gamma = \alpha$ splits up into two sums with $\beta_1 = 1, \beta_i = 0, \gamma_j = 0$ and $\beta_j = 0, \gamma_1 = 1, \gamma_i = 0 \ \forall \ j = 1, 2, ..., d, \ i = 2, 3, ..., d$.

So, RHS is $\frac{\partial f}{\partial x_1} g + f \frac{\partial g}{\partial x_1}$ which is just the product rule in one-variable (keeping all the other variables fixed). Hence, base case of induction is true.

Now suppose that the result holds for all $|\alpha| \leq k$

• $|\alpha| = k+1$: We can write $\alpha = \alpha' + \lambda$ with $|\alpha'| = k, |\lambda| = 1$. Thus, by definition, $\partial^\alpha(fg) = \partial^\lambda(\partial^{\alpha'}(fg))$.

$$\partial^\lambda(\partial^{\alpha'}(fg)) = \partial^\lambda\Big(\sum_{\beta',\gamma' \in \mathbb{N}_0^d} \mathbb{1}\{\beta' + \gamma' = \alpha'\} \frac{\alpha'!}{\beta'!\,\gamma'!} (\partial^{\beta'} f)(\partial^{\gamma'} g)\Big) \text{(by induction hypothesis)}$$

$$= \sum_{\beta',\gamma' \in \mathbb{N}_0^d} \mathbb{1}\{\beta' + \gamma' = \alpha'\} \frac{\alpha'!}{\beta'!\,\gamma'!} \Big[(\partial^{\beta'+\lambda} f)(\partial^{\gamma'} g) + (\partial^{\beta'} f)(\partial^{\gamma'+\lambda} g)\Big]$$

$$\text{(using base case)}$$

$$= \sum_{\beta',\gamma' \in \mathbb{N}_0^d} \mathbb{1}\{\beta' + \gamma' = \alpha'\} \frac{\alpha'!}{\beta'!\,\gamma'!} (\partial^{\beta'+\lambda} f)(\partial^{\gamma'} g)$$

$$+ \sum_{\beta',\gamma' \in \mathbb{N}_0^d} \mathbb{1}\{\beta' + \gamma' = \alpha'\} \frac{\alpha'!}{\beta'!\,\gamma'!} (\partial^{\beta'} f)(\partial^{\gamma'+\lambda} g)$$

$$= \sum_{\beta,\gamma \in \mathbb{N}_0^d} \mathbb{1}\{\beta + \gamma = \alpha\} \frac{(\alpha-\lambda)!}{(\beta-\lambda)!\,\gamma!} (\partial^{\beta} f)(\partial^{\gamma} g)$$

$$+ \sum_{\beta,\gamma \in \mathbb{N}_0^d} \mathbb{1}\{\beta + \gamma = \alpha\} \frac{(\alpha-\lambda)!}{\beta!\,(\gamma-\lambda)!} (\partial^{\beta} f)(\partial^{\gamma} g)$$

(using $\beta = \beta' + \lambda, \gamma = \gamma'$ in the first sum; $\beta = \beta', \gamma = \gamma' + \lambda$ in the second sum; $\alpha = \alpha' + \lambda$)

$$= \sum_{\beta,\gamma \in \mathbb{N}_0^d} \mathbb{1}\{\beta + \gamma = \alpha\} \Big[\frac{(\alpha-\lambda)!}{(\beta-\lambda)!\,\gamma!} + \frac{(\alpha-\lambda)!}{\beta!\,(\gamma-\lambda)!}\Big] (\partial^{\beta} f)(\partial^{\gamma} g)$$

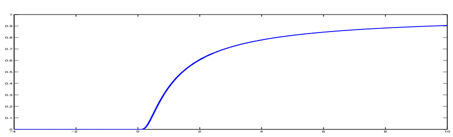
$$= \sum_{\beta,\gamma \in \mathbb{N}_0^d} \mathbb{1}\{\beta + \gamma = \alpha\} \frac{\alpha!}{\beta!\,\gamma!} (\partial^{\beta} f)(\partial^{\gamma} g)$$

using the fact that $\dfrac{(\alpha-\lambda)!}{(\beta-\lambda)!\,\gamma!} + \dfrac{(\alpha-\lambda)!}{\beta!\,(\gamma-\lambda)!} = \dfrac{\alpha!}{\beta!\,\gamma!}$ for $|\gamma| = 1$. □

1.2 Bump Functions & Smooth Tensor-like Partitions of Unity

The key tool for this section will be the following function:

$$h(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$



Proposition 1.2.1. *h is C^∞.*

Proof. On $(0,\infty)$, $h(x) = g(f(x))$ where $g(x) = e^x, f(x) = -\frac{1}{x}$. Observe that the derivatives look like $g^{(n)}(x) = e^x, f^{(n)}(x) = (-1)^{n-1}n!x^{-(n+1)}$. So, using the Faà di Bruno formula, we have

$$\begin{aligned} h^{(k)}(x) &= k! \sum_{n\geq 0} \sum_{\alpha\in\mathbb{N}_0^n} \mathbb{1}\Big\{|\alpha| = k, \alpha_i \geq 1 \;\forall\; i\Big\} \frac{e^{-\frac{1}{x}}}{n!\,\alpha!} \prod_{i=1}^{n} \Big[(-1)^{\alpha_i-1}\alpha_i!\,x^{-\alpha_i-1}\Big] \\ &= e^{-\frac{1}{x}} k!\,(-1)^k x^{-k} \sum_{n\geq 0} \sum_{\alpha\in\mathbb{N}_0^n)} \mathbb{1}\Big\{|\alpha| = k, \alpha_i \geq 1 \;\forall\; i\Big\} \frac{1}{n!}(-1)^n x^{-n} \end{aligned}$$

Now impose $k \geq 1$.

$$h^{(k)}(x) = e^{-\frac{1}{x}} k!\,\frac{(-1)^k}{x^k} \sum_{n=1}^{k} \frac{1}{n!}\Big(-\frac{1}{x}\Big)^n \sum_{\alpha\in\mathbb{N}_0^n} \mathbb{1}\Big\{|\alpha| = k, \alpha_i \geq 1 \;\forall\; i\Big\}.$$

Note that if $\beta_i = \alpha_i - 1$, then as the middle quantity is precisely counting all possible ways of getting $\sum_{i=1}^{n} \beta_i = k - n$ with the condition $\beta_i \geq 0$ for each i.

So, we get

$$\begin{aligned} h^{(k)}(x) &= e^{-\frac{1}{x}} k!\,\frac{(-1)^k}{x^k} \sum_{n=1}^{k} \frac{1}{n!}\Big(-\frac{1}{x}\Big)^n \binom{k-1}{n-1} \\ &= e^{-\frac{1}{x}} \frac{1}{x^{2k}} \mathbb{P}_k(x) \end{aligned}$$

where $\mathbb{P}_k(x) = k!(-1)^k \sum_{n=1}^{k} \frac{(-1)^n}{n!} \binom{k-1}{n-1} x^{k-n}$ is a polynomial of degree $k-1$.

Observe that for all $n \in \mathbb{Z}$, $\lim_{x\to 0^+} e^{-\frac{1}{x}} x^n = 0$. This shows that for all k, $\lim_{x\to 0^+} h^{(k)}(x) = 0$, which in turn implies that h is C^∞.

□

Definition 1.2.2. Let I be an open interval in $\mathbb{R}$ and $\sigma \geq 1$. A function $f : I \to \mathbb{K}$ is called **σ-Gevrey** if for all compact sets $C \subset I$, there exist $A_C, B_C > 0$ such that

$$\Big|f^{(k)}(x)\Big| \leq A_C B_C^k (k!)^\sigma$$

for all $x \in \mathbb{C}$ and $k \geq 0$.

Remark 1.2.3. $\sigma = 1$ if and only if f is analytic. Indeed, if f is σ-Gevrey with $\sigma = 1$, then the Taylor series for f centered at any $x \in I$ has a positive radius of convergence, and hence we can find a domain (open, connected set) $U \subset \mathbb{C}$ with $U \cap \mathbb{R} = I$ and a holomorphic function $g : U \to \mathbb{C}$ such that $f = g|_I$.

Remark 1.2.4. Since an analytic function has discrete zero set, our h is clearly not analytic.

Theorem 1.2.5. *For all $k \geq 0$ and $x \in \mathbb{R}$ we have $|h^{(k)}(x)| \leq 8^k (k!)^2$. Thus h is σ-Gevrey with $\sigma = 2$.*

Proof. Let $k \geq 1, x > 0$. Following the above discussion, we have

$$\begin{aligned}
|h^{(k)}(x)| &\leq e^{-\frac{1}{x}} k! \sum_{n=1}^{k} \frac{1}{n!} \frac{1}{x^{n+k}} \frac{(k-1)!}{(n-1)!\,(k-n)!} \\
&\leq k! \sum_{n=1}^{k} \frac{(n+k)!\,(k-1)!}{n!\,(n-1)!\,(k-n)!} && \text{using } \frac{1}{x^{n+k}} \leq (n+k)! e^{\frac{1}{x}} \\
&= k! \sum_{n=1}^{k} \frac{(n+k)!\,k!\,(k-1)!}{n!\,k!\,(n-1)!\,(k-n)!} \\
&= (k!)^2 \sum_{n=1}^{k} \binom{n+1}{k} \frac{(k-1)!}{(n-1)!\,(k-n)!} \\
&\leq (k!)^2 \sum_{n=1}^{k} 2^{n+k} 2^{k-1} && \text{using } \binom{m}{i} \leq 2^m \\
&= (k!)^2 2^{2k-1} \sum_{n=1}^{k} 2^n \\
&\leq 8^k (k!)^2
\end{aligned}$$

□

Remark 1.2.6.

$$\frac{d^k}{dx^k}\left(e^{-\frac{1}{x}}\right) = \frac{k!}{2\pi i} \oint_\gamma \frac{e^{-\frac{1}{z}}}{(z-x)^{k+1}} dz$$

where γ is a the boundary of some disk centred at 0.

Proof. Note that $f(z) = e^{-\frac{1}{z}}$ is a holomorphic function on $\mathbb{C} \setminus \{0\}$. So, by the Cauchy integral formula applied to the holomorphic function $f^{(k)}(z)$ on the same domain, we have for any $x > 0$,

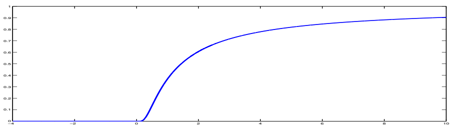
$$f^{(k)}(x) = \frac{d^k}{dx^k}\left(e^{-\frac{1}{x}}\right) = \frac{k!}{2\pi i} \oint_\gamma \frac{e^{-\frac{1}{z}}}{(z-x)^{k+1}} dz$$

.

□

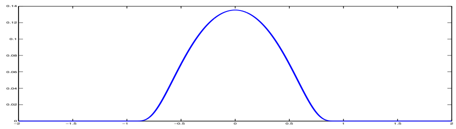
In the previous lecture we proved that the function

$$h(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$



is infinitely differentiable (i.e. smooth) everywhere. Now consider

$$\varphi_0(x) = h(x+1)h(1-x)$$

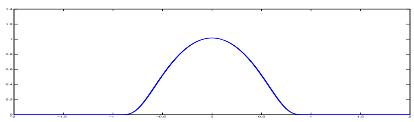


By the construction of φ_0 we see that

- $\varphi_0 \in C^\infty$
- $\varphi_0 \geq 0$
- φ_0 is even
- $\text{supp}(\varphi_0) \subset [-1, 1]$
- $\varphi_0 \neq 0$ when $x \in (-1, 1)$

Our goal in this section is to construct a grid-like partition of unity on $\mathbb{R}^d$ based on such φ_0. First, let

$$\varphi_1(x) = \left(\int_{\mathbb{R}} \varphi_0(t) dt \right)^{-1} \varphi_0(x)$$

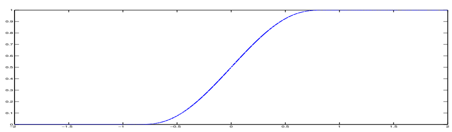


Then, besides the above 5 properties φ_1 also satisfies

- $\displaystyle\int_{\mathbb{R}} \varphi_1(x) dx = 1$

In order to construct a "buffer function" between constants 0 and 1, we let

$$\varphi_2(x) = \int_{-\infty}^{x} \varphi_1(t) dt$$

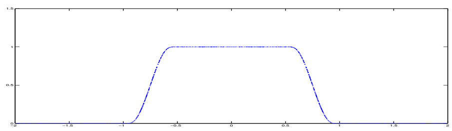


From those properties of φ_1 we know

- $\varphi_2 \in C^\infty$
- $\varphi_2 = 0$ when $x \leq -1$
- $\varphi_2 = 1$ when $x \geq 1$
- $0 < \varphi_2 < 1$ when $-1 < x < 1$
- φ_2 is strictly increasing on $(-1, 1)$
- φ_2 is symmetric at $(0, 1/2)$

Now we squish φ_2 horizontally by $1/4$, translate it to the left by $3/4$ and multiply the reflection by y-aixs:

$$\varphi_3(x) = \varphi_2(4x+3)\varphi_2(3-4x)$$

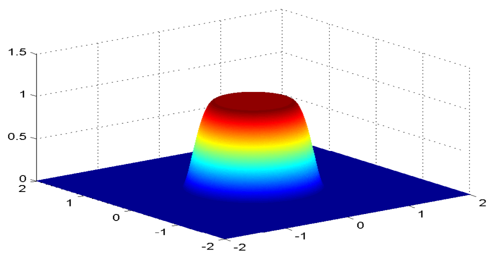


From the properties of φ_2 we know

- $\varphi_3 \in C^\infty$
- φ_3 is even
- $\varphi_3 = 0$ if $x \in (-\infty, -1] \cup [1, \infty)$
- $\varphi_3 = 1$ if $x \in [-\frac{1}{2}, \frac{1}{2}]$
- $0 < \varphi_3 < 1$ if $x \in (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$
- φ_3 is strictly increasing on $(-1, -\frac{1}{2})$ and strictly decreasing on $(\frac{1}{2}, 1)$
- $\varphi_3|_{(-1,-\frac{1}{2})}$ is symmetric at $(-\frac{3}{4}, \frac{1}{2})$ and $\varphi_3|_{(\frac{1}{2},1)}$ is symmetric at $(\frac{3}{4}, \frac{1}{2})$

To make a "bump function" on $\mathbb{R}^d$ (the picture below is a bump function on $\mathbb{R}^2$) we can take

$$\Phi(x) = \varphi_3(|x|)$$



which "inherits" most of the features of φ_3:

- $\Phi \in C^\infty$
- Φ is radially symmetric
- $\Phi = 0$ if $x \in \mathbb{R}^d \setminus B(0,1)$
- $\Phi = 1$ if $x \in \overline{B(0,\frac{1}{2})}$
- $0 < \Phi < 1$ if $x \in B(0,1) \setminus \overline{B(0,\frac{1}{2})}$

For the first property: one can write out explicitly all higher order partial derivatives of Φ except at $x = 0$, but obviously Φ is also smooth at $x = 0$ since it is a constant on a neighborhood.

Definition 1.2.7. For any subset $\Omega \subset \mathbb{R}^d$ we define

$$\mathscr{D}(\Omega, \mathbb{K}) := \{f \in C^\infty(\Omega) \mid \overline{\text{supp}(f)} \subset \Omega\}.$$

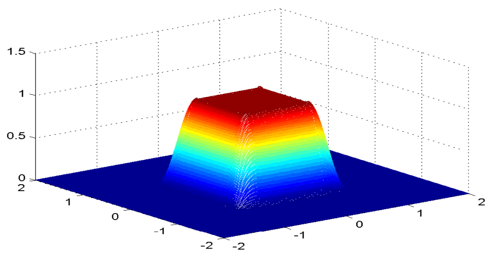
where the closure is taken in $\mathbb{R}^d$.

Proposition 1.2.8. *If $\Omega \subset \mathbb{R}^d$ has non-empty interior then $\mathscr{D}(\Omega, \mathbb{K}) \neq \{0\}$.*

Proof. The interior being non-empty means there is a ball $B_r(x_0) \subset \Omega$. Then from the above construction we know $\Phi(|x - x_0|/r) \in \mathscr{D}(\Omega, \mathbb{K})$. □

Definition 1.2.9 (Tensor-like bump function)**.** To make a "cube-supported" bump function we let

$$\Psi(x) = \varphi_3(x_1)\varphi_3(x_2)\ldots\varphi_3(x_d)$$



Similar to what Φ does, this gives a transition map from a large cube $\{x \mid |x_i| \leq 1\}$ to a smaller cube $\{x \mid |x_i| \leq \frac{1}{2}\}$.

To construct a partition of unity on $\mathbb{R}^d$, we define a copy of tensor-like bump function on each "integer tile" so that they add up to 1. First let us consider this in one dimension: we want a sequence $\{\psi_n\}_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}} \psi_n(x) = 1$. Let

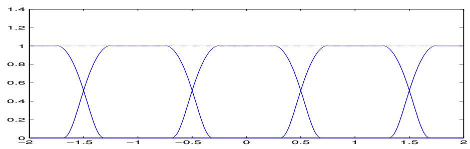
$$\phi(x) = \varphi_2(4x + 2)\varphi_2(2 - 4x)$$

which satisfies

- $\phi \in C^\infty$
- ϕ is even
- $\phi = 0$ if $x \in (-\infty, -\frac{3}{4}] \cup [\frac{3}{4}, \infty)$
- $\phi = 1$ if $x \in [-\frac{1}{4}, \frac{1}{4}]$
- $0 < \phi < 1$ if $x \in (-\frac{3}{4}, -\frac{1}{4}) \cup (\frac{1}{4}, \frac{3}{4})$
- $\phi\big|_{(-\frac{3}{4},-\frac{1}{4})}$ is symmetric at $(-\frac{1}{2}, \frac{1}{2})$ and $\phi\big|_{(\frac{1}{4},\frac{3}{4})}$ is symmetric at $(\frac{1}{2}, \frac{1}{2})$

It is clear that $\sum_{n\in\mathbb{Z}} \phi(x-n) = 1$ when $x \in [-\frac{1}{4}+n, \frac{1}{4}+n]$. The symmetry properties (the second one and the last one) also garuantee that on the transition areas two adjacent $\phi(x-n)$'s add up to 1. Therefore

$\sum_{n\in\mathbb{Z}} \phi(x-n) = 1$ for all $x \in \mathbb{R}$



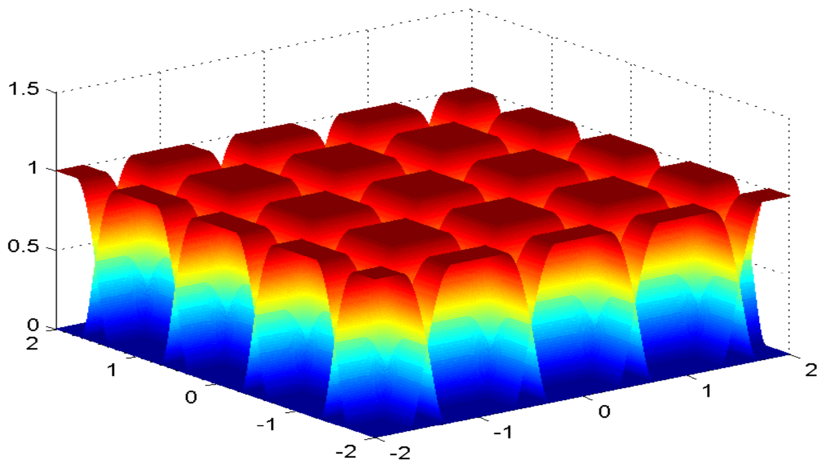
Proposition 1.2.10. *Define index $N = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$ and let*

$$\psi(x) = \phi(x_1)\phi(x_2)\ldots\phi(x_d)$$

Then $\sum_{N\in\mathbb{Z}^d} \psi(x-N) = 1$ for all $x \in \mathbb{R}^d$.

Proof. By Tonelli's theorem (in discrete measure), for each fixed x

$$\begin{aligned}\sum_{N\in\mathbb{Z}^d} \psi(x-N) &= \sum_{N\in\mathbb{Z}^d} \phi(x_1-n_1)\phi(x_2-n_2)\ldots\phi(x_d-n_d)\\ &= \left(\sum_{n_1\in\mathbb{Z}} \phi(x_1-n_1)\right)\left(\sum_{n_2\in\mathbb{Z}} \phi(x_2-n_2)\right)\ldots\left(\sum_{n_d\in\mathbb{Z}} \phi(x_d-n_d)\right)\\ &\equiv 1\cdot 1\ldots 1 = 1\end{aligned}$$



□

1.3 The Local/Nonlinear Change of Variable Formula

Recall from Real Analysis I that an affine map is a map on $\mathbb{R}^d$ of the form

$$x \mapsto c + Tx$$

where $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear invertible map and $c \in \mathbb{R}^d$. Recall also the Global/Affine Change of Variable Formula

Theorem 1.3.1 (Global/Affine Change of Variable Formula)**.** *If $f : \mathbb{R}^d \to [0, \infty]$ is Borel measurable or $f : \mathbb{R}^d \to \mathbb{K}$ is Borel measurable and Lebesgue integrable, then*

$$\int_{\mathbb{R}^d} f(c + Tx) \, |\det T| \, d^d x = \int_{\mathbb{R}^d} f(x) d^d x.$$

Recall that a domain is a nonempty open connected subset of $\mathbb{R}^d$.

Corollary 1.3.2. *Let $U \subseteq \mathbb{R}^d$ be a domain. If $f : U \to [0, \infty]$ is Borel measurable or $f : U \to \mathbb{K}$ is Borel measurable and Lebesgue integrable, then*

$$\int_U f(c + Tx) \, |\det T| \, d^d x = \int_U f(x) d^d x.$$

Proof Sketch. Extend f by zero:

$$\tilde{f}(x) = \begin{cases} f, & x \in U \\ 0, & x \notin U \end{cases}$$

Then apply Theorem 1.3.1 to $\tilde{f}$. □

1.3.1 Local Change of Variables, Version 1.0

Definition 1.3.3. Let $U, V \subseteq \mathbb{R}^d$ be open and nonempty, and let $f : U \to V$. We call f **Frechét differentiable** on U if for all $x \in U$ there exists an $L : \mathbb{R}^d \to \mathbb{R}^d$ linear such that for $y \neq x$,

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y-x)|}{|y-x|} = 0.$$

L called the **Frechét derivative**, and denote it by $D_x f$. We say f is C^1 if the function $U \subseteq \mathbb{R}^d \to \mathrm{Hom}(\mathbb{R}^d, \mathbb{R}^d)$ given by $x \mapsto D_x f$ is continuous, and we say that f is a **diffeomorphism** if f is bijective and $f^{-1} : V \to U$ is also C^1.

Notation. Let $x \in \mathbb{R}^d$.

- The determinant of the matrix of $D_x f$ in the canonical basis,

$$J_x f = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \le i,j \le d},$$

  is called the **Jacobian determinant** of f at x.

- $|\cdot|$ will denote the Euclidean norm.

- If $A : \mathbb{R}^d \to \mathbb{R}^d$ is linear, then we denote the operator norm on A by

$$\|A\| = \sup_{x \in \mathbb{R}^d \setminus \{0\}} \frac{|Ax|}{|x|}.$$

- For $r > 0$, the open ball of radius r centered at x is denoted $B(x, r)$.

- The closed ball of radius r centered at x is denoted $\overline{B}(x, r)$.

Theorem 1.3.4 (Local Change of Variables, Version 1.0)**.** *Let $U, V \subseteq \mathbb{R}^d$ be open and nonempty. Let $f : U \to V$ be a C^1 diffeomorphism and $g \in \mathcal{D}(V, \mathbb{K})$. Then*

$$\int_U g(f(x)) \; |J_x f| \; d^d x = \int_V g(z) \; d^d z.$$

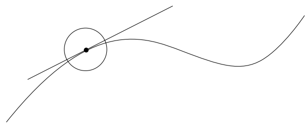
The proof we choose will be complicated, but we choose it in order to

- learn some important techniques, and

- use grid (tensor)-like partitions of unity developed in the last section.

Idea/Intuition of Proof

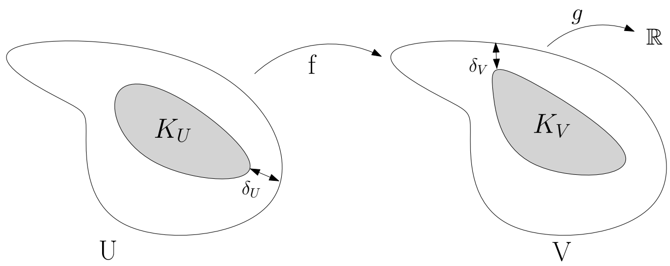
In calculus, nice maps locally look like affine maps.



So, we will

1. decompose the integral into local pieces (because the map $f \mapsto \int f$ is linear),
2. use the affine change of variables on the pieces, and
3. collect.

Prep Work



Let $K_V := \text{supp}(g)$, and $K_U := f^{-1}(K_V) = \text{supp}(g \circ f)$. Note that both K_U and K_V are compact. Define δ_U to be the distance between K_U and U^c and δ_V to be the distance between K_V and V^c. Note that both are positive values. Indeed, because U^c is closed and K_U is compact,

$$\begin{aligned}\delta_U &= d(K_U, U^c) \\ &= \inf_{x \in K_U} d(x, U^c) \\ &= \inf_{x \in K_U} \inf_{y \in U^c} |x - y| \\ &= \min_{x \in K_U} \inf_{y \in U^c} |x - y|.\end{aligned}$$

Similarly $\delta_V > 0$.

Remark: If $g = 0$, result follows almost immediately so assume g is nonzero. Hence, K_U and K_V are nonempty. In addition, if $U = \mathbb{R}^2$, then distance between K_U and U^c is infinite so take $\delta_U = 1$

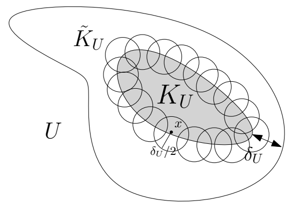
Let

$$\widetilde{K_U} := \{x + y \in \mathbb{K}^d : (x, y) \in K_U \times \overline{B}(0, \delta_U/2)\}$$

and

$$\widetilde{K}_V := \{x + y \in \mathbb{K}^d : (x, y) \in K_V \times \overline{B}(0, \delta_V/2)\}$$

be **thickenings** for K_U and K_V, respectively.



Note that $\widetilde{K_U}$ is also compact because it is the continuous image of the compact set $K_U \times \overline{B}(0, \delta_U/2)$, and similarly $\widetilde{K_V}$ is compact.

Remark 1.3.5. Note that

$$\mathrm{vol}(\widetilde{K_U}) := \int_{\widetilde{K_U}} 1 d^d x < \infty,$$

and similarly, $\widetilde{K_V}$ has finite volume.

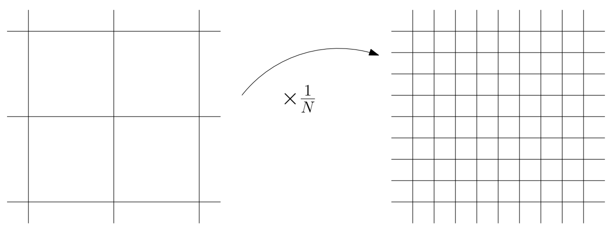
Recall our partition of unity from Section 1.2: Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a partition of unity, i.e. it satisfies the following:

- For all $x \in \mathbb{R}^d$, $\sum_{y \in \mathbb{Z}^d} \psi(x - y) = 1$.
- ψ is C^∞
- $\psi \equiv 1$ on $[-1/4, 1/4]^d$
- $\mathrm{supp}(\psi) \subseteq [-3/4, 3/4]^d \subset \bar{B}(0, 3\sqrt{d}/4)$

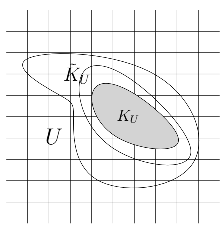
Let $N \geq 1$ be an integer, and define $\psi_N := \psi(Nx)$. Then for all $x \in \mathbb{R}^d$,

$$\sum_{y \in (\frac{1}{N}\mathbb{Z})^d} \psi_n(x - y) = 1.$$

This "shrinks" the grid from before.



Note that $\mathrm{supp}(\psi_N) \subset \bar{B}(0, 3\sqrt{d}/4N)$.



Define

$$L_N := \{y \in \left(\frac{1}{N}\mathbb{Z}\right)^d : B(y, \sqrt{d}/N) \cap K_U \neq \emptyset\}.$$

This will collect the points in the small grid that are within $\sqrt{d}/N$ of K_U.

In order to prove Theorem 1.3.4, we need the following five lemmas.

Lemma 1.3.6. *There exists an $N_1 \in \mathbb{N}_0$ such that for all $N \geq N_1$,*

$$|L_N| \leq N^d \times Vol(\widetilde{K_U}).$$

Proof. Note that L_N is at most countably infinite since it is counting points on the grid.

$$\begin{aligned}
\sum_{y\in L_N} 1 &= \sum_{y\in L_N} \int_{\mathbb{R}^d} \psi(x) d^d x \\
&= \sum_{y\in L_N} \int_{\mathbb{R}^d} \psi(N(z-y)) d^d z \qquad \text{Change of variables, } x = N(z-y) \\
&= \sum_{y\in L_N} \int_{\mathbb{R}^d} \psi_N(z-y) d^d z \\
&= N^d \int_{\mathbb{R}^d} \left(\sum_{y\in L_N} \psi_N(z-y) \right) d^d z \quad \text{Tonelli}
\end{aligned}$$

The integrand is nonzero and since $\mathrm{supp}(\psi_N) \subset \bar{B}(0, 3\sqrt{d}/4N)$, there exists $y \in L_N$ such that $|z-y| \leq 3\sqrt{d}/4N$. Then by construction of L_N, there is some $w \in K_U$ such that $|y-w| \leq \sqrt{d}/N$. By application of triangle inequality, we get

$$|z-w| \leq \frac{7\sqrt{d}}{4N} \leq \frac{\delta_U}{2}$$

where the last inequality is true provided

$$N \geq N_1 := \left\lceil \frac{7\sqrt{d}}{2\delta_U} \right\rceil.$$

It follows that $z \in \widetilde{K_U}$ and therefore

$$\sum_{y\in L_N} 1 \leq N^d \int_{\mathbb{R}^d} 1 \times \mathbb{1}\{z \in \widetilde{K_U}\} d^d z$$

where the integral is the volume of $\widetilde{K_U}$. This completes the proof. □

The next lemma essentially proves g is uniformly Lipshitz on space V.

Lemma 1.3.7. *There exists a constant $c_1 > 0$ such that for any $z_1, z_2 \in V$, $|g(z_1) - g(z_2)| \leq c_1|z_1 - z_2|$.*

Proof. Define $\tilde{g} : \mathbb{R} \to \mathbb{K}$ to be the extension of g by 0 outside V, i.e.,

$$\tilde{g}(z) := \begin{cases} g(z) & z \in V \\ 0 & \text{otherwise} \end{cases}.$$

Since $g \in \mathscr{D}(V, \mathbb{K})$ and V is compact, it follows that $\tilde{g}$ is C^∞ on $\mathbb{R}^d$. For any $z_1, z_2 \in \mathbb{R}^d$, we can write $\tilde{g}(z_1)$ in the form of interpolation between z_1 and z_2.

$$\begin{aligned} \tilde{g}(z_1) &= \tilde{g}(z_2 + t(z_1 - z_2))\Big|_{t=1} \\ &= \tilde{g}(z_2) + \int_0^1 \frac{d}{dt}\Big(\tilde{g}(z_2 + t(z_1 - z_2))\Big)\, dt && \text{FTC} \\ &= \tilde{g}(z_2) + \int_0^1 \Big(D_{z_2+t(z_1-z_2)}\tilde{g}\Big)(z_1 - z_2)\, dt && \text{M.V. Chain Rule} \end{aligned}$$

It follows that

$$\begin{aligned} |\tilde{g}(z_1) - \tilde{g}(z_2)| &\leq \sup_{0 \leq t \leq 1} \Big| D_{z_2+t(z_1-z_2)}\tilde{g}\Big)(z_1 - z_2)\Big| \\ &\leq |z_1 - z_2| \sup_{0 \leq t \leq 1} \Big\| D_{z_2+t(z_1-z_2)}\tilde{g}\Big\| \\ &\leq |z_1 - z_2| \sup_{z \in \mathbb{R}} \|D_z \tilde{g}\| \end{aligned}$$

Set $c_1 := \sup_{z \in K_V} \|D_z \tilde{g}\| + 1$. Note that $c_1 < \infty$ because its bounded for $z \in V$ (since compact) and 0 on its complement. This gives us the desired inequality and completes the proof. □

Note that the extension was done because V is not necessarily convex so we could interpolate between two points in V which takes us outside the set. The following lemma is similar in flavor.

Lemma 1.3.8. *There exists a constant $c_2 > 0$ such that for any $x_1, x_2 \in \mathbb{R}^d$, $|\Psi(x_1) - \Psi(x_2)| \leq c_2|x_2 - x_1|$.*

Proof. Same. □

The next lemma is a uniform version of Frechet differentiation.

Lemma 1.3.9. *For any $\epsilon > 0$, there exists $\eta > 0$ such that for any $x, y \in \tilde{K}_U$ where $|x - y| \leq \eta$, we have $|f(x) - f(y) - D_y f(x - y)| \leq \epsilon|x - y|$.*

Proof. For any $\alpha > 0$, define

$$D_\alpha := \{(x,y) \in \tilde{K_U}^2 \,:\, |x-y| \leq \alpha\} \subset (\mathbb{R}^d)^2$$

to be the set of pairs of points in $\tilde{K_U}$ within a distance of α of each other. Because it is a product of compact sets, D_α is also compact. Consider the map

$$D_\alpha \times [0,1] \longrightarrow \mathbb{R}^d$$

sending

$$(x,y,t) \longmapsto y + t(x-y) \quad \in [x,y].$$

Denote $\tilde{K}_{U,\alpha}$ as the image of this map. Observe this set is also compact and it contains $\tilde{K_U}$ for if $x \in \tilde{K_U}$, then $(x,x) \in D_\alpha$ and so $(x,x,0) \mapsto x \in \tilde{K}_{U,\alpha}$.

Claim: for small enough $\alpha > 0$, $\tilde{K}_{U,\alpha} \subset U$. It is enough to set $\alpha < \delta_U/2$ to get this result (see image below).

Fix $\alpha = \delta_U/4$ and $\epsilon > 0$. Consider the map

$$\begin{cases} \tilde{\tilde{K_U}} \longrightarrow \mathscr{L}(\mathbb{R}^d, \mathbb{R}^d) \\ z \longmapsto D_z f \end{cases}$$

(recall $f : U \mapsto V$ is a C^1 diffeomorphism.) Since $wide\tilde{\tilde{K_U}}$ is compact, this map is uniformly continuous so there exists $\eta > 0$ such that for all $x, y \in \tilde{\tilde{K_U}}$ where $|x-y| \leq \eta$,

$$\|D_x f - D_y f\| < \epsilon.$$

Note that there is no harm in taking $\eta \leq \delta_U/4$.

Fix $x, y \in \tilde{K_U}$ such that $|x-y| \leq \eta$. Note that $|x-y| \leq \eta \leq \alpha$ implies $[x,y] \in \tilde{\tilde{K_U}} \subset U$ and so we can interpolate between x and y. By Fundamental Theorem of Calculus and chain rule, we have

$$f(x) = f(y) + \int_0^1 \Big(D_{y+t(x-y)} f\Big)(x-y)\, dt.$$

Therefore

$$f(x) - f(y) - D_y f(x-y) = \int_0^1 \Big[D_{y+t(x-y)} f(x-y) - D_y f(x-y)\Big]\, dt$$

whose norm is bounded by

$$|x-y| \sup_{t\in[0,1]} \Big\|D_{y+t(x-y)} f - D_y f\Big\| \leq |x-y|\epsilon$$

which completes the proof. □

We have an analogous lemma and proof over the space $\tilde{K_V}$. Here however, we'll use the inverse of f on V (since f is a diffeomorphism).

Lemma 1.3.10. *For any $\epsilon > 0$, there exists $\eta > 0$ such that for any $z, w \in \tilde{K}_V$ where $|z-w| \leq \eta$, we have $|f^{-1}(z) - f^{-1}(w) - D_w(f^{-1})(z-w)| \leq \epsilon|z-w|$.*

Proof. Same. □

We are now ready to prove Theorem 1.3.4.

Proof of Theorem 1.3.4. Set $N \geq N_1$ where N_1 is taken from Lemma 1.3.6. Apply the partition of unity on Ψ_N to left side of desired equation.

$$\begin{aligned}\int_U g(f(x))\,|J_x f|\,d^d x &= \int_U 1 \cdot g(f(x))\,|J_x f|\,d^d x \\ &= \int_U \left(\sum_{y \in (\frac{1}{N}\mathbb{Z})^d} \Psi_N(x-y) \right) g(f(x))\,|J_x f|\,d^d x\end{aligned}$$

Notice that if $\Psi_N(x-y) \neq 0$, then it follows

$$x - y \in \frac{1}{N}\left[-\frac{3}{4}, \frac{3}{4}\right]^d \subset \overline{B}\left(0, \frac{3\sqrt{d}}{4N}\right) \subset B\left(0, \frac{\sqrt{d}}{N}\right)$$

and if we assume $g(f(x)) \neq 0$, then it follows $x \in K_U$. Futhermore, since $|x-y| < \sqrt{d}/N$, then $y \in L_N$ by definition. Therefore, we can replace the sum with one over L_N, i.e.,

$$\int_U g(f(x))|\,|J_x f|\,d^d x = \int_U \left(\sum_{y \in L_N} \Psi_N(x-y) \right) g(f(x))\,|J_x f|\,d^d x.$$

Since this is a finite sum, we can interchange the integral and sum.

$$\int_U g(f(x))|J_x f| d^d x = \sum_{y \in L_N} \int_U \Psi_N(x-y) g(f(x))\,|J_x f|\,d^d x$$

Approximation. Since x and y are "close" to each other, we want to approximate $|J_x f|$ by $|J_y f|$. If we can do that, then we pick up an error term E_1 and so

$$\int_U g(f(x))\,|J_x f|\,d^d x = E_1 + \sum_{y \in L_N} \int_U \Psi_N(x-y) g(f(x))\,|J_y f|\,d^d x.$$

We just need to verify that $|J_y f|$ is well defined (potential issue is that y could be outside K_U and so we need to check y isn't outside U.) Since $y \in L_N$, the distance between y and K_U is bounded by $\sqrt{d}/N$. Assume now that $N \geq N_2 := \left\lceil 4\sqrt{d}/\delta_U \right\rceil$. Because $4\sqrt{d}/N \leq \delta_U/2$, it follows that

$$B\left(y, \frac{\sqrt{d}}{N}\right) \subset \tilde{K_U} \subset U$$

and so $y \in U$. Therefore $|J_y f|$ is well defined.

end of lecture 10 □

We left off where the LHS was equal to $\int_U g(f(x))|J_x f|d^d x$ and noted that $|J_x(f)| \approx |J_y(f)|$. With that approximation in mind, we define an error term E_1 such that the LHS is

$$E_1 + \sum_{y \in L_N} \int_U \psi_N(x-y) g(f(x))|J_y f|d^d x$$

where we consider N such that
$N \geq N_2 := \lceil * \rceil \frac{4\sqrt{d}}{\delta_v} \Rightarrow \forall y \in L_N, y \in B\Big(y, \frac{\sqrt{d}}{N}\Big) \subset \widetilde{K_U} \subset U$
to address well-definedness. Now we claim that the LHS is in fact

$$E_1 + \sum_{y \in L_N} \int_{B(y, \frac{\sqrt{d}}{N})} \psi_N(x-y) g(f(x))|J_y f|d^d x$$

which follows from the fact that $x \in U \setminus B(y, \frac{\sqrt{d}}{N}) \Rightarrow \psi_N(x-y) = 0$. Now we attempt to approximate the LHS again by noting that $f(x) \approx f(y) + D_y f(x-y)$. Recall first that $\text{supp}\psi_N \subset B\Big(0, \frac{3\sqrt{d}}{4N}\Big) \subset B\Big(0, \frac{\sqrt{d}}{N}\Big)$, so our approximation suggests we define a new error term E_2 such that the LHS is

$$E_1 + E_2 + \int_{B(y, \frac{\sqrt{d}}{N})} \psi_N(x-y) g(f(y) + D_y f(x-y))|J_y f|d^d x,$$

so of course we must ensure that $\int_{B(y, \frac{\sqrt{d}}{N})} \psi_N(x-y) g(f(y)+D_y f(x-y))|J_y f|d^d x$ is well-defined in order to proceed. We need that $f(y) + D_y f(x-y) \in V$. We will show a stronger statement: for N sufficiently large, for all $y \in L_N$, we have $f(y) + D_y f B\Big(0, \frac{\sqrt{d}}{N}\Big) \subset \widetilde{K_V} \subset V$.

We start by applying lemma 4, in particular let $\epsilon_1 = 1$, so the lemma implies there exists $\nu_1 > 0$ such that for all $v, w \in \widetilde{K_U}$, we have $|v-w| \leq \nu_1 \Rightarrow |f(v) - f(w) - D_w(v-w)| \leq 1 * |v-w|$. Now let $y \in L_N$, so there exists $u \in K_U$ such that $|u-y| < \frac{\sqrt{d}}{N}$. Furthermore, we let $x \in B\Big(y, \frac{\sqrt{d}}{N}\Big) \Rightarrow v, y \in \widetilde{K_U}$. Note that $u \in K_U \Rightarrow u \in \widetilde{K_U}$ and $y \in \widetilde{K_U}$ from the previous assumption that $N \geq N_\epsilon$. Note that if $N \geq N_3 := \lceil * \rceil \frac{\sqrt{d}}{\nu_1}$, then $|u-y| < \sqrt{d}/N \leq \nu_1 \Rightarrow |f(u) - f(y) - D_y f(u-y)| \leq |u-y|$ but $|D_y f(x-u)| \leq ||D_y f|| * |x-u|$. Therefore $|f(u) - f(y) - D_y f(x-y)| \leq |f(u) - f(y) - D_y f(u-y) - D_y f(x-u)| \leq |u-y| + c_3|x-u|$ where $c_3 := sup_{v \in \widetilde{K_V}} ||D_U f|| < \infty$ (by compactness). Note that $|u-y| < \sqrt{d}/N$ and $|x-u|$ is at most $2\sqrt{d}/N$. Therefore, $d(f(y) + D_y f(x-y), K_v) \leq \frac{(1+2c_3)\sqrt{d}}{N} \leq \delta_v/2$.

We now impose the restriction that $N \geq N_4 := \lceil * \rceil \frac{2\sqrt{d}(1+2c_3)}{\delta_v}$, so $f(y) + D_y f(x-y) \in \widetilde{K_V}$. Therefore, we proved the claim that for N large, then for all $y \in L_N$, $f(y) + D_y f B\Big(0, \frac{\sqrt{d}}{N}\Big) \subset \widetilde{K_V} \subset V$. We go back to our previous concern: the LHS is equal to

$$E_1 + E_2 + \int_{B(y, \frac{\sqrt{d}}{N})} \psi_N(x-y) g(f(y) + D_y f(x-y)) |J_y f| d^d x.$$

Now apply the affine change of variables formula for a new variable $z := f(y) + D_y f(x-y)$ where y is fixed. The LHS then becomes

$$E_1 + E_2 + \sum_{y \in L_N} \int_{f(y)+D_y f B\left(0, \frac{\sqrt{d}}{N}\right)} \psi_N((D_y f)^{-1}(z - f(y))) g(z) d^d x.$$

Now we use the approximation $(D_y f)^{-1}(z - f(y)) \approx f^{-1}(z) - y$ to consider a new error term E_3 defined by the claim that the LHS is

$$E_1 + E_2 + E_3 + \sum_{y \in L_N} \int_{f(y)+D_y f B\left(0, \frac{\sqrt{d}}{N}\right)} \psi_N(f^{-1}(z) - y) g(z) d^d x.$$

This expression defines E_3 so long as the integral above is well-defined. Note that $f(y) + D_y f B\Big(0, \frac{\sqrt{d}}{N}\Big) \subset \widetilde{K_V} \subset V$; our next step is that we want to change $f(y) + D_y f B\Big(0, \frac{\sqrt{d}}{N}\Big)$ into V. To do so, we need $\psi_N(f^{-1}(z) - y) = 0$ on the complement. Note that $\psi_N(f^{-1}(z) - y) \neq 0 \to f^{-1} - y \in \Big[\frac{-3}{4N}, \frac{3}{4N}\Big]^d$. Therefore $z \in f(y + \Big[\frac{-3}{4N}, \frac{3}{4N}\Big]^d) \subset f(\bar{B}(y, 3\sqrt{d}/N)) \subset f(B(y, \sqrt{d}/N)) \subset U$. We now need to verify that $f(\bar{B}(y, 3\sqrt{d}/N)) \subset f(y) + D_y f(B(0, \sqrt{d}/N))$. Note that $|x-y| \leq \sqrt{d}/4N \Rightarrow |(D_y f)^{-1}(f(x) - f(y))| < \sqrt{d}/N$. Notice that

$$(D_y f)^{-1}(f(x) - f(y)) = (D_y f)^{-1}(f(x) - f(y) - D_y f(x-y)) + (D_y f)^{-1} D_y f(x-y)$$

$$\Rightarrow |(D_y f)^{-1}(f(x) - f(y))| \leq ||D_{f(y)}(f^{-1})|| * |f(x) - f(y) - D_y f(x-y)| + |x-y|$$

where $||D_{f(y)}(f^{-1})|| \leq c_4 := sup_{v \in \widetilde{K_V}} ||D_v f^{-1}|| < \infty$ as $\widetilde{K_V}$ is compact. Not that $g \neq 0 \Rightarrow K_v \subset \widetilde{K_V} \neq \emptyset$, $c_4 > 0$, and f is C^1 differentiable. Now, we use lemma 4 with $\epsilon_2 = \frac{1}{4c_4} \Rightarrow \exists \nu_2 > 0$ such that

$$|f(x) - f(y) - D_y f(x-y)| < \frac{1}{4c_4} |x-y|$$

provided that $y \in \widetilde{K_U}, x \in \widetilde{K_U}, |x-y| \leq \nu_2$. However, we note that $y \in L_N$ and $x \in \bar{B}\Big(y, \frac{3\sqrt{d}}{4N}\Big) \subset B\Big(y, \frac{\sqrt{d}}{N}\Big) \subset \widetilde{K_U}$ as $N \geq N_2$. Furthermore, we now enforce the requirement that

$$N \geq N_5 := \lceil * \rceil \frac{3\sqrt{d}}{4\nu_2} \Rightarrow |x-y| \leq \frac{3\sqrt{d}}{4N} \leq \nu_2$$

$\Rightarrow |(D_y f)^{-1}(f(x) - f(y))| \leq (c_4 * \frac{1}{c_4} + 1)|x-y| \leq \frac{5}{4}|x-y| \leq \frac{5*3}{4*4}\frac{\sqrt{d}}{N} < \frac{\sqrt{d}}{N}$.

We've now proven the claim that for N sufficiently large,

$$f(\bar{B}\Big(y, \frac{3\sqrt{d}}{4N}\Big) \subset f(y) + D_y f B\Big(0, \frac{\sqrt{d}}{N}\Big),$$

so that the LHS is indeed

$$E_1 + E_2 + E_3 + \sum_{y \in L_N} \int_V \psi_N(f^{-1}(z) - y) g(z) d^d x.$$

We wish to prove that

$$\text{LHS} := \int_U g(f(x)) \, |J_x f| \; \mathrm{d}^d x = \int_V g(z) \, \mathrm{d}^d z := \text{RHS}.$$

We have already shown that $N \geq N_1,,, N_5$ implies

$$\text{LHS} = E + \sum_{y \in L_N} \int_V \psi_N(f^{-1}(z) - y) g(z) \, \mathrm{d}^d z$$

where $E = E_1 + E_2 + E_3$, the sum of three error terms that are the result of three approximations:

(E_1): $|J_x f| \approx |J_y f|$

(E_2): $g\Big(f(x)\Big) \approx g\Big(f(y) + D_y f(x-y)\Big)$

(E_3): $\psi_N\Big((D_y f)^{-1}(z - f(y))\Big) \approx \psi_N\Big(f^{-1}(z) - y\Big)$.

L_N is finite, so

$$\text{LHS} = E + \int_V \left(\sum_{y \in L_N} \psi_N(f^{-1}(z) - y) \right) g(z) \, \mathrm{d}^d z.$$

Suppose that $g(z) \neq 0$ and $\psi_N(f^{-1}(z) - y) \neq 0$.

- $g(z) \neq 0 \implies z \in K_V := \text{Supp}(g)$
  $\implies f^{-1}(z) \in K_U := f^{-1}(K_V) = \text{Supp}(g \circ f)$.

- $\psi_N(f^{-1}(z) - y) \neq 0 \implies \Big|f^{-1}(z) - y\Big| \leq \frac{3\sqrt{d}}{4N} \leq \frac{\sqrt{d}}{N}$.

The above two facts combine to get

$$f^{-1}(z) \in K_U \cap B\Big(y, \tfrac{\sqrt{d}}{N}\Big) \neq \varnothing$$
$$\implies y \in L_N \text{ (by the definition of } L_N).$$

This allows us to conclude

$$\begin{aligned}
\text{LHS} &= E + \int_V \left(\sum_{y \in L_N} \psi_N(f^{-1}(z) - y) \right) g(z) \, \mathrm{d}^d z \\
&= E + \int_V \left(\sum_{y \in (\frac{1}{N}\mathbb{Z})^d} \psi_N(f^{-1}(z) - y) \right) g(z) \, \mathrm{d}^d z \\
&= E + \int_V (1) g(z) \, \mathrm{d}^d z \\
&= \text{RHS}.
\end{aligned}$$

Yayy!!!

Now that we have proved LHS = RHS, let's get some bounds on the error $E = E_1 + E_2 + E_3$ by bounding each error term. Fix $\varepsilon > 0$.

E_1 **error estimate:**

By definition,

$$E_1 \coloneqq \sum_{y \in L_N} \int_U \psi_N(x - y) g(f(x)) \Big(|J_x f| - |J_y f| \Big) \, \mathrm{d}^d x.$$

We know that f is C^1-differentiable and $|J_x f|$ is a polynomial in the 1^{st} partial derivatives of f. Therefore, $|J_x f|$ is uniformly continuous on $\widetilde{K_U} \subset U$.

$$\exists \eta_3 > 0, \forall x, y \in \widetilde{K_U}, \left(|x - y| \leq \eta_3 \implies \Big| |J_x f| - |J_y f| \Big| < \varepsilon \right).$$

This gives us the bound

$$|E_1| \leq \sum_{y \in L_N} \int_U \psi_N(x - y) \, |g(f(x))| \Big| |J_x f| - |J_y f| \Big| \, \mathrm{d}^d x.$$

We already know that

- $y \in L_N \implies y \in \widetilde{K_U}$,
- $g(f(x)) \neq 0 \implies x \in K_U \subset \widetilde{K_U}$,
- and $\psi_N(x - y) \neq 0 \implies |x - y| \leq \frac{3\sqrt{d}}{4N} \leq \eta_3$ under the condition that $N \geq N_6 \coloneqq \left\lceil \frac{3\sqrt{d}}{4\eta_3} \right\rceil$.

This gives

$$\begin{aligned}
|E_1| &\leq \sum_{y\in L_N} \int_U \psi_N(x-y)\,|g(f(x))| \Big| |J_x f| - |J_y f| \Big| \,\mathrm{d}^d x \\
&\leq \sum_{y\in L_N} \int_U \psi_N(x-y)\,|g(f(x))|\,\varepsilon\,\mathrm{d}^d x \\
&\leq \sum_{y\in L_N} \int_U \psi_N(x-y)\,\|\cdot\|_{L^\infty} g \mathbb{1}\{x\in K_U\}\,\varepsilon\,\mathrm{d}^d x \\
&\leq \varepsilon\,\|\cdot\|_{L^\infty} g \sum_{y\in L_N} \int_U \psi_N(x-y) \mathbb{1}\{x\in K_U\}\,\mathrm{d}^d x \\
&\leq \varepsilon\,\|\cdot\|_{L^\infty} g \sum_{y\in L_N} \int_{K_U} \psi_N(x-y)\,\mathrm{d}^d x \\
&\leq \varepsilon\,\|\cdot\|_{L^\infty} g \int_{K_U} \sum_{y\in L_N} \psi_N(x-y)\,\mathrm{d}^d x \\
&\leq \varepsilon\,\|\cdot\|_{L^\infty} g \int_{K_U} 1\,\mathrm{d}^d x \\
&\leq \varepsilon\,\|\cdot\|_{L^\infty} g\, \mathrm{Vol}(K_U)\,.
\end{aligned}$$

E_2 **error estimate:**

By definition,

$$E_2 \coloneqq \sum_{y\in L_N} \int_{B\left(y,\frac{\sqrt{d}}{N}\right)} \psi_N(x-y) \Big(g(f(x)) - g(f(y) - D_y f(x-y)) \Big) |J_y f| \,\mathrm{d}^d x$$

$$|E_2| \leq \sum_{y\in L_N} \int_{B\left(y,\frac{\sqrt{d}}{N}\right)} \psi_N(x-y) \Big| g(f(x)) - g(f(y) - D_y f(x-y)) \Big| |J_y f| \,\mathrm{d}^d x$$

We know that $y \in L_N \subset \widetilde{K}_U$ when $N \geq N_2$, so we have the bound $|J_y f| \leq c_5 \coloneqq \sup_{u\in\widetilde{K}_U} |J_u f| < \infty$.

$$|E_2| \leq c_5 \sum_{y\in L_N} \int_{B\left(y,\frac{\sqrt{d}}{N}\right)} \psi_N(x-y) \Big| g(f(x)) - g(f(y) - D_y f(x-y)) \Big| \,\mathrm{d}^d x$$

Also, by Lemma 2 (or Lemma 1.3.7), we have $|g(f(x)) - g(f(y) - D_y f(x-y))| \leq c_1\,|f(x) - f(y) - D_y f(x-y)|$. Lemma 4 states that $\exists \eta_4 > 0$ such that $\forall u, v \in \widetilde{K}_U$, we have $(|u-v| \leq \eta_4 \implies |f(x) - f(y) - D_y f(x-y)| \leq \varepsilon |u-v|)$.

Therefore, since $y \in \widetilde{K}_U$, choosing $N \geq N_4 \coloneqq \left\lceil \frac{\sqrt{d}}{\eta_4} \right\rceil$, and because $x \in B\left(y, \frac{\sqrt{d}}{N}\right)$, we have $|x-y| < \frac{\sqrt{d}}{N} \leq \eta_4$ and thus $|f(x) - f(y) - D_y f(x-y)| \leq$

$\varepsilon|x-y| \leq \varepsilon\frac{\sqrt{d}}{N}$. Putting this result back in our error bound, we compute

$$\begin{aligned}
|E_2| &\leq \varepsilon c_1 c_5 \frac{\sqrt{d}}{N} \sum_{y\in L_N} \int_{B\left(y,\frac{\sqrt{d}}{N}\right)} \psi_N(x-y)\,\mathrm{d}^d x \\
&\leq \varepsilon c_1 c_5 \frac{\sqrt{d}}{N} \int_{B\left(y,\frac{\sqrt{d}}{N}\right)} \sum_{y\in L_N} \psi_N(x-y)\,\mathrm{d}^d x \\
&\leq \varepsilon c_1 c_5 \frac{\sqrt{d}}{N} \int_{B\left(y,\frac{\sqrt{d}}{N}\right)} 1\,\mathrm{d}^d x \\
&\leq \varepsilon c_1 c_5 \frac{\sqrt{d}}{N} \operatorname{Vol}\left(B\left(y,\frac{\sqrt{d}}{N}\right)\right) \\
&\leq \varepsilon c_1 c_5 \frac{\sqrt{d}}{N} \operatorname{Vol}\left(\widetilde{K_U}\right).
\end{aligned}$$

E_3 **error estimate:**

By definition,

$$E_3 := \sum_{y\in L_N} \int_{f(y)+D_y f B\left(0,\frac{\sqrt{d}}{N}\right)} \left[\psi_N\left((D_y f)^{-1}(z-f(y)\right) - \psi_N(f^{-1}(z)-y)\right] g(z)\,\mathrm{d}^d z.$$

We use Lemma 3, the definition $\psi_N(\cdot) := \psi(N\cdot)$, and the fact that ψ is Lipshitz (with constant c_2) to get

$$\begin{aligned}
E_3 &:= \sum_{y\in L_N} \int_{f(y)+D_y f B\left(0,\frac{\sqrt{d}}{N}\right)} \left[\psi_N\left((D_y f)^{-1}(z-f(y)\right) - \psi_N(f^{-1}(z)-y)\right] g(z)\,\mathrm{d}^d z \\
|E_3| &\leq \|\cdot\|_{L^\infty g} \sum_{y\in L_N} \int_{f(y)+D_y f B\left(0,\frac{\sqrt{d}}{N}\right)} \left|\psi_N\left((D_y f)^{-1}(z-f(y)\right) - \psi_N(f^{-1}(z)-y)\right| \mathrm{d}^d z \\
|E_3| &\leq c_2 N \|\cdot\|_{L^\infty g} \sum_{y\in L_N} \int_{f(y)+D_y f B\left(0,\frac{\sqrt{d}}{N}\right)} \left|(D_y f)^{-1}(z-f(y)-(f^{-1}(z)-y)\right| \mathrm{d}^d z
\end{aligned}$$

We already showed for $N \geq N_1,,,N_4$, $\forall y \in L_N$, we have $f(y)+D_y f B\left(0,\frac{\sqrt{d}}{N}\right) \subset \widetilde{K_V} \subset V$.

For such $y \in L_N$, denote $w := f(y)$. Let $z \in f(y)+D_y f B\left(0,\frac{\sqrt{d}}{N}\right)$. Then

$$\begin{aligned}
&\left|(D_y f)^{-1}(z-f(y)-(f^{-1}(z)-y)\right| \\
&= \left|(D_y f)^{-1}(z-f(y)) - f^{-1}(z)+y\right| \\
&= \left|(D_y f)^{-1}(z-w) - f^{-1}(z)+f^{-1}(w)\right| \\
&= \left|D_w(f^{-1})(z-w) - f^{-1}(z)+f^{-1}(w)\right|
\end{aligned}$$

because $(D_y f)^{-1}$ and $D_w(f^{-1})$ represent the same linear map, but going in opposite directions (this is a more general version of the formula $[f^{-1}(w)]' = \frac{1}{f'(y)}$ where $w = f(y)$).

Recall that Lemma 5 states $\exists \eta_5 > 0$ such that $\forall u, v \in \widetilde{K_V}$, we have $(|u - v| \leq \eta_5 \implies |f^{-1}(u) - f^{-1}(v) - D_u(f^{-1})(u - v)| \leq \varepsilon |u - v|)$.

In our case, $z, w \in \widetilde{K_V}$.

$z - w \in D_y f B\left(0, \frac{\sqrt{d}}{N}\right)$

$\implies |z - w| \leq \| \cdot \| \, D_y f \frac{\sqrt{d}}{N} \leq c_3 \frac{\sqrt{d}}{N} \leq \eta_5$ when $N \geq N_8 := \left\lceil c_3 \frac{\sqrt{d}}{\eta_5} \right\rceil$. Now

$$
\begin{aligned}
|E_3| &\leq \| \cdot \|_{L^\infty g} c_2 N \sum_{y \in L_N} \int_{f(y) + D_y f B\left(0, \frac{\sqrt{d}}{N}\right)} \varepsilon c_3 \frac{\sqrt{d}}{N} \, \mathrm{d}^d z \\
&\leq \varepsilon c_2 c_3 \sqrt{d} \, \| \cdot \|_{L^\infty g} \sum_{y \in L_N} \int_{f(y) + D_y f B\left(0, \frac{\sqrt{d}}{N}\right)} \mathrm{d}^d z
\end{aligned}
$$

We perform an affine change of variables replacing z with x using the equation $z = f(y) + D_y f(x)$, getting

$$
|E_3| \leq \varepsilon c_2 c_3 \sqrt{d} \, \| \cdot \|_{L^\infty g} \sum_{y \in L_N} \int_{B\left(0, \frac{\sqrt{d}}{N}\right)} |J_y f| \, \mathrm{d}^d x.
$$

Again since $y \in \widetilde{K_U}$, we use the bound c_5 to get

$$
\begin{aligned}
|E_3| &\leq \varepsilon c_2 c_3 c_5 \sqrt{d} \, \| \cdot \|_{L^\infty g} \sum_{y \in L_N} \int_{B\left(0, \frac{\sqrt{d}}{N}\right)} 1 \, \mathrm{d}^d x \\
&\leq \varepsilon c_2 c_3 c_5 \sqrt{d} \, \| \cdot \|_{L^\infty g} \, |L_N| \int_{B\left(0, \frac{\sqrt{d}}{N}\right)} 1 \, \mathrm{d}^d x
\end{aligned}
$$

Yet another affine change of variables, this time using the substitution $x = \frac{\sqrt{d}}{N} u$, gives

$$
\begin{aligned}
|E_3| &\leq \varepsilon c_2 c_3 c_5 \sqrt{d} \, \| \cdot \|_{L^\infty g} \, |L_N| \int_{B(0,1)} \left(\frac{\sqrt{d}}{N}\right)^d \mathrm{d}^d u \\
&\leq \varepsilon c_2 c_3 c_5 \sqrt{d}^{d+1} \, \| \cdot \|_{L^\infty g} \, |L_N| \left(\frac{1}{N}\right)^d \int_{B(0,1)} 1 \, \mathrm{d}^d u \\
&\leq \varepsilon c_2 c_3 c_5 \sqrt{d}^{d+1} \, \| \cdot \|_{L^\infty g} \, |L_N| \left(\frac{1}{N}\right)^d \mathrm{Vol}(B(0,1))
\end{aligned}
$$

Finally, requiring $N \geq N_1$, Lemma 1 guarantees

$$
|E_3| \leq \varepsilon c_2 c_3 c_5 \sqrt{d}^{d+1} \, \| \cdot \|_{L^\infty g} \, \mathrm{Vol}\left(\widetilde{K_U}\right) \mathrm{Vol}(B(0,1)) \,.
$$

By taking $\varepsilon \to 0$ and $N \to \infty$, we see that all of the error terms vanish, completing the proof of the Theorem. ◻

Remark 1.3.11. We note a few points about the technique of the above proof.

- Compactness was used heavily; it provided us with uniform estimates (constant bounds) and uniform continuity.
- It is important to keep track of dependencies, e.g., many of our estimates depended on N.
- In proofs like that of Theorem 1.3.4, it is common to employ "big O" (O) or "little o (o) notations, or even the notation $f \lesssim g$ to mean $f \leq cg$ for some constant $c > 0$. One should exercise extreme caution when employing such notations because it is quite easy to make a mistake which is then particularly difficult for the reader to self-correct.

Theorem 1.3.12 (Smooth Urysohn Lemma)**.** *Let $\varnothing \neq K \subset U \subset \mathbb{R}^d$ with K compact and U open. Then there exists a smooth function $\rho : \mathbb{R}^d \to \mathbb{R}$ such that $\rho|_K \equiv 1$ and supp$(\rho) \subset U$.*

Proof. For $\delta > 0$, define

$$\widetilde{K_\delta} := \bigcup_{x \in K} \overline{B}(x, \delta)$$

By choosing $\delta < \min\{d(K, U^c), 1\}$, we can ensure that $\widetilde{K_\delta} \subset U$. As before, for $N \in \mathbb{N}$, define

$$L_N := \left\{ y \in \frac{1}{N}\mathbb{Z}^d \;\Big|\; \exists\, x \in K \text{ such that } |x - y| < \frac{\sqrt{d}}{N} \right\}$$

We claim that for N sufficiently large,

$$\rho(x) := \sum_{y \in L_N} \psi_N(x - y) = \sum_{y \in L_N} \psi(N(x - y))$$

is such a function. If $N \geq N_0 := \lceil \frac{7\sqrt{d}}{4\delta} \rceil$, then supp$(\rho) \subset \widetilde{K_\delta} \subset U$. Indeed, let $y \in L_n$, and suppose that $\psi_N(x - y) \neq 0$ for some $x \in \mathbb{R}^d$, so that $|x - y| \geq \frac{3\sqrt{d}}{4N}$. Choose $x_0 \in K$ such that $|y - x_0| < \sqrt{d}/N$. Then

$$|x - x_0| \leq |x - y| + |y - x_0| < \frac{7\sqrt{d}}{4N} \leq \delta \implies x \in \widetilde{K_\delta} \subset U$$

It remains to show that $\rho(x) = 1$ for any $x \in K$. Because

$$1 = \sum_{y \in \frac{1}{N}\mathbb{Z}^d} \psi_N(x - y),$$

it suffices to show that $\psi_N(x - y) = 0$ for any $y \in \frac{1}{N}\mathbb{Z}^d \setminus L_N$. Indeed, for any such y,

$$|x - y| \geq \frac{\sqrt{d}}{N} > \frac{3\sqrt{d}}{4N} \implies \psi_N(x - y) = 0$$

□

Theorem 1.3.13 (Smooth Functions Approximate Continuous Functions)**.** *Let K_1 and K_2 be compact sets contained in an open set $U \subseteq \mathbb{R}^d$ with $\varnothing \neq K_1 \subset \mathring{K}_2$. Let $f : U \to \mathbb{K}$ be a continuous function with supp$(f) \subset K_1$. Then there exists a sequence of functions $f_N \in \mathfrak{D}(U, \mathbb{K})$ such that supp$(f_N) \subseteq K_2$ for all N and*

$$\lim_{N\to\infty} \|f - f_N\|_\infty = 0$$

Proof. Let $\widetilde{f} = f \cdot \mathbb{1}\{x \in U\}$ be the extension of f by 0. Then $\widetilde{f}$ is uniformly continuous on $\mathbb{R}^d$. Indeed, $\widetilde{f}|_{K_2} = f|_{K_2}$ is uniformly continuous, so given $\varepsilon > 0$ there exists $\delta > 0$ such that

(i) $|\widetilde{f}(x) - \widetilde{f}(y)| < \varepsilon$ for all $x, y \in K_2$ with $|x - y| < \delta$; and

(ii) $|x - y| < \delta$ implies that $x, y \notin K_1$ or $x, y \in K_2$ (take $\delta < d(K_1, \mathring{K}_2^c)$).

Recall that there exists a C^∞ function $\psi : \mathbb{R}^d \to \mathbb{R}$ with the following properties:

- ψ has compact support.
- $\psi(\mathbb{R}^d) \subseteq [0, 1]$ and $\psi(0) > 0$.
- $\int_{\mathbb{R}^d} \psi(x)\ d^dx = 1$.

That is to say, ψ is a **mollifier**. Define $\rho_N(x) := N^d\psi(Nx)$ for $N \in \mathbb{N}$. Note that

$$\int_{\mathbb{R}^d} \rho_N(x)\ d^dx = \int_{\mathbb{R}^d} \psi(x)\ d^dx = 1,$$

so $\rho_N(x) \to \delta^d(x)$ as $N \to \infty$. Now define

$$f_N(x) := (\rho_N * \widetilde{f})(x) = \int_{\mathbb{R}^d} \rho_N(x - y)\widetilde{f}(y)\ d^dy$$

f_N can be thought of as the "local average" of $\widetilde{f}$ with respect to the weight ρ_N near x. Note that $\|\widetilde{f}\|_\infty \cdot \mathbb{1}\{y \in K_1\}$ dominates (in the sense of the Dominated Convergence Theorem and its corollaries) the above integrand. More generally,

$$\|\widetilde{f}\|_\infty \mathbb{1}\{y \in K_1\} \sup_{z\in\mathbb{R}^d} |\partial^\alpha \rho_N(z)|$$

dominates $\frac{\partial^\alpha}{\partial x}\left(\rho_N(x - y)\widetilde{f}(y)\right)$ for all $\alpha \in \mathbb{N}_0^k$ and $k \in \mathbb{N}_0$. Thus we can write

$$\partial^\alpha f_N(x) = \int_{\mathbb{R}^d} \partial^\alpha \rho_N(x - y)\widetilde{f}(y)\ d^dy,$$

the point here being that each f_N is C^∞. Choose $\delta > 0$ small enough that $B(x, \delta) \subset \mathring{K}_2$ for all $x \in K_1$. Then choose $N_0 \in \mathbb{N}$ large such that supp$(\rho_N) \subset B(0, \delta)$. Then for all $N \geq N_0$, we have supp$(f_N) \subseteq K_2$. To show uniform convergence, choose $\delta_N > 0$ such that $\operatorname{supp}(\rho_N) \subset B(0, \delta_N)$. For all $x \in U$ and $N \geq N_0$, we have

$$\begin{aligned} |f(x) - f_N(x)| &= \left| \int_{\mathbb{R}^d} \rho_N(x-y) f(x) \; d^d y - \int_{\mathbb{R}^d} \rho_N(x-y) \widetilde{f}(y) \; d^d y \right| \\ &= \left| \int_{B(x,\delta_N)} \rho_N(x-y)(\widetilde{f}(x) - \widetilde{f}(y)) \; d^d y \right| \\ &\leq \sup_{z \in B(x,\delta_N)} |\widetilde{f}(x) - \widetilde{f}(z)| \end{aligned}$$

The latter term tends to 0 uniformly by the uniform continuity of $\widetilde{f}$. □

Proposition 1.3.14. *Let U be a nonempty open subset of $\mathbb{R}^d$. There exists a continuous function $\varphi : U \to [0, \infty)$ such that $\varphi^{-1}([0, R])$ is compact for all $R > 0$.*

Proof. If $U = \mathbb{R}^d$, let $\varphi(x) = |x|$. Otherwise, let

$$\varphi(x) = \max \left\{ |x|, \frac{1}{d(x, U^c)} \right\}$$

□

Proposition 1.3.15. *Let U be a nonempty open subset of $\mathbb{R}^d$. Then there exists a sequence $(K_N)_{N \geq 1}$ of compact subsets of U such that $K_1 \neq \varnothing$, $K_N \subseteq K_{N+1}$ for all $N \geq 1$, and $U = \bigcup_{N=1}^{\infty} K_N$.*

Proof. Fix a point $x_0 \in U$, let $\varphi : U \to [0, \infty)$ be as in Proposition 1.3.14, and let $K_N := \varphi^{-1}([0, \varphi(x_0) + N])$. Note that $x_0 \in K_1 = \varphi^{-1}([0, \varphi(x_0) + 1])$, so $K_1 \neq \varnothing$. □

Theorem 1.3.16 (Change of Variable Formula, Version 2.0)**.** *Let $f : U \to V$ be a C^1 diffeomorphism between open subsets U, V of $\mathbb{R}^d$.*

1. *A function $g : V \to [0, \infty]$ is $(\mathscr{B}_V, \mathscr{B}_{[0,\infty]})$-measurable if and only if the map*

$$U \to [0, \infty], \quad x \mapsto g(f(x)) \; |J_x f|$$

is $(\mathscr{B}_U, \mathscr{B}_{[0,\infty]})$-measurable. In this case, we have

$$\int_U g(f(x)) \; |J_x f| \; d^d x = \int_V g(z) \; d^d z$$

2. *Let* $g : V \to \mathbb{K}$ *be* $(\mathscr{B}_V, \mathscr{B}_{\mathbb{K}})$*-measurable. Then* g *is Lebesgue-integrable if and only if the map*

$$U \to [0, \infty], \quad x \mapsto g(f(x)) \; |J_x f|$$

is Lebesgue-integrable. In this case, we have

$$\int_U g(f(x)) \; |J_x f| \; d^d x = \int_V g(z) \; d^d z$$

Proof. We first make a series of reductions.

- Regarding the second statement, the case $\mathbb{K} = \mathbb{C}$ follows from the case $\mathbb{K} = \mathbb{R}$ by setting $g = \Re g + i \Im g$. The case $\mathbb{K} = \mathbb{R}$ follows from the first statement by setting $g = g^+ - g^-$, where $g^+ = \max\{0, g\}$ and $g^- = \min\{0, -g\}$. Thus we reduce to proving the first statement.

- The statement about measurablity is clear: if $h(x) = g(f(x))|J_x f|$, then, because f is a diffeomorphism, f and f^{-1} are in particular measurable, so

$$\begin{aligned} g \text{ measurable} \quad &\Longrightarrow \quad g(f(x)) \; |J_x f| \text{ measurable} \\ &\Longrightarrow \quad g(f(f^{-1}(y))) \; |J_{f^{-1}(y)} f| \; |J_y f^{-1}| = g(y) \text{ measurable} \end{aligned}$$

- The change of variables formula for general g follows from the case in which g is a simple function by applying the Monotone Convergence Theorem. By linearity of the integral, we thus reduce to the case in which $g = \mathbb{1}\{z \in A\}$ for some $A \in \mathscr{B}_V$.

- We can reduce further to the case in which $A \subseteq \mathring{K}$ for some compact subset K of V. Indeed, take an exhausting sequence $(K_N)_{N \geq 1}$ for V as in Proposition 1.3.15. Then $\mathbb{1}\{z \in A \cap K_N\}$ is a sequence of measurable functions which converges to $\mathbb{1}\{a \in A\}$ from below, so the Monotone Convergence Theorem applies.

Now, suppose that $A \in \mathscr{B}_V$, $K \subset V$ is compact, and $\varnothing \neq A \subseteq \mathring{K}$. For any $B \in \mathscr{B}_{\mathbb{R}^d}$, let

$$\mu(B) := \int_U \mathbb{1}\{f(x) \in B \cap \mathring{K}\} \; |J_x f| \; d^d x$$

Then μ is a finite Borel measure on $\mathbb{R}^d$. Indeed, if $B = \bigcup_{j=1}^{\infty} B_j$ (disjoint union) with $B_j \in \mathscr{B}_{\mathbb{R}^d}$, then $B \cap \mathring{K} = \bigcup_{j=1}^{\infty} B_j \cap \mathring{K}$ (disjoint union), and

$$\begin{aligned}
\mu(B) &= \int_U \left(\sum_{j=1}^{\infty} \mathbb{1}\{f(x) \in B_j \cap \mathring{K}\} \right) |J_x f| \; d^d x \\
&= \sum_{j=1}^{\infty} \int_U \mathbb{1}\{f(x) \in B_j \cap \mathring{K}\} \; |J_x f| \; d^d x \qquad \text{(MCT)} \\
&= \sum_{j=1}^{\infty} \mu(B_j)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mu(\mathbb{R}^d) &= \int_U \mathbb{1}\{f(x) \in \mathring{K}\} \; |J_x f| \; d^d x \\
&\leq \mathrm{Vol}(f^{-1}(K)) \sup_{x \in f^{-1}(K)} |J_x f| \\
&< \infty
\end{aligned}$$

Any such measure is both inner and outer regular (see Homework 8, MATH 7310, Spring 2017). Combining this with the inner and outer regularity of the Lebesgue measure, we have that for any given $\varepsilon > 0$, there exists a compact set $C \subseteq A$ and an open set W with $A \subseteq W \subseteq \mathring{K}$ such that

$$\mu(A) - \varepsilon \leq \mu(C) \leq \mu(A) \leq \mu(W) \leq \mu(A) + \varepsilon$$

and

$$\mathrm{Vol}(A) - \varepsilon \leq \mathrm{Vol}(C) \leq \mathrm{Vol}(A) \leq \mathrm{Vol}(W) \leq \mathrm{Vol}(A) + \varepsilon$$

By applicaiton of the smooth Urysohn lemma, we can choose a C^∞ function $\rho : \mathbb{R}^d \to \mathbb{R}$ with $0 \leq \rho \leq 1$, $\rho|_C \equiv 1$, and $\mathrm{supp}(\rho) \subset W$. We now apply the change of variable formula version 1.0 to ρ:

$$\int_U \rho(f(x)) \; |J_x f| \; d^d x = \int_V \rho(z) \; d^d z$$

This gives us

$$\begin{aligned}
\mu(A) &\leq \varepsilon + \mu(C) \\
&= \varepsilon + \int_V \mathbb{1}\{f(x) \in C\} \; |J_x f| \; d^d x \\
&\leq \varepsilon + \int_U \rho(f(x)) \; |J_x f| \; d^d x \\
&= \varepsilon + \int_V \rho(z) \; d^d z \\
&\leq \varepsilon + \mathrm{Vol}(W) \\
&\leq 2\varepsilon + \mathrm{Vol}(A)
\end{aligned}$$

Likewise,

$$\begin{aligned}\mu(A) &\geq -\varepsilon + \mu(W) \\ &= -\varepsilon + \int_U \mathbb{1}\{f(x) \in W\} \; |J_x f| \; d^d x \\ &\geq -\varepsilon + \int_V \rho(f(x)) \; |J_x f| \; d^d x \\ &= -\varepsilon + \int_V \rho(z) \; d^d z \\ &\geq -\varepsilon + \mathrm{Vol}(C) \\ &\geq \mathrm{Vol}(A) - 2\varepsilon\end{aligned}$$

Taking $\varepsilon \to 0$, we see that $\mu(A) = \mathrm{Vol}(A)$, i.e.,

$$\int_U g(f(x)) \; |J_x f| \; d^d x = \int_V g(z) \; d^d z$$

where $g(z) = \mathbb{1}\{z \in A\}$. □

1.4 Spherical Coordinates

In this section, we present a generalization of the familiar polar coordinate formulas $x = r\cos\theta$, $y = r\sin\theta$ to higher dimensions.

Let $d \geq 2$, and set $U := (0,\infty) \times (0,\pi)^{d-2} \times (0, 2\pi) \subset \mathbb{R}^d$. We define $f : U \to \mathbb{R}$ by $f(r, \theta_1, \ldots, \theta_d) := (x_1, \ldots, x_d)$, where each x_i is defined as follows:

$$\begin{aligned}x_1 &= r\cos\theta_1, \\ x_2 &= r\sin\theta_1\cos\theta_2, \\ x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3, \\ &\vdots \\ x_{d-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2}\cos\theta_{d-1}, \\ x_d &= r\sin\theta_1\cdots\sin\theta_{d-1}.\end{aligned}$$

For $2 \leq i \leq d-1$, we obtain x_i from x_{i-1} by changing the last cosine in x_{i-1} to a sine and appending a new cosine. For $i = d$, no new cosine is appended.

We set $V := f(U)$ to be the range of this coordinate chart. In this section, we build towards the result that f is a C^∞-diffeomorphism onto V, which is almost all of $\mathbb{R}^d$.

Example 1.4.1. When $d = 2$, we have the standard polar coordinate formulas $x_1 = r\cos\theta_1$ and $x_2 = r\sin\theta_2$. The restriction $r \in (0, \infty)$ and $\theta_1 \in (0, 2\pi)$ removes the positive x-axis from V.

Example 1.4.2. When $d = 3$, we have $x_1 = r\cos\theta_1$, $x_2 = r\sin\theta_1\cos\theta_2$, and $x_3 = r\sin\theta_1\sin\theta_2$. These are the familiar spherical coordinates, possibly with different angle convention than other branches of the natural sciences. Here, θ_1 is the angle from $(1, 0, 0)$ to the projection of $x = (x_1, x_2, x_3)$ onto the x_1x_3 plane. And θ_2 is the angle from $(0, 1, 0)$ to the projection of x onto the x_2x_3 plane.

Proposition 1.4.3. *For $1 \le i \le d - 1$, we have*

$$\sum_{j=i}^{d} x_j^2 = r^2 \cdot \prod_{j=1}^{i-1} \sin^2\theta_j.$$

Proof. We use descending induction on i. If $i = d - 1$, the proposition's statement is that $x_{d-1}^2 + x_d^2 = r^2 \prod_{j=1}^{d-2} \sin^2\theta_j$. This is the case, since by definition of x_{d-1} and x_d, and some simple factoring, we have,

$$x_{d-1}^2 + x_d^2 = \left((r \prod_{j=1}^{d-2} \sin\theta_j) \cdot \cos\theta_{d-1} \right)^2 + \left((r \prod_{j=1}^{d-2} \sin\theta_j) \cdot \sin\theta_{d-1} \right)^2$$

$$= \left(r \prod_{j=1}^{d-2} \sin\theta_j \right) \cdot \left(\cos^2\theta_{d-1} + \sin^2\theta_{d-1} \right) = r^2 \prod_{j=1}^{d-2} \sin^2\theta_j.$$

Now suppose that the statement holds for some i between 2 and $d - 1$. We claim that the statement holds for $i - 1$, which will prove the proposition. By assumption, we have:

$$\sum_{j=i-1}^{d} x_j^2 = x_{i-1}^2 + r^2 \prod_{j=1}^{i-1} \sin^2\theta_j.$$

By definition of x_{i-1} and factoring, we obtain

$$\sum_{j=i-1}^{d} x_j^2 = r^2 \cdot \left(\prod_{j=1}^{i-2} \sin^2\theta_j \right) \cdot \cos^2\theta_{i-1} + r^2 \left(\prod_{j=1}^{i-1} \sin^2\theta_j \right)$$

$$= r^2 \cdot \left(\prod_{j=1}^{i-2} \sin^2\theta_j \right) \cdot (\cos^2\theta_{i-1} + \sin^2\theta_{i-1}) = r^2 \cdot \left(\prod_{j=1}^{i-2} \sin^2\theta_j \right).$$

□

In particular, we have $x_1^2 + \ldots + x_d^2 = r^2$, since our proof of Proposition 1.4.3 is consistent with the convention that the empty product is equal to 1.

Proposition 1.4.4. *The image $V = f(U)$ is all of $\mathbb{R}^d$ except for a half-hyperplane (which has measure zero). We have*

$$V = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d \neq 0 \text{ or } (x_d = 0 \text{ and } x_{d-1} < 0)\}.$$

Proof. To show V is contained in the right-hand set, we first observe that by 1.4.3, we have $x_{d-1} = \rho \cos\theta_{d-1}$ and $x_d = \rho \sin\theta_{d-1}$, where $\rho = \sqrt{x_{d-1}^2 + x_d^2} = r \sin\theta_1 \cdots \sin\theta_{d-2}$. By definition, $\theta_i \in (0, \pi)$ for $1 \leq i \leq d-2$. Hence $\rho > 0$ since sin is positive on $(0, \pi)$. If $x_d = 0$, then $\theta_{d-1} = \pi$, and therefore $x_{d-1} = \rho\cos\theta_{d-1} = -\rho < 0$.

To show the opposite inclusion, let $x = (x_1, \ldots, x_d)$ be a member of the right-hand set. We explicitly construct a preimage of x under f by descending induction on i. By assumption, we have $\rho = \sqrt{x_{d-1}^2 + x_d^2} > 0$. Hence, for the case $i = d-1$, there exists $\theta_{d-1} \in (0, \pi)$ such that $x_{d-1} = \rho\cos\theta_{d-1}$ and $x_d = \rho\sin\theta_{d-1}$. Now suppose that for we have chosen $\theta_{d-1}, \ldots, \theta_i$ for $2 \leq i \leq d-1$ such that $\cos\theta_{d-2} = \frac{x_{d-2}}{\sqrt{x_i^2 + \ldots + x_d^2}}$ and $\sin\theta_{d-2} > 0$. Then, by our observation that $x_{d-1}^2 + x_d^2 > 0$, we have that $\frac{x_{i-1}}{\sqrt{x_{i-1}^2 + \ldots + x_d^2}} \in (-1, 1)$. So there exists $\theta_{i-1} \in (0, \pi)$ such that $\cos\theta_{i-1} = \frac{x_{i-1}}{\sqrt{x_{i-1}^2 + \ldots + x_d^2}}$ and $\sin\theta_i > 0$.

Let $r := \sqrt{x_1^2 + \ldots + x_d^2}$. We claim that the tuple $\tau := (r, \theta_1, \ldots, \theta_{d-1})$ maps to x under f. By construction, $\tau \in U$. Let $y = (y_1, \ldots, y_d) := f(\tau)$.

By ascending induction on i with $1 \leq i \leq d$, we show that (a) $x_i^2 + \ldots + x_d^2 = r^2 \sin^2\theta_1 \cdots \sin^2\theta_{i-1}$ and that (b) $x_i = y_i$.

For $i = 1$, we have $\sum_{i=1}^d x_i^2 = r^2$ by definition. And by construction, we have

$$y_1 = r\cos\theta_1 = \sqrt{x_1^2 + \ldots + x_d^2} \cdot \frac{x_1}{\sqrt{x_1^2 + \ldots + x_d^2}} = x_1.$$

If the statement holds for $1 \leq i \leq d-2$, then it holds for $i+1$. To see this, we apply the inductive assumption, in particular, our formula for $x_i = y_i$. We have:

$$x_{i+1}^2 + \ldots + x_d^2 = (x_i^2 + \ldots + x_d)^2 - x_i^2$$
$$= r^2 \sin^2\theta_1 \cdots \sin^2\theta_{i-1} - r^2 \sin^2\theta_1 \cdots \sin^2\theta_{i-1} \cos^2\theta_i$$
$$= r^2 \sin^2\theta_1 \cdots \sin^2\theta_{i-1}(1 - \cos^2\theta_i) = r^2 \sin^2\theta_1 \cdots \sin^2\theta_{i-1} \sin\theta_i.$$

Hence (a) holds. And by the inductive assumption and by construction, we have

$$y_{i+1} = (r \sin \theta_1 \cdots \sin \theta_i) \cdot \cos \theta_{i+1} = \sqrt{x_i^2 + \ldots + x_d^2} \cdot \frac{x_{i+1}}{\sqrt{x_i^2 + \ldots + x_d^2}} = x_{i+1}.$$

Hence (b) holds.

Finally, suppose the claim holds for i with $1 \leq i \leq d-1$. The same argument for (a) in the previous paragraph applies here to show that (a) holds for the case $i = d$. And by definition of θ_{d-1}, we have $x_d = \rho \sin \theta_{d-1} = r \sin \theta_1 \cdots \sin \theta_{d-1} = y_d$. Hence (b) holds. $\square$

Proposition 1.4.5. *$f : U \to V$ is injective.*

Proof. Let $x = (x_1, \ldots, x_d) \in V$ and $y = (r, \theta_1, \ldots, \theta_{d-1}) \in U$. We show that y is uniquely determined by x. By Proposition 1, $r^2 = x_1^2 + \ldots + x_d^2 > 0$. Since $r \in (0, \infty)$, then $r = \sqrt{x_1^2 + \ldots + x_d^2}$. Define $x_0 := 0$. Then by Proposition 2, for i with $1 \leq i \leq d-2$, we have

$$\cos \theta_i = \frac{x_i}{\sqrt{x_i^2 + \ldots + x_d^2}}.$$

Since $x_{d-1}^2 + x_d^2 > 0$ by assumption, the right-hand expression makes sense and is contained in $(-1, 1)$. Hence, there is a unique θ_i in $(0, \pi)$ which makes the above hold, namely, $\theta_i = \arccos\left(\frac{x_i}{\sqrt{x_i^2 + \ldots + x_d^2}}\right)$. Hence, y is unique such that $f(y) = x$. $\square$

We are ready to prove that $f : U \to V$ is a C^∞-diffeomorphism. This follows by finding an expression for f^{-1} which is evidently infinitely differentiable. This can also be shown by explicitly computing the Jacobian of f and showing it is non-zero everywhere. Since we will need to compute the Jacobian anyways for change of variables in integrals, we end up with two proofs of this following theorem.

Theorem 1.4.6. *$f : U \to V$ is a C^∞-diffeomorphism.*

Proof. Let $x = (x_1, \ldots, x_d) \in V$. By the previous proposition, we have

$$r = \sqrt{x_1^d + \ldots + x_d^2}.$$

For $1 \leq i \leq d-2$, we have

$$\theta_i = \arccos\left(\frac{x_i}{\sqrt{x_i^2 + \ldots + x_d^2}}\right).$$

We observe that $\sqrt{\cdot}$ is C^∞ on $(0,\infty)$, that arccos is C^∞ on $(-1,1)$, that $x \mapsto x_1^d + \ldots + x_d^2$ is a C^∞ function mapping into V into $(0,\infty)$, and that $x \mapsto \frac{x_i}{\sqrt{x_i^2+\ldots+x_d^2}}$ is a C^∞ function mapping V into $(-1,1)$. Hence, to show f^{-1} is C^∞ is only remains to show that θ_{d-1} is a C^∞ function of x. We have $\cos(\theta_{d-1}) = \frac{x_{d-1}}{\sqrt{x_{d-1}^2+x_d^2}}$ and $\sin(\theta_{d-1}) = \frac{x_d}{\sqrt{x_{d-1}^2+x_d^2}}$. Hence $e^{i\theta_{d-1}} = \frac{x_{d-1}+ix_d}{\sqrt{x_{d-1}^2+x_d^2}}$ and so $\theta_{d-1} = \frac{1}{i}\log\left(\frac{x_{d-1}+ix_d}{\sqrt{x_{d-1}^2+x_d^2}}\right)$. We choose log to be the branch cut of the logarithm with the non-negative real axis removed. Since $\log(\cdot)$ is holomorphic on this domain and $\frac{x_{d-1}+ix_d}{\sqrt{x_{d-1}^2+x_d^2}}$ is never contained in the non-negative real axis by assumption, we conclude that θ_{d-1} is also a C^∞ function of x. □

Proposition 1.4.7. *Let $A_\epsilon = \{x \in \mathbb{R}^d \mid |x| \in [1, 1+\epsilon]\}$. Then*

$$Vol_{d-1}(\$^{d-1}) = \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\ Vol_d(A_\epsilon).$$

Proof.

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \frac{1}{\epsilon} Vol_d(A_\epsilon) &= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d} d^d x\ \mathbb{1}\{x \in A_\epsilon\}\mathbb{1}\{1 \le x \le 1+\epsilon\} \\
&= \lim_{\epsilon\to 0^+} [Vol_{d-1}(\$^{d-1}) \times \int_1^{1+\epsilon} r^{d-1} dr] \text{ (change of variable, Fubini)} \\
&= Vol_{d-1}(\$^{d-1}) \times \lim_{\epsilon\to 0^+} \int_1^{1+\epsilon} r^{d-1} dr \\
&= Vol_{d-1}(\$^{d-1}) \text{ (Fundamental Theorem of Calculus)}
\end{aligned}$$

□

Theorem 1.4.8. $\forall\ d \ge 2, \quad Vol_{d-1}(\$^{d-1}) = \dfrac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.

Proof. Applying change of variable and Fubini, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{-x.x} d^d x &= Vol_{d-1}(\$^{d-1}) \times \int_0^\infty e^{-r^2} r^{d-1} dr \\
\Longrightarrow \Big(\int_{\mathbb{R}} e^{-t^2} dt\Big)^d &= Vol_{d-1}(\$^{d-1}) \times \int_0^\infty \frac{dt}{2\sqrt{t}} e^{-t} t^{\frac{d-1}{2}} \\
\Longrightarrow (\sqrt{\pi})^d &= Vol_{d-1}(\$^{d-1}) \times \frac{1}{2}\Gamma(\frac{d}{2}) \\
\Longrightarrow Vol_{d-1}(\$^{d-1}) &= \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}
\end{aligned}$$

□

Remark 1.4.9. If $d = 1$ in the formula, we have $Vol_0(\$^0) = 2$ which is consistent with the notion of measure (here the cardinality) of the two point set.

Theorem 1.4.10. *Let $d \geq 1$.*
(i) $\int_{\{0<|x|\leq 1\}} \frac{1}{|x|^\alpha} d^d x < \infty \iff \alpha < d.$
(ii) $\int_{|x|>1} \frac{1}{|x|^\alpha} < \infty \iff \alpha > d.$

Proof.

$$(i) \qquad \int_{\{0<|x|\leq 1\}} \frac{1}{|x|^\alpha} d^d x = Vol_{d-1}(\$^{d-1}) \int_0^1 r^{d-1-\alpha} dr < \infty \iff \alpha < d$$

.

$$(ii) \qquad \int_{\{|x|>1\}} \frac{1}{|x|^\alpha} d^d x = Vol_{d-1}(\$^{d-1}) \int_1^\infty r^{d-1-\alpha} dr < \infty \iff \alpha > d$$

□

Remark 1.4.11. Recall the solution to the poisson equation: $\int_{\mathbb{R}^3\setminus\{x\}} \frac{\rho(y)}{|x-y|^\alpha} d^3 y$— this converges because $\alpha = 1 < 3$.

More generally, $\rho \in S(\mathbb{R}^d)$— the latter will be discussed later and we shall have a new function $\phi(x) = \int_{\mathbb{R}^d\setminus\{x\}} \frac{\rho(y)}{|x-y|^\alpha} d^d y$.

- Integrals where convergence happens for $\alpha < d$ are called *fractional integrals.*
- Integrals where convergence happens for $\alpha = d$ are called *singular integrals* which play an important role in harmonic analysis.
- Integrals where convergence haapens for $\alpha > d$ are called *hypersingular integrals.*

Notation: We will use $\approx$ to denote that the ratio of L.H.S. and R.H.S. is bounded away from 0 and ∞.

Theorem 1.4.12. *Let $d \geq 1$.*
1. $\alpha > d$, $\int_{\{R\leq|x|\leq 1\}} \frac{1}{|x|^\alpha} d^d x \approx \frac{1}{R^{\alpha-d}}$ *when* $R \to 0^+$.
2. $\alpha < d$, $\int_{\{1\leq|x|\leq R\}} \frac{1}{|x|^\alpha} d^d x \approx R^{d-\alpha}, R \to \infty$.
3. $\int_{\{R\leq|x|\leq 1\}} \frac{1}{|x|^d} d^d x \approx \log\left(\frac{1}{R}\right), R \to 0^+$.
4. $\int_{\{1\leq|x|\leq R\}} \frac{1}{|x|^d} d^d x \approx \log(R), R \to \infty$.

Proof. All of the above follow from change of variable to spherical coordinates and following the same method as we did in the earlier proofs. □

2

Topological Vector Spaces

2.1 Abstract Metrics

Definition 2.1.1. Let $(V, +, \cdot)$ be a $\mathbb{K}$-vector space with topology $\mathcal{T}$ ($\mathbb{K}$ is equipped with some topology $\mathcal{T}'$ also; particularly if $\mathbb{K}$ is an Euclidean space then we would just take $\mathcal{T}'$ to be the standard topology induced by the Euclidean norm). It is called a topological vector space (TVS) if the two maps defined by

$$\begin{aligned} V \times V &\to V & \qquad \mathbb{K} \times V &\to V \\ (x, y) &\mapsto x + y & \qquad (\lambda, v) &\mapsto \lambda v \end{aligned}$$

are continuous with respect to $\mathcal{T}$. Here the products $V \times V$ and $\mathbb{K} \times V$ are equipped with product topology.

Definition 2.1.2. Let $(V, +, \cdot)$ be a $\mathbb{K}$-vector space ($\mathbb{K}$ is either real or complex). We call $\rho : V \to [0, \infty)$ a seminorm if

- $\rho(\lambda x) = |\lambda| \rho(x)$;
- $\rho(x + y) \leq \rho(x) + \rho(y)$.

Additionally, if

- $\rho(x) = 0 \Rightarrow x = 0$,

then we call ρ a norm.

Notations: Generally if we use (X, d) to denote a metric space and $\tau(d)$ to denote the topology induced by metric d. And we use $\mathcal{N}_{all}(V)$ to denote the set of all seminorms on V.

Definition 2.1.3. Let $(V, +, \cdot)$ be a $\mathbb{K}$-vector space. For a subset $\mathcal{N} \subset \mathcal{N}_{all}(V)$ and an element $\rho \in \mathcal{N}_{all}(V)$, we say ρ is continuous relative to $\mathcal{N}$ if there exist $c > 0$ and a finite subset $\{T_1, \dots, T_n\} \subset \mathcal{N}$ such that

$$\rho(x) \leq c(T_1(x) + \ldots + T_n(x)) \quad \text{for all } x \in V.$$

Remark 2.1.4. Every $\rho \in \mathcal{N}$ is continuous relative to $\mathcal{N}$. This is because $\rho(x) \leq 1(\rho(x))$ for all $x \in V$.

Definition 2.1.5. Let $(V, +, \cdot)$ be a $\mathbb{K}$-vector space and let $F = \{\rho_1, \dots, \rho_n\} \subset \mathcal{N}_{all}(V)$. A multi-ball $B(x, F, \epsilon)$ is defined to be

$$B(x, F, \epsilon) := \{y \in V \mid \rho_1(y - x) < \epsilon, \dots, \rho_n(y - x) < \epsilon\}.$$

Definition 2.1.6. Let $(V, +, \cdot)$ be a $\mathbb{K}$-vector space. For a subset $\mathcal{N} \subset \mathcal{N}_{all}(V)$, the topology induced by $\mathcal{N}$ is denoted $\tau(\mathcal{N})$ and defined by

$$\{U \subset V \mid \forall x \in U, \exists \epsilon > 0 \ \&\ \text{finite set } F \subset \mathcal{N} \text{ s.t. } B(x, F, \epsilon) \subset U\}.$$

Proposition 2.1.7. *The $\tau(\mathcal{N})$ defined above is indeed a topology. Moreover, if ρ is continuous relative to $\mathcal{N}$, then $B(x, \{\rho\}, \epsilon) \in \tau(\mathcal{N})$ for any x and ϵ.*

Proof. $\tau(\mathcal{N})$ is a topology: It is trivial that $\emptyset \in \tau(\mathcal{N})$ and $\tau(\mathcal{N})$ is closed under arbitrary union. If $U_1, U_2 \in \tau(\mathcal{N})$ and $U_1 \cap U_2 \neq \emptyset$ (otherwise $U_1 \cap U_2 = \emptyset \in \tau(\mathcal{N})$), for any $x \in U_1 \cap U_2$ there exist $\epsilon_1, \epsilon_2, F_1, F_2$ such that $B(x, F_1, \epsilon_1) \subset U_1$ and $B(x, F_2, \epsilon_2) \subset U_2$, then

$$B(x, F_1 \cup F_2, \min\{\epsilon_1, \epsilon_2\}) \subset B(x, F_1, \epsilon_1) \cap B(x, F_2, \epsilon_2) \subset U_1 \cap U_2.$$

By definition $U_1 \cap U_2 \in \tau(\mathcal{N})$.

$B(x, \{\rho\}, \epsilon) \in \tau(\mathcal{N})$: For any $y \in B(x, \{\rho\}, \epsilon)$, let $\epsilon' = (\epsilon - \rho(y - x))/2$. Then for any $z \in B(y, \{\rho\}, \epsilon')$

$$\rho(z - x) \leq \rho(z - y) + \rho(y - x) \leq \epsilon' + \rho(y - x) < \epsilon$$

so $B(y, \{\rho\}, \epsilon') \subset B(x, \{\rho\}, \epsilon)$. Since ρ is continuous relative to $\mathcal{N}$ there are c and $F = \{\rho_1, \dots, \rho_n\} \subset \mathcal{N}$ such that $\rho \leq c(\rho_1 + \ldots + \rho_n)$. If we take $\epsilon'' = \frac{\epsilon'}{(n+1)c}$, then $\rho_k(z - y) < \epsilon''$ for all $1 \leq k \leq n$ implies $\rho(z - y) < \epsilon'$, thus

$$B(y, F, \epsilon'') \subset B(y, \{\rho\}, \epsilon') \subset B(x, \{\rho\}, \epsilon).$$

□

Remark 2.1.8. Since for every ρ continuous wrt $\mathcal{N}$, $B(x, \{\rho\}, \epsilon) \in \tau(\mathcal{N})$, it follows that $\tau(\mathcal{N}_{\text{crt}}) = \tau(\mathcal{N})$, where $\mathcal{N}_{\text{crt}}$ is the set of all seminorms continuous wrt $\mathcal{N}$. Therefore, it is more convenient to use $\mathcal{N}_{\text{crt}}$ as the defining set of seminorms for V as opposed to using $\mathcal{N}$.

Proposition 2.1.9 (Single Seminorm Openness Criterion)**.** *Let $U \subset V$. Then $U \in \tau(\mathcal{N})$ if and only if for any $x \in U$ there exists $\epsilon > 0$ and seminorm ρ continuous relative to $\mathcal{N}$ such that $B(x, \{\rho\}, \epsilon) \subset U$.*

Proof. ($\Longrightarrow$): If $x \in U$, there exists ϵ and $F = \{\rho_1, \ldots, \rho_n\} \subset \mathcal{N}$ such that $B(x, F, \epsilon) \subset U$. Let $\rho = \rho_1 + \ldots + \rho_n$ which is continuous relative to $\mathcal{N}$, then

$$B(x, \{\rho\}, \epsilon) \subset B(x, F, \epsilon) \subset U$$

($\Longleftarrow$): Now suppose for any $x \in U$, $B(x, \{\rho\}, \epsilon) \subset U$ with some $\epsilon > 0$ and seminorm ρ continuous relative to $\mathcal{N}$. By Proposition 2.1.7 we know every such $B(x, \{\rho\}, \epsilon) \in \tau(\mathcal{N})$ therefore

$$U = \bigcup_{x \in U} B(x, \{\rho\}, \epsilon) \in \tau(\mathcal{N}).$$

Note that ρ depends on x. □

Proposition 2.1.10. *Let $(V, +, \cdot)$ be a $\mathbb{K}$-vector space and $\mathcal{N} \subset \mathcal{N}_{all}(V)$. Then $(V, \tau(\mathcal{N}))$ is a topological vector space. Such a TVS is called a* <u>*locally convex TVS*</u> *(LCTVS).*

Proof. We've shown $(V, \tau(\mathcal{N}))$ is a topology in Proposition 2.1.7.

"+" is a continuous map: We'll show preimage of open set is open. Given $U \in \tau(\mathcal{N})$ and $x + y \in U$ $(x, y \in V)$, there exist ϵ and $F = \{\rho_1, \ldots, \rho_n\} \subset \mathcal{N}$ such that $B(x + y, F, \epsilon) \subset U$. Because ρ_k's are seminorms, triangular inequality guarantees that

$$B(x, F, \epsilon/2) + B(y, F, \epsilon/2) \subset B(x + y, F, \epsilon) \subset U.$$

Thus $B(x, F, \epsilon/2) \times B(y, F, \epsilon/2)$ is an open neighbourhood of (x, y) contained in the preimage of U.

"$\cdot$" is a continuous map: Given $U \in \tau(\mathcal{N})$ and $\lambda x \in U$ $(\lambda \in \mathbb{K}, x \in V)$, by Proposition 2.1.9 there exist ϵ and ρ continuous relative to $\mathcal{N}$ such that $B(\lambda x, \{\rho\}, \epsilon) \subset U$. Now for any $(\mu, y) \in B(\lambda, \epsilon') \times B(x, \{\rho\}, \epsilon')$ with small enough ϵ' we have

$$\begin{aligned} \rho(\mu y - \lambda x) &= \rho(\mu y - \mu x + \mu x - \lambda x) \\ &\leq |\mu| \rho(y - x) + |\mu - \lambda| \rho(x) \\ &\leq (|\lambda| + \epsilon')\epsilon' + \epsilon' \rho(x) < \epsilon \end{aligned}$$

so $B(\lambda, \epsilon') \times B(x, \{\rho\}, \epsilon')$, which is an open neighbourhood of (λ, x) (by Proposition 2.1.7), is contained in the preimage of U. □

Proposition 2.1.11. *Let $(V, +, \cdot)$ be a $\mathbb{K}$-vector space, $\rho \in \mathcal{N}_{all}(V)$ and $\mathcal{N} \subset \mathcal{N}_{all}(V)$. Then ρ is continuous relative to $\mathcal{N}$ if and only if $\rho : (V, \tau(\mathcal{N})) \to [0, \infty)$ is continuous.*

Proof.

($\Longrightarrow$): Only need to observe that $\rho^{-1}([0, r)) = B(0, \{\rho\}, r) \in \tau(\mathcal{N})$ by Proposition 2.1.7.

($\Longleftarrow$): In particular, ρ is continuous at $0 \in V$. So there exist ϵ_0 and $F = \{\rho_1, \ldots, \rho_n\} \subset \mathcal{N}$ such that $\rho(B(0, F, \epsilon_0)) \subset [0, 1]$. Fix an $x \in V$. If $\rho_k(x) \neq 0$ for some k, let $\lambda = 2(\rho_1(x) + \ldots + \rho_n(x))/\epsilon_0$. Then for all $1 \leq k \leq n$

$$\rho_k(\lambda^{-1}x) = \lambda^{-1}\rho_k(x) \leq \epsilon_0/2 < \epsilon_0$$

so $\lambda^{-1}x \in B(0, F, \epsilon_0)$. Then $\rho(\lambda^{-1}x) \leq 1$ and

$$\rho(x) = \lambda\rho(\lambda^{-1}x) \leq \lambda = \frac{2}{\epsilon_0}(\rho_1(x) + \ldots + \rho_n(x)).$$

If $\rho_k(x) = 0$ for all k, then $\mu^{-1}x \in B(0, F, \epsilon_0)$ for any $\mu > 0$ thus $\rho(\mu^{-1}x) \leq 1$ and

$$\rho(x) = \mu\rho(\mu^{-1}x) \leq \mu$$

for any $\mu > 0$, so $\rho(x) = 0$ and it is trivial that

$$\rho(x) \leq \frac{2}{\epsilon_0}(\rho_1(x) + \ldots + \rho_n(x)).$$

Since x is arbitrary and ϵ_0 only depends on ρ, by letting $c = 2/\epsilon_0$ we prove ρ is continuous relative to $\mathcal{N}$.

□

Proposition 2.1.12. *Let $(V, +, \cdot)$ be a $\mathbb{K}$-vector space and $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{N}_{all}(V)$. Then $\tau(\mathcal{N}_1) \subset \tau(\mathcal{N}_2)$ if and only if every $\rho \in \mathcal{N}_1$ is continuous relative to $\mathcal{N}_2$.*

Proof.

($\Longrightarrow$): Suppose $\rho \in \mathcal{N}_1$. Then ρ is continuous relative to $\mathcal{N}_1$. Then by Proposition 2.1.11, ρ is continuous with respect to $\tau(\mathcal{N}_1)$. $\tau(\mathcal{N}_1)$ is contained in $\tau(\mathcal{N}_2)$, so ρ is also continuous with respect to $\tau(\mathcal{N}_2)$. By Proposition 2.1.11 again, ρ is continuous relative to $\mathcal{N}_2$.

($\Longleftarrow$): If $\rho \in \mathcal{N}_1$ then ρ is continuous relative to $\mathcal{N}_2$ and by Proposition 2.1.7 we know $B(x, \{\rho\}, \epsilon) \in \tau(\mathcal{N}_2)$ for all x and ϵ. But $\tau(\mathcal{N}_1)$ is generated by $\{B(x, \{\rho\}, \epsilon) \mid \rho \in \mathcal{N}_1\}$, so $\tau(\mathcal{N}_1) \subset \tau(\mathcal{N}_2)$.

□

Corollary 2.1.13. *Let* $(V, +, \cdot)$ *be a* $\mathbb{K}$*-vector space and* $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{N}_{all}(V)$*. Then* $\tau(\mathcal{N}_1) = \tau(\mathcal{N}_2)$ *if and only if every* $\rho \in \mathcal{N}_1$ *is continuous relative to* $\mathcal{N}_2$ *and vice versa.*

Definition 2.1.14. Let (V, τ) be a LCTVS. We call $\mathcal{N} \subset \mathcal{N}_{all}(V)$ a defining collection of seminorms for (V, τ) if $\tau(\mathcal{N}) = \tau$.

Proposition 2.1.15. $(V, \tau(\mathcal{N}))$ *is Hausdorff if and only if* $\mathcal{N}$ *is a separating collection of seminorms, i.e. for any* $x \in V \setminus \{0\}$ *there is* $\rho \in \mathcal{N}$ *such that* $\rho(x) > 0$.

Proof. See the lecture notes in Math 7310, proposition 3.8. □

Proposition 2.1.16. *Let* $(V, \tau(\mathcal{N}))$ *be a Hausdorff LCTVS. Then the followings are equivalent:*

1. *V is metrizable.*
2. *V is first countable.*
3. *There exists a countable* $\mathcal{N}_c \subset \mathcal{N}$ *such that* $\tau(\mathcal{N}) = \tau(\mathcal{N}_c)$.

Proof. See the lecture notes in Math 7310. □

Definition 2.1.17. A LCTVS (V, τ) is called a Fréchet space if and only if there exists a sequence of continuous seminorms $\{\rho_n\}_{n\geq 1}$ such that

$$d(x, y) := \sum_{n\geq 1} 2^{-n} \min\{1, \rho_n(x - y)\}$$

is a distance which induces the topology τ and (V, d) is a complete metric space.

Remark 2.1.18. The $\{\rho_n\}_{n\geq 1}$ above is automatically defining and separating. Any single or finite collection of seminorms is included by the sequences $\{\rho_n\}_{n\geq 1}$ with repeating elements.

Definition 2.1.19. A LCTVS (V, τ) is called a Banach space if and only if there exists a norm $\|\cdot\|$ that induces its topology (that is, $\tau(\|\cdot\|) = \tau$) and V is a complete metric space under the induced metric d from $\|\cdot\|$.

Remark 2.1.20. Every Banach space is a Fréchet space because the norm $\|\cdot\|$ from the Banach space creates the distance d needed in the Fréchet space. On the other hand, a distance d does not necessarily induce a norm, and hence a Fréchet space is not necessarily a Banach space.

Theorem 2.1.21. *Let $V_1, \ldots, V_n, W$ be LCTVS's and let $h : V_1 \times \ldots \times V_n \to W$ be a $\mathbb{K}$-multilinear map. Then h is continuous if and only if for any continuous seminorm ρ on W there exist continuous seminorms $T_1, \ldots, T_n$ on $V_1, \ldots, V_n$ respectively, such that*

$$\rho(h(x_1, \ldots, x_n)) \leq T_1(x_1) \cdots T_n(x_n)$$

for all $(x_1, \ldots x_n) \in V_1 \times \cdots \times V_n$.

Proof. "$\Longrightarrow$": For any continuous seminorm ρ on W, by Proposition 2.1.7 and Proposition 2.1.11, $B(0, \{\rho\}, 1)$ is open in W. Because h is continuous, there exist open sets $U_k \subset V_k$ which all contain 0 such that

$$h(U_1 \times \cdots \times U_n) \subset B(0, \{\rho\}, 1)$$

By Proposition 2.1.9 and Proposition 2.1.11 there exist ϵ_k and continuous seminorms σ_k on V_k such that

$$B(0, \{\sigma_k\}, \epsilon_k) \subset U_k \quad \text{for all } 1 \leq k \leq n.$$

Fix an $(x_1, \ldots, x_n) \in V_1 \times \cdots \times V_n$.

CASE: $\sigma_k(x_k) \neq 0$ for all k.

Let $\lambda_k = 2\sigma_k(x_k)/\epsilon_k$. Then for all $1 \leq k \leq n$

$$\sigma_k(\lambda_k^{-1} x_k) = \lambda_k^{-1} \sigma_k(x_k) \leq \epsilon_k/2 < \epsilon_k$$

so $\lambda_k^{-1} x_k \in U_k$. Then $\rho(h(\lambda_1^{-1} x_1, \ldots, \lambda_n^{-1} x_n)) < 1$ and

$$\begin{aligned} \rho(h(x_1, \ldots, x_n)) &= \lambda_1 \cdots \lambda_n \rho(h(\lambda_1^{-1} x_1, \ldots, \lambda_n^{-1} x_n)) \\ &< \lambda_1 \cdots \lambda_n \\ &= \frac{2\sigma_1(x_1)}{\epsilon_1} \cdots \frac{2\sigma_n(x_n)}{\epsilon_n}. \end{aligned}$$

CASE: $\sigma_j(x_j) = 0$ for some j.

Then $\mu^{-1} x_j \in B(0, \{\sigma_j\}, \epsilon_j)$ for any $\mu > 0$ thus

$$\rho(h(\lambda_1^{-1} x_1, \ldots, \mu^{-1} x_j, \ldots, \lambda_n^{-1} x_n)) \leq 1$$

where λ_k's are defined as above. So, similarly

$$\rho(h(x_1,\ldots,x_n)) \le \left(\prod_{k\neq j} \lambda_k\right)\mu$$

for any $\mu > 0$, so $\rho(h(x_1,\ldots,x_n)) = 0$ and it is trivial that

$$\rho(h(x_1,\ldots,x_n)) \le \frac{2\sigma_1(x_1)}{\epsilon_1}\cdots\frac{2\sigma_n(x_n)}{\epsilon_n}.$$

Since x_k's are arbitrary and ϵ_k's only depend on ρ, by letting $T_k = 2\sigma_k/\epsilon_k$, we prove the right direction.

"$\Longleftarrow$": To prove the other direction, let $(x_1,...,x_n) \in V_1 \times ... \times V_n$. Assume Ω is an open set in W such that $h(x_1,\cdots,x_n \in \Omega$. By Proposition 2.1.9, there exists a continuous seminorm ρ on W such that that

$$\rho(z - h(x_1,\cdots,x_n)) < \epsilon$$

for any $z \in W$. By hypothesis, there exist continuous seminorms $\tau_1,...,\tau_n$ on $V_1,\cdots,V_n$ satisfying the hypothesis inequality. Let

$$M = \left(\max_{1\le i\le n} \tau_i(x_i)\right) + 1$$

and fix $\alpha > 0$. Consider $(y_1,...,y_n) \in V_1,...,V_n$ such that $\tau_i(y_i - x_i) < \alpha$ for all i.

Observe we have the following telescopic sum

$$\begin{aligned} h(y_1,...,y_n) - h(x_1,...,x_n) &= \sum_{j=1}^{n} \Big[h(x_1,...,x_{j-1},y_j,,...,y_n) - \\ &\qquad - h(x_1,...,x_j,y_{j+1},...,y_n)\Big] \\ &= \sum_{j=1}^{n} h(x_1,...,x_{j-1},y_j - x_j,y_{j+1},...,y_n) \end{aligned}$$

where the last equality is because h is multilinear. Therefore,

$$\begin{aligned} \rho\Big(h(y_1,...,y_n) - h(x_1,...,x_n)\Big) &\le \sum_{j=1}^{n} \tau_1(x_1)\cdots\tau_{j-1}(x_{j-1})\tau_j(y_j - x_j)\cdot \\ &\qquad \cdot \tau_{j+1}(y_{j+1})\cdots\tau_n(y_n) \\ &\le n\alpha(M+\alpha)^{n-1} \qquad \text{(by triangle inequality)} \\ &< \epsilon \qquad \text{(for small enough } \alpha > 0\text{)} \end{aligned}$$

And so $h(\prod_{i=1}^{n} B(x_i,\{\tau_i\},\alpha)) \subset \Omega$. Hence h is continuous. □

We'll denote the space of continuous multilinear maps $V_1 \times \ldots \times V_n \to W$ by $\mathscr{L}_n(V_1, \ldots, V_n; W)$. This is a vector space and we'll give it a topology at a later time.

Definition 2.1.22. A morphism from a TVS (V_1, τ_1) to a TVS (V_2, τ_2) is both a morphism of topological spaces (continuous function) and a morphism of vector spaces (linear transformation).

Remark 2.1.23. The definition of an isomorphism of TVS's follows immediately.

Definition 2.1.24. Two topological vector spaces (V_1, τ_1) and (V_2, τ_2) are isomorphic as topological vector spaces if and only if there exists a linear bijection $f : V_1 \to V_2$ such that both f and f^{-1} are continuous.

Definition 2.1.25. Let (V, τ) be a locally convex topological vector space. Then $V' := \mathscr{L}_1(V; \mathbb{K})$ is the topological dual space of V.

Note that we can consider the dual space of the dual space: $\mathscr{L}_1(V'; \mathbb{K})$ which we call the double dual space. We denote this by V''.

Definition 2.1.26. Let (V, τ) be a TVS. A subset $A \subset V$ is a bounded subset if and only if for any open set Ω in V containing 0, there exists $\lambda > 0$ such that $\lambda A \subset \Omega$.

In other words, a bounded subset can always be rescaled to fit inside any open set as long as the open set contains 0. In general, bounded sets are much smaller than multiballs and open sets.

insert $\mathbb{R}^2$ example

Proposition 2.1.27. *Let $(V, \tau(\mathscr{N}))$ be a LCTVS and $A \subset V$. Then the following are equivalent.*

1. *A is bounded.*
2. *For every continuous seminorm ρ, $\rho(A)$ is bounded (in $\mathbb{R}$)*
3. *For every $\rho \in \mathscr{N}$, $\rho(A)$ is bounded.*

Proof. We'll show 2 is equivalent to 3 and 1 is equivalent to 2.

($2 \Rightarrow 3$) Trivial because $\rho \in \mathcal{N}$ is automatically continuous relative to $\mathcal{N}$.

($3 \Rightarrow 2$) If ρ is continuous, then $\rho \leq C(\rho_1 + \cdots \rho_n)$ where $\rho_i \in \mathcal{N}$. By assumption, there exists $M_i > 0$ (for $1 \leq i \leq n$), such that for all $x \in A$, $\rho_i(x) \leq M_i$. Therefore $\rho(x) \leq C(M_1 + ... + M_n)$ for all $x \in A$ and we have $\rho(A)$ is bounded.

($1 \Rightarrow 2$) Assume ρ is a continuous seminorm. Since $B(0, \{\rho\}, 1)$ is an open set, A bounded implies $\lambda A \subset B(0, \{\rho\}, 1)$ for some $\lambda > 0$. Therefore, $\rho(\lambda x) < 1$ for all $x \in A$ which is equivalent to $\rho(x) < \lambda^{-1}$ for all $x \in A$. Hence $\rho(A)$ is bounded.

($2 \Rightarrow 1$) Let $\Omega \subset V$ be an open set containing 0. We want to show that there exists $\lambda > 0$ such that $\lambda x \in \Omega$ for all $x \in A$. By Proposition 2.1.9, there exists a continuous seminorm ρ and $\epsilon > 0$ such that $B(0, \{\rho\}, \epsilon) \subset \Omega$. By hypothesis, $\rho(A)$, is bounded so there exists $M > 0$ such that $\rho(x) \leq M$ for all $x \in A$. Fix $\lambda = \epsilon/2M > 0$ and $x \in A$. It follows that

$$\rho(\lambda x) \leq \left(\frac{\epsilon}{2M}\right) M < \epsilon \quad \rightarrow \quad \lambda x \in \Omega$$

and so A is bounded. This completes the proof. □

Proposition 2.1.28. *Let* (V, τ) *be a LCTVS with topological dual* V'. *Fix a bounded subset* $A \subset V$. *For functional* $L \in V'$, *define*

$$\|L\|_A := \sup_{x \in A} |L(x)|.$$

Then $\| \cdot \|_A$ *is a seminorm on* V'.

Proof. First check that $\| \cdot \|_A$ is well defined over V'. Let $L \in V'$ meaning $L : V \to \mathbb{K}$ is a continuous linear form. Note that absolute value map $| \cdot |$ on $\mathbb{K}$ is a seminorm. By Theorem 2.1.21, there exists a continuous seminorm ρ on V such that $|L(y)| \leq \rho(y)$ for all $y \in V$. Since A is bounded, $\rho(A)$ is bounded by Proposition 2.1.27 and so we get $|L(x)|$ is finite for all $x \in A$. Hence, $\| \cdot \|_A$ is well defined.

It is easy to check that $\| \cdot \|_A$ preserves scaling and is subadditive. Fix $\lambda \in \mathbb{K}$ and $L_1, L_2 \in V'$.

$$\|\lambda L\|_A = \sup_{x \in A} |\lambda L(x)| = \sup_{x \in A} |\lambda| \, |L(x)| = |\lambda| \, \|L\|_A$$

$$\begin{aligned}\|L_1+L_2\|_A &= \sup_{x\in A}|L_1(x)+L_2(x)| \\ &\le \left(\sup_{x\in A}|L_1(x)|\right)+\left(\sup_{x\in A}|L_2(x)|\right) \\ &= \|L_1\|_A+\|L_2\|_A\end{aligned}$$

and so $\|\cdot\|_A$ is a seminorm on V'. □

Definition 2.1.29. The topology on V' which we will use is

$$\tau(\{\|\cdot\|_A \,:\, A\subset V \text{ bounded}\}).$$

We call this topology the strong topology and V' equipped with the strong topology is called the strong dual.

Be aware that the definition of the strong topology is done over all bounded subsets A of V. Observe that this is the topology of uniform convergence.

Another topology which is common is to only consider singleton sets, i.e., $\tau(\{|\cdot(x)| \,:\, x\in V\})$. This topology is known as the weak-* topology which corresponds with the topology of pointwise convergence. These lecture notes will focus soley on the strong topology.

Proposition 2.1.30. *Let (V,τ) be a LCTVS. There exists a natural linear map*

$$ev\begin{cases}V\to V''\\ x\mapsto ev(x)\end{cases}$$

where for all $L\in V'$, $ev(x)(L)=L(x)$. This map is called the evaluation map.

Proof. Fix $x\in V$. The map $ev(x)$ is a well defined function since $ev(x)(L)=L(x)$ is finite for all $L\in V'$. Furthermore, $ev(x)$ is also linear (in V and in V'). Indeed,

$$ev(x+y)(L)=L(x+y)=L(x)+L(y)=ev(x)(L)+ev(y)(L)$$

and

$$ev(x)(L+M)=(L+M)(x)=L(x)+M(x)=ev(x)(L)+ev(x)(M).$$

We need to show $ev(x)$ is continuous for strong topology on V'.

Take $A=\{x\}\subset V$. Since a singleton set, A is automatically a bounded set in V. Since

$$|L(x)|=\sup_{y\in A}|L(y)|=\|L\|_A$$

for all $L \in V'$ it follows that

$$|ev(x)(L)| \leq \|L\|_A$$

for all $L \in V'$. But $\|\cdot\|_A$ is a generator for the strong topology on V' and so it follows that $|ev(x)|$ (a seminorm on V') is continuous. Hence $ev(x)$ is also continuous which allows us to say $ev(x) \in V''$. This completes the proof. □

Notice that we in fact showed that for any $x \in V$, $ev(x) : V' \to \mathbb{K}$ is continuous with respect to the weak-* topology. Indeed the proof only uses a singleton set $\{x\}$ in V which is exactly a generator for the weak-* topology.

Definition 2.1.31. Let (V, τ) be a LCTVS. It is reflexive if and only if $V \xrightarrow{ev} V''$ is an isomorphism of TVS, i.e., bijective with ev and it's inverse continuous.

2.2 Multisequences With Fast Decay

Definition 2.2.1. A Fréchet space (this is equivalent to the original definition of Fréchet space)

1. has a topology induced by a countable number of seminorms
2. is Hausdorff
3. is a TVS with respect to that topology
4. is complete with respect to the metric induced by that topology

Definition 2.2.2. The space of multisequences with fast decay is the TVS

$$\begin{aligned} \mathfrak{s}(\mathbb{N}_0^d, \mathbb{K}) &:= \{\text{multisequences with fast decay}\} \\ &:= \{(x_\alpha)_{\alpha \in \mathbb{N}_0^d} \in \mathbb{K}^{\mathbb{N}_0^d} \mid \forall k \in \mathbb{N}_0, \|x\|_{\infty,k} < \infty\} \end{aligned}$$

where $\|x\|_{\infty,k} := \sup_{\alpha \in \mathbb{N}_0^d} \langle\alpha\rangle^k |x_\alpha| \in [0, \infty]$ is a norm, and the topology is $\mathcal{T}(\{\|\cdot\|_{\infty,k} \mid k \in \mathbb{N}_0\})$.

Remark 2.2.3. The above is a well-defined TVS.

Proposition 2.2.4. $\mathfrak{s}(\mathbb{N}_0^d, \mathbb{K})$ *is Fréchet.*

Proof. By definition, $\mathfrak{s} = \mathfrak{s}(\mathbb{N}_0^d, \mathbb{K})$ is already a TVS with its topology induced by a countable number of seminorms. It can be verified that this space is Hausdorff. The only thing left to verify is that the metric is complete.

We define

$$d(x,y) \coloneqq \sum_{n\geq 1} 2^{-n} \min\{1, \|x-y\|_{\infty,n-1}\}.$$

Now $\mathcal{T}(\{\|\cdot\|_{\infty,k} \mid k \in \mathbb{N}_0\}) = \mathcal{T}(d)$ is guaranteed as long as d is a distance (but we will not prove d is a distance here. We will assume that is true). We can now show that $\mathfrak{s}$ is complete with respect to d instead of working directly with the seminorms.

Let $(x^{(m)})_{m\geq 1} \coloneqq \left(\left((x_\alpha)_{\alpha\in\mathbb{N}_0^d}\right)^{(m)}\right)_{m\geq 1}$ be a Cauchy sequence of sequences $(x_\alpha)_{\alpha\in\mathbb{N}_0^d}$. By definition of Cauchy, this means that

$$\forall \varepsilon > 0, \exists M \geq 0, \forall p, q \geq M,\ d(x^{(p)}, x^{(q)}) < \varepsilon.$$

But $2^{-1}\min\{1, \|x-y\|_{\infty,0}\} = 2^{-1}\min\{1, \|x-y\|_{\infty,1-1}\} \leq d(x^{(p)}, x^{(q)})$, so $(x^{(m)})_{m\geq 1}$ is Cauchy for $\|\cdot\|_{\infty,0}$ too.

So for any fixed α, $\left(x_\alpha^{(m)}\right)_{m\geq 1}$ is Cauchy in $\mathbb{K}$. Because $\mathbb{K}$ is complete, $\lim_{m\to\infty} x_\alpha^{(m)}$ exists. So we define

$$x_\alpha \coloneqq \lim_{m\to\infty} x_\alpha^{(m)}.$$

Since this happens for each α, we now define

$$x \coloneqq (x_\alpha)_{\alpha\in\mathbb{N}_0^d}.$$

Remember, we are trying to prove that $(x^{(m)})_{m\geq 1}$ converges. We now prove that $(x^{(m)})_{m\geq 1}$ converges to x. But first, a lemma:

Lemma 2.2.5.

1. $\forall k, \forall p,\ \left\|x^{(p)} - x\right\|_{\infty,k} < \infty$

2. $\forall k, \lim_{p\to\infty} \left\|x^{(p)} - x\right\|_{\infty,k} = 0$

Proof. We know, for any fixed $k \geq 0$,

$$2^{-(k+1)} \min\left\{1,\ \left\|x^{(p)} - x^{(q)}\right\|_{\infty,k}\right\} \leq d(x^{(p)}, x^{(q)}).$$

Since $(x^{(m)})_{m\geq 1}$ is Cauchy, $\forall \varepsilon, \exists M, \forall p, q \geq M$,

$$2^{-(k+1)} \min\left\{1,\ \left\|x^{(p)} - x^{(q)}\right\|_{\infty,k}\right\} \leq d(x^{(p)}, x^{(q)}) < \varepsilon$$

$$\min\left\{1,\ \left\|x^{(p)} - x^{(q)}\right\|_{\infty,k}\right\} < 2^{k+1}\varepsilon$$

Now as long as we chose our ε to be smaller than $2^{-(k+1)}$ in the first place, we have $\min\left\{1, \left\|x^{(p)} - x^{(q)}\right\|_{\infty,k}\right\} < 1$, so that $\min\left\{1, \left\|x^{(p)} - x^{(q)}\right\|_{\infty,k}\right\} = \left\|x^{(p)} - x^{(q)}\right\|_{\infty,k}$. Consequently, we no longer have to bother with the $\min\{1, \cdot\}$ part of our expression:

$$\left\|x^{(p)} - x^{(q)}\right\|_{\infty,k} < 2^{k+1}\varepsilon$$
$$\sup_{\alpha\in\mathbb{N}_0} \langle\alpha\rangle^k \left|x_\alpha^{(p)} - x_\alpha^{(q)}\right| < 2^{k+1}\varepsilon$$

Since this is true for all $q \geq M$, we can let $q \to \infty$ while leaving the other variables alone:

$$\sup_{\alpha\in\mathbb{N}_0} \langle\alpha\rangle^k \left|x_\alpha^{(p)} - x_\alpha\right| < 2^{k+1}\varepsilon$$
$$\left\|x^{(p)} - x\right\|_{\infty,k} < 2^{k+1}\varepsilon$$
$$\left\|x^{(p)} - x\right\|_{\infty,k} < \infty$$

To summarize, we have shown that $\exists M, \forall p \geq M, \ \left\|x^{(p)} - x\right\|_{\infty,k} < \infty$.

This proves item (a).

((this next part is unclear:

and $x^{(p)} \in \mathfrak{s} \implies \|x\|_{\infty,k} < \infty$ for all $x \in \mathfrak{s}$, for all $k \in \mathbb{N}_0^d$.

))

Now since $\forall\varepsilon, \exists M, \forall p \geq M, \ \left\|x^{(p)} - x\right\|_{\infty,k} < 2^{k+1}\varepsilon$, we have that

$$\forall k, \lim_{p\to\infty} \left\|x^{(p)} - x\right\|_{\infty,k} = 0,$$

so we have proven item (b). □

We now continue with our main proof, using the lemma to show that $d(x^{(p)}, x) \to 0$...

d is a summation, so the idea is to chop off a tail from d that is small enough.

Let $(x^{(m)})_{m\geq 1}$ be Cauchy for d. It follows that for all $p, q \geq 1$, for all

$r \in \mathbb{N}$,

$$\begin{aligned} d(x^{(p)}, x^{(q)}) &:= \sum_{n=1}^{\infty} 2^{-n} \min\left\{1, \left\|x^{(p)} - x^{(q)}\right\|_{\infty,n-1}\right\} \\ &\leq \sum_{n=1}^{r} 2^{-n} \min\left\{1, \left\|x^{(p)} - x^{(q)}\right\|_{\infty,n-1}\right\} + \sum_{n=r+1}^{\infty} 2^{-n} \\ &= \sum_{n=r+1}^{\infty} 2^{-n} + \sum_{n=1}^{r} 2^{-n} \min\left\{1, \left\|x^{(p)} - x^{(q)}\right\|_{\infty,n-1}\right\} \\ &= 2^{-r} + \sum_{n=1}^{r} 2^{-n} \min\left\{1, \left\|x^{(p)} - x^{(q)}\right\|_{\infty,n-1}\right\} \end{aligned}$$

We will now pick an r so that each piece of the above is bounded by $\frac{\varepsilon}{2}$ and thus $d(x^{(p)}, x^{(q)}) < \varepsilon$.

Let $\varepsilon > 0$. Pick r s.t. $2^{-r} < \frac{\varepsilon}{2}$. This bounds one piece.

As for $\sum_{n=1}^{r} 2^{-n} \min\{1, \left\|x^{(p)} - x^{(q)}\right\|_{\infty,n-1}\}$, note first that this is a finite sum, and second that for all n, for any ε, there exists an M s.t. for all $p, q \geq M$, $\left\|x^{(p)} - x^{(q)}\right\|_{\infty,n-1} < 2^n \varepsilon$. So we choose $\frac{\varepsilon}{2r}$ as our ε, and then for $p, q \geq M_1, , , M_r$, we have

$$\begin{aligned} &\sum_{n=1}^{r} 2^{-n} \min\left\{1, \left\|x^{(p)} - x^{(q)}\right\|_{\infty,n-1}\right\} \\ &\leq \sum_{n=1}^{r} 2^{-n} \left\|x^{(p)} - x^{(q)}\right\|_{\infty,n-1} \\ &< \sum_{n=1}^{r} 2^{-n} \left(2^n \frac{\varepsilon}{2r}\right) \\ &= \sum_{n=1}^{r} \frac{\varepsilon}{2r} \\ &= \frac{\varepsilon}{2} \end{aligned}$$

So we have $\forall \varepsilon, \exists M, \forall p, q \geq M$, $d(x^{(p)}, x^{(q)}) < \varepsilon$ as desired. Again we can hold all variables constant while letting $q \to \infty$, so that

$$\forall \varepsilon, \exists M, \forall p, q \geq M, d(x^{(p)}, x) < \varepsilon,$$

and therefore $d(x^{(p)}, x) \to 0$ and so $(x^{(m)})_{m \geq 1}$ converges to x. Yay! ((unsure if what i did immediately above was legal or not))

We conclude that $(x^{(m)})_{m \geq 1}$ is complete with respect to d, and thus with respect to $\mathcal{T}(d)$, and thus with respect to $\mathcal{T}(\{ \| \cdot \|_{\infty,k} \mid k \in \mathbb{N}_0\})$.

All of the properties of Fréchet have been satisfied, so we conclude that $\mathfrak{s} = \mathfrak{s}(\mathbb{N}_0^d, \mathbb{K})$ is Fréchet. $\square$

Proposition 2.2.6 ("one c to bound them all")**.** *Let $V_1, , , V_n, W$ be normed spaces and $h\colon V_1 \times \times \times V_n \to W$ be multilinear. Then h is continuous iff $\exists c > 0, \forall (x_1, , , x_n) \in V_1 \times \times \times V_n$, $\|h(x_1, , , x_n)\|_W \leq c\,\|x_1\|_{V_1} \cdots \|x_n\|_{V_n}$.*

Proof. Please recall theorem 2.1.21.

- " $\implies$ "

  Given h is continuous, then because $\|\cdot\|_W$ itself is continuous with respect to $\{\|\cdot\|_W\}$, theorem 2.1.21 gives us that

  $$\|h(x_1, , , x_n)\|_W \leq T_1(x_1) \cdots T_n(x_n)$$

  for some T_k continuous with respect to $\{\|\cdot\|_{V_k}\}$. But T_k continuous with respect to $\{\|\cdot\|_{V_k}\}$ means that there exists some c_k such that $T_k \leq c_k \|\cdot\|_{V_k}$. So we have

  $$\begin{aligned}\|h(x_1, , , x_n)\|_W &\leq T_1(x_1) \cdots T_n(x_n)\\ &\leq c_1 \|x_1\|_{V_1} \cdots c_n \|x_n\|_{V_n}\\ &= (c_1 \ldots c_n)\,\|x_1\|_{V_1} \cdots \|x_n\|_{V_n}\\ &= c\,\|x_1\|_{V_1} \cdots \|x_n\|_{V_n}\end{aligned}$$

  as desired, as long as we choose $c = c_1 \ldots c_n$.

- "$\impliedby$"

  In order to apply theorem 2.1.21 to our situation, we need only show that not just $\|\cdot\|_W$, but ALL seminorms ρ continuous with respect to $\{\|\cdot\|_W\}$ satisfy the inequality $\rho(h(x_1, , , x_n)) \leq T_1(x_1) \cdots T_n(x_n)$ for some continuous seminorms T_k on V_k.

  But since ρ is continuous with respect to $\{\|\cdot\|_W\}$, then $\rho \leq c_\rho \|\cdot\|_W$ for some constant c_ρ. It follows immediately that

  $$\begin{aligned}\rho(h(x_1, , , x_n)) &\leq c_\rho \|h(x_1, , , x_n)\|_W\\ &\leq c_\rho c \|x_1\|_{V_1} \cdots \|x_n\|_{V_n}\\ &= T_1(x_1) \cdots T_n(x_n)\end{aligned}$$

  as long as we choose $T_1 = c_\rho c \|\cdot\|_{V_1}$ and $T_k = \|\cdot\|_{V_k}$ for all $k \neq 1$, which are certainly all continuous seminorms on V_k.

  We have now satisfied the conditions of theorem 2.1.21, so we get that h is continuous as desired.

□

Proposition 2.2.7. *Let $(V, \|\cdot\|)$ be a normed space and $A \subset V$. Then A is bdd by the TVS definition of bdd iff A is bdd by the normed space definition of bdd.*

Proof. Please recall proposition 2.1.27.

Remember that $(V, \|\cdot\|)$ is a LCTVS because its topology is the topology $\mathcal{T}(\{\|\cdot\|\})$ induced from the seminorm $\|\cdot\|$.

Therefore proposition 2.1.27 gives us the following equivalence:

$$\begin{aligned}
&A \text{ is bdd by the TVS definition of bdd} \\
\text{iff } &\forall \rho \in \{\|\cdot\|\},\ \rho(A) \text{ is bdd in } \mathbb{R} \\
\text{iff } &\|A\| \text{ is bdd in } \mathbb{R} \\
\text{iff } &\sup_{x \in A} \|x\| \text{ is finite} \\
\text{iff } &A \text{ is bdd by the normed space definition of bdd.}
\end{aligned}$$

□

Proposition 2.2.8. *Let $(V, \|\cdot\|)$ be a normed space. Then $V' = V'$, where the LHS is the strong dual of V as a LCTVS and the RHS is the dual of V as a normed space defined using the operator norm $\|\cdot\|_{\mathcal{L}_1(V,\mathbb{K})}$.*

Proof.

- " $\implies$ "

  For any linear form $L \colon V \to \mathbb{K}$, we know that

  $$\|L\|_{V'} := \sup_{x \in V \setminus \{0\}} \frac{|L(x)|}{\|x\|}$$

  is the operator norm of L. We wish to show that $\mathcal{T}(\{\|\cdot\|_{V'}\}) = \mathcal{T}(\{\|\cdot\|_A \mid A \subset V, A \text{ bounded}\})$.

  If we choose cleverly a specific $A := \{x \in V \mid \|x\| \leq 1\}$ (note that A is bounded in the LCTVS sense), then we have that

  $$\|L\|_{V'} := \sup_{x \in A} |L(x)|\,.$$

  Since L was arbitrary, this implies that $\|\cdot\|_{V'}$ is continuous with respect to $\{\|\cdot\|_A \mid A \subset V, A \text{ bounded}\}$.

- "$\Longleftarrow$"

  Conversely, let $A \subset V$ be bounded (and ignore the trivial cases $A = \varnothing$

and $A = \{0\}$). Then $\sup_{x\in A} \|x\| = B < \infty$. It follows that

$$\begin{aligned}\|L\|_A &= \sup_{x\in A} |L(x)| \\ &= \sup_{x\in A\setminus\{0\}} |L(x)| \\ &= \sup_{x\in A\setminus\{0\}} \frac{|L(x)|}{\|x\|} \cdot \|x\| \\ &\le \sup_{x\in V\setminus\{0\}} \frac{|L(x)|}{\|x\|} \cdot \sup_{x\in A} \|x\| \\ &\le \|L\|_{V'} \cdot B \\ &< \infty\end{aligned}$$

Since L was arbitrary, this implies that for every A, $\|\cdot\|_A$ is continuous with respect to $\|\cdot\|_{V'}$. This implies that $\{\|\cdot\|_A \mid A \subset V, A \text{ bounded}\}$ is continuous with respect to $\|\cdot\|_{V'}$.

- "="

  Putting it all together, we have shown that $\|\cdot\|_{V'}$ and $\{\|\cdot\|_A \mid A \subset V, A \text{ bounded}\}$ are continuous with respect to each other. Therefore, they induce the same topology. This gives $\mathscr{T}(\{\|\cdot\|_{V'}\}) = \mathscr{T}(\{\|\cdot\|_A \mid A \subset V, A \text{ bounded}\})$ as desired.

□

Corollary 2.2.9. *Let $(V, \|\cdot\|)$ be a normed space. Then V is reflexive in the normed space sense iff V is reflexive in the LCTVS sense.*

Example 2.2.10. $(V, \|\cdot\|) = (L^2(X, \mathscr{A}, \mu, \mathbb{K}), \|\cdot\|_{L^2})$ is reflexive.

For $x = (x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \mathbb{K}^{\mathbb{N}_0^d}$, we defined for $k \in \mathbb{N}_0$,

$$\|x\|_{\infty,k} = \sup_{\alpha\in\mathbb{N}_0^d} (\langle\alpha\rangle^k |x_\alpha|)$$

where for $z \in \mathbb{R}^d \supset \mathbb{N}_0^d$,

$$\langle z\rangle = \sqrt{1 + |z|^2}.$$

Define

$$\mathfrak{s}(\mathbb{N}_0^d, \mathbb{K}) = \{x \in \mathbb{K}^{\mathbb{N}_0^d} \mid \forall\, k \ge 0, \|x\|_{\infty,k} < \infty\}$$

with topology $\tau(\{\|\cdot\|_{\infty,k} \mid k \ge 0\})$.

Theorem 2.2.11. *$\mathfrak{s}(\mathbb{N}_0^d, \mathbb{K})$ is Fréchet. (Yes, this is redundant, and redundancy is okay. This happened because the theorem was incorrectly proved the first time. But the first version of the proof in the notes has been fixed, so now we have two versions of the correct proof.)*

Proof. Define
$$d(x,y) = \sum_{k\geq 1} 2^{-k} \min(1, \|x-y\|_{\infty,k-1}).$$
It is easy to see that d defines the topology, and so $\mathfrak{s}$ is metrizable. It remains to show that it is complete.

Suppose $(x^{(m)})_{m\geq 1}$ is Cauchy in $\mathfrak{s}$ for d. First, we show that for all $k \geq 0$, $(x^{(m)})_{m\geq 1}$ is Cauchy for $\|\cdot\|_{\infty,k}$. Let $k \geq 0$ and $\epsilon > 0$. Then there exists an $M \geq 0$ such that for $p,q \geq M$,
$$d(x^{(p)}, x^{(q)}) < 2^{-(k+1)} \min(1/2, \epsilon).$$
Then
$$2^{-(k+1)} \min(1, \|x^{(p)} = x^{(q)}\|_{\infty,k}) \leq d(x^{(p)}, x^{(q)}) < 2^{-(k+1)} \min(1/2, \epsilon).$$
Since $d(x^{(p)}, x^{(q)}) \leq 1$, we have that
$$\min(1, \|x^{(p)} - x^{(q)}\|_{\infty,k}) = \|x^{(p)} - x^{(q)}\|_{\infty,k} < \epsilon.$$

Now, since $\|\cdot\|_{\infty,0}$ is the norm of uniform convergence on $\mathbb{N}_0^d$, since $(x^{(m)})_{m\geq 1}$ is Cauchy in $\|\cdot\|_{\infty,0}$, it is pointwise Cauchy, i.e. for all $\alpha \in \mathbb{N}_0^d$, if $(x_\alpha^{(m)})_{m\geq 1}$ is Cauchy in $\mathbb{K}$, then it converges to x_α. This defines $x = (x_\alpha)_{\alpha\in\mathbb{N}_0^d}$.

Claim: For all $k \geq 0$, $\|x^{(m)} - x\|_{\infty,k}$ is finite and converges to 0 as $m \to \infty$.

Fix k, and suppose $(x^{(m)})_{m\geq 1}$ is Cauchy for $\|\cdot\|_{\infty,k}$. Then for all $\epsilon > 0$, there exists an M such that for all $p,q \geq M$,
$$\|x^{(p)} - x^{(q)}\|_{\infty,k} = \sup_{\alpha\in\mathbb{N}_0^d} \langle\alpha\rangle^k |x_\alpha^{(p)} - x_\alpha^{(q)}| \leq \epsilon.$$
Hence, for all $p,q \geq M$ and all α,
$$\langle\alpha\rangle^k |x_\alpha^{(p)} - x_\alpha^{(q)}| \leq \epsilon.$$
Taking the limit $q \to \infty$, we have that for all $p \geq M$ and $\alpha \in \mathbb{N}_0^d$,
$$\langle\alpha\rangle^k |x_\alpha^{(p)} - x_\alpha| \leq \epsilon,$$
i.e. for all $p \geq M$, $\|x^{(p)} - x\|_{\infty,k} \leq \epsilon$. Hence $x \in \mathfrak{s}$ (is we choose $x^{(p)} \in \mathfrak{s}$).

Moreover, for all $k \geq 0$,
$$\lim_{m\to\infty} \|x^{(m)} - x\|_{\infty,k} = 0.$$
So, let $\epsilon > 0$, and choose $K \geq 0$ so that $2^{-K} \leq \epsilon/2$. Then
$$d(x^{(m)}, x) \leq 2^{-K} + \sum_{k=0}^{K} 2^{-k} \min(1, \|x^{(m)} - x\|_{\infty,k-1})$$

where the first term is smaller than $\epsilon/2$ and the second term limits to 0 as $m \to \infty$. Hence
$$\lim_{m\to\infty} d(x^{(m)}, x) = 0,$$
which implies that $(\mathfrak{s}, d)$ is a complete metric space, which then implies that it is Fréchet. □

Definition 2.2.12. For all $p \in [1, \infty)$, $x \in \mathfrak{s}$, and $k \geq 0$, let
$$\|x\|_{p,k} = \left[\sum_{\alpha\in\mathbb{N}_0^d} \langle\alpha\rangle^k |x_\alpha|^p\right]^{1/p}.$$

Remark 2.2.13. This is $\|\cdot\|_{L^p}$ for $(\mathbb{N}_0^d, \mathcal{P}(\mathbb{N}_0^d), \mu_k)$ where μ_k is defined on singletons by $\mu_k(\{\alpha\}) = \langle\alpha\rangle^k$.

Note that $\|\cdot\|_{L^\infty} \neq \|\cdot\|_{\infty,k}$.

Proposition 2.2.14. *For all $p \in [1, \infty)$, $\{\|\cdot\|_{p,k} : k \in \mathbb{N}_0\}$ is a defining collection of seminorms.*

Corollary 2.2.15. *For each $k \geq 0$, let $\mathcal{H}_k$ denote the Hilbert space $L^2(\mathbb{N}_0^d, \mathcal{P}(\mathbb{N}_0^d), \mathbb{K})$. Then $\mathcal{H}_k \subset \mathcal{H}_{k+1}$ and*
$$\mathfrak{s} = \bigcap_{k\geq 0} \mathcal{H}_k.$$

Remark 2.2.16. For all $p \in [1, \infty]$, $\|\cdot\|_{p,k} \leq \|\cdot\|_{p,k+1}$ because the dependence in k for μ_k is based on the weight.

Before we prove Propostion 2.2.14, we will need a lemma.

Lemma 2.2.17. *Let $\lambda \in \mathbb{R}$. Then if $\lambda > d$, we have*
$$\sum_{\alpha\in\mathbb{N}_0^d} \langle\alpha\rangle^{-\lambda} < \infty.$$

Proof. Recall that for $a \in \mathbb{R}$, $2a \leq 1 + a^2$ (because $(a-1)^2 \geq 0$. Then for $x, y \in \mathbb{R}^d$,
$$\begin{aligned}\langle x+y\rangle^2 &= 1 + |x+y|^2\\ &\leq 1 + |x|^2 + |y|^2 + 2|x||y|\\ &\leq 1 + |x|^2 + |y|^2 + 1 + |x|^2|y|^2\\ &\leq 2 + (1+|x|^2)(1+|y|^2)\end{aligned}$$

Hence $\langle x+y\rangle \leq \sqrt{2}\langle x\rangle\langle y\rangle$.

Now, if $u \in [0,1)^d$, then $|u| \leq \sqrt{d}$, and so $\langle u \rangle \leq \sqrt{d+1}$. Let $\lambda > d$, $\alpha \in \mathbb{N}_0^d$, and $u \in [1,0)^d$. We have $\langle \alpha + u \rangle \leq \sqrt{2}\langle \alpha \rangle \sqrt{d+1}$. Then

$$\begin{aligned}\langle \alpha \rangle^{-\lambda} &\leq (2(d+1))^{\lambda/2} \langle \alpha + u \rangle^{-\lambda} \\ &\leq (2(d+1))^{\lambda/2} \int_{[0,1)^d} d^d u \langle \alpha + u \rangle^{-\lambda},\end{aligned}$$

and so, using countable additivity of measure and then switching to spherical coordinates, we have

$$\begin{aligned}\sum_{\alpha \in \mathbb{N}_0^d} \langle \alpha \rangle^{-\lambda} &\leq (2(d+1))^{\lambda/2} \int_{[0,\infty)^d} d^d x \langle x \rangle^{-\lambda} \\ &\leq (2(d+1))^{\lambda/2} \int_{\mathbb{R}^d} d^d x \langle x \rangle^{-\lambda} \\ &= (2(d+1))^{\lambda/2} Vol_{d-1}(\mathbb{S}^{d-1}) \int_0^\infty \frac{r^{d-1}}{(1+r^2)^{\lambda/2}} dr\end{aligned}$$

Since the integral is finite, we have proved the lemma. □

Now, we are ready to prove Proposition 2.2.14.

Proof. Suppose $p \in [1,\infty)$ and $k \in \mathbb{N}_0$ We first show that $\|\cdot\|_{p,k}$ is finite.

If $x \in \mathfrak{s}$, then $\|x\|_{p,k}^p = \sum_{\alpha \in \mathbb{N}_0^d} \langle \alpha \rangle^k |x_\alpha|^p$. Pick $m > \frac{k+d}{p}$. Then $k - mp < d$, and by the preceeding lemma,

$$\begin{aligned}\|x\|_{p,k}^p &= \sum_{\alpha \in \mathbb{N}_0^d} \langle \alpha \rangle^k (\langle \alpha \rangle^{-m} \|x\|_{\infty,m})^p \\ &= \|x\|_{\infty,m} \sum_{\alpha \in \mathbb{N}_0^d} \langle \alpha \rangle^{k-mp} < \infty\end{aligned}$$

Hence $\|x\|_{p,k} < \infty$. Moreover, $\|\cdot\|_{p,k} \leq$ (constant) $\|\cdot\|_{\infty,m}$, i.e. $\|\cdot\|_{p,k}$ is continuous relative to the $\|\cdot\|_{\infty,k}$ seminorms.

It remains to show that $\|\cdot\|_{\infty,k}$ is continuous relative to $\|\cdot\|_{p,k}$. For $m \geq kp$,

$$\begin{aligned}(\|x\|_{\infty,k})^p &= \sup_\alpha (\langle \alpha \rangle^k |x_\alpha|)^p \\ &\leq \sum_\alpha \langle \alpha \rangle^{kp} |x_\alpha|^p \\ &\leq \sum_\alpha \langle \alpha \rangle^m |x_\alpha|^p.\end{aligned}$$

Hence $\|\cdot\|_{\infty,k} \leq \|\cdot\|_{p,k}$, which implies $\|\cdot\|_{\infty,k}$ is continuous relative to $\|\cdot\|_{p,k}$. So, $\mathcal{N} = \{\|\cdot\|_{p,m} | m \in \mathbb{N}_0\}$. □

A variant: One can also define for $d \geq 1$

$$\mathfrak{s}(\mathbb{Z}^d, \mathbb{K}) = \{(x_\alpha)_{\alpha\in\mathbb{Z}} \in \mathbb{K}^{\mathbb{Z}^d} | \;\forall\; k \in \mathbb{N}_0, \|x\|_{\infty,k} < \infty\}$$

where $\|x\|_{\infty,k} = \sup_{\alpha\in\mathbb{Z}^d}\langle\alpha\rangle^k|x_\alpha|$.

Let τ be the topology defined by these seminorms. [1]

Theorem 2.2.18. *For all $d \geq 1$, $\mathfrak{s}(\mathbb{Z}^d, \mathbb{K}) \simeq \mathfrak{s}(\mathbb{N}_0^d, \mathbb{K})$ as TVSs.*

Proposition 2.2.19. *$\mathfrak{s}(\mathbb{Z}^d, \mathbb{K})$ is isomorphic as a topological vector space to $\mathfrak{s}(\mathbb{N}_0^d, \mathbb{K})$.*

Proof. Define a bijection $\tau : \mathbb{N}_0 \to \mathbb{Z}$ by $(\tau(0), \tau(1), \tau(2), \dots) = (0, 1, -1, 2, -2, 3, \dots)$, i.e. for $n \in \mathbb{N}_0$,

$$\tau(n) = \begin{cases} -n/2 & n \equiv 0 \mod 2 \\ \frac{n+1}{1} & n \equiv 1 \mod 2 \end{cases}.$$

Next define

$$\sigma : \mathbb{N}_0^d \to \mathbb{Z}^d$$
$$\alpha = (\alpha_1, \dots, \alpha_d) \mapsto \sigma(\alpha) := (\tau(\alpha_1), \dots, \tau(\alpha_d)).$$

Furthermore, define

$$L : \mathfrak{s}(\mathbb{Z}^d) \to \mathfrak{s}(\mathbb{N}_0^d)$$
$$L(x) = x * \sigma$$
$$x = (x_\alpha)_{\alpha\in\mathbb{Z}^d} \mapsto L(x) = (x_{\sigma(\alpha)})_{\alpha\in\mathbb{N}_0^d}.$$

Note,

$$\langle\sigma(\alpha)\rangle^2 = 1 + \sum_{i=1}^d \tau(\alpha_i)^2 \leq 1 + \sum_{i=1}^d \Big(\frac{\alpha_i + 1}{2}\Big)^2$$

so that for all $a, b \in \mathbb{R}$, $(a - b)^2 \geq 0 \Rightarrow (a + b)^2 \leq 2(a^2 + b^2)$,

$$\langle\sigma(\alpha)\rangle^2 \leq 1 + \sum_{i=1}^d \frac{1}{2}(\alpha_i^2 + 1) \leq (1 + d/2) + \frac{1}{2}\sum_{i=1}^d \alpha_i^2 \leq (1 + d/2)\langle\alpha\rangle^2$$

$$\Rightarrow \exists c_1 > 0, \forall d \in \mathbb{N}_0^d, \langle\sigma(\alpha)\rangle \leq c_1\langle\alpha\rangle,$$

e.g., we can let $c_1 = \sqrt{1 + d/2}$.

[1] One can, in the preceding proposition, replace $\mathbb{N}_0^d$ with $\mathbb{Z}^d$ everywhere with no change except the step establishing $\langle\alpha\rangle^{-\lambda} \leq \dots \int_{\mathbb{R}^d} d^dx\langle x\rangle^{-\lambda}$.

If $n \in \mathbb{N}_0$ is even, $n = 2|\tau(n)|$.

If $n \in \mathbb{N}_0$ is odd, $n < n+1 = 2|\tau(n)|$. Therefore,

$$\forall n, n \leq 2|\tau(n)|$$

$$\Rightarrow \forall \alpha \in \mathbb{N}_0^d, \langle \alpha \rangle^2 = 1 + \sum_{i=1}^d \alpha_i^2 \leq 1 + \sum_{i=1}^d 4\tau(\alpha_i)^2 \leq 4\langle \sigma(\alpha) \rangle^2$$

$$\Rightarrow \exists c_2 > 0, \forall \alpha \in \mathbb{N}_0^d, \langle \alpha \rangle \leq c_2 \langle \sigma(\alpha) \rangle,$$

e.g. we can let $c_2 = 2$.

We verify that L is well-defined. If $x \in \mathfrak{s}(\mathbb{Z}^d)$, $L(x) \in \mathbb{K}^{\mathbb{N}_0^d}$, let $k \geq 0$, note

$$||L(x)||_{\alpha,k} = \sup_{\beta \in \mathbb{N}_0^d} \langle \beta \rangle^k |x_{\sigma(\beta)}| \leq c_2^k \sup_{\beta \in \mathbb{N}_0^d} \langle \sigma(\beta) \rangle^k |x_{\sigma(\beta)}| = c_2^k \sup_{\alpha \in \mathbb{Z}^d} \langle \alpha \rangle^k |x_\alpha|,$$

so L is well-defined and continuous, as the final expression is $||x||_{\infty,k} < \infty$. Note we used the bijectivity of σ at this step.

Conversely: let

$$R : \mathbb{N} \to \mathfrak{s}(\mathbb{Z}^d)$$
$$y \mapsto y \circ \sigma^{-1}$$

so $\forall \alpha \in \mathbb{Z}^d, (y \circ \sigma^{-1})_\alpha = y_{\sigma^{-1}(\alpha)}$. We now note that

$$k \geq 0, ||R(y)||_{\infty,k} = \sum_{\alpha \in \mathbb{Z}^d} \langle \alpha \rangle^k |y_{\sigma^{-1}(\alpha)}|$$

and using the bijectivity of our map, this expression is equal to

$$\sup_{\beta \in \mathbb{N}_0^d} \langle \sigma(\beta) \rangle^k |y_\beta| \leq c_1^k \sup_{\beta \in \mathbb{N}_0^d} \langle \beta \rangle^k |y_\beta|$$

$$\Rightarrow \forall k \in \mathbb{N}_0, \forall y \in \mathfrak{s}(\mathbb{N}_0^d), ||R(y)||_\infty \leq c_1^k ||y||_{\infty,k}$$
$$\Rightarrow R(y) \in \mathfrak{s}(\mathbb{Z}^d),$$

and therefore R is continuous using the criterion for the continuity of multilinear maps. Furthermore, L and R are inverse to each other by construction, so our work is done.

□

Theorem 2.2.20. *$\forall d \geq 2$, $\mathfrak{s}(\mathbb{N}_0^d, \mathbb{K})$ is isomorphic as a topological vector space to $\mathfrak{s}(\mathbb{N}_0, \mathbb{K}) =: \mathfrak{s}(\mathbb{K})$ (also denoted $\mathfrak{s}$).*

Proof. Recall $\mathbb{N}_0^2 \simeq \mathbb{N}_0$ as sets. We employ the bijection of these sets that enumerates the elements of $\mathbb{N}_0^2$ in the following way:

$$(0,0),(0,1),(1,0),(0,2),(1,1),(2,0),\dots$$

. This enumeration corresponds to the general construction for $d \geq 2$:

$$\rho_d : \mathbb{N}_0 \to \mathbb{N}_0^d$$

where $\forall k \geq 0,\ s_k := \{\alpha \in \mathbb{N}_0^d : \alpha_1 + \cdots + \alpha_d = k\}$; in our initial enumeration, s_2 would corresponds to $\{(0,2),(1,1),(2,0)\}$, for example. Then putting s_k's consecutively, we obtain $(\rho_d(0), \rho_d(1), \dots)$, where inside each s_k block, we use lexicographic order. Therefore, in our original construction, $\rho_2(0) = (0,0), \rho_2(1) = (0,1), \rho_2(1) = (1,0)$, and so on.

We note that in this process, we consider a dictionary where $0 = A, 1 = B, \dots$, e.g., for $d = 3, k = 2$, the ordering inside s_2 is:

$$(0,0,2),(0,1,1),(0,2,0),(1,0,1),(1,1,0),(2,0,0).$$

We have that

$$|s_k| = \binom{k+d-1}{d-1} = \frac{(k+1)(k+2)\dots(k+d-1)}{(d-1)!}.$$

The key remark here is that $|s_k|$ grows polynomially in k ($\approx k^{d-1}$).

Lemma 2.2.21. *$\exists c_1, c_2 > 0$ such that $\forall n \in \mathbb{N}_0$,*

$$\langle \rho_d(n) \rangle \leq c_1 \langle n \rangle$$

$$\langle n \rangle \leq c_2 \langle \rho_d(n) \rangle.$$

Proof. Let $\rho_d(n) = (\alpha_1, \dots, \alpha_d) \in s_k$ so that $|\alpha| = k$. Our proof of the lemma begins with the observation that if $\rho_d(n) \in s_k$, then $k \leq n$ because

$$n \geq |s_0| + \cdots + |s_{k-1}| \Rightarrow n \geq k.$$

If $\rho_d(n) = (\alpha_1, \dots, \alpha_d)$, then

$$\begin{aligned}
\langle \rho_d(n) \rangle^2 &= 1 + \alpha_1^2 + \cdots + \alpha_d^2 \\
&\leq 1 + (\alpha_1 + \cdots + \alpha_d)^2 \\
&= 1 + k^2 \\
&\leq 1 + n^2 \\
&= \langle n \rangle^2,
\end{aligned}$$

so $c_1 = 1$ suffices.

Next, we note that

$$\begin{aligned} N &= |s_0| + \cdots + |s_k| \\ &= |\{\alpha \in \mathbb{N}_0^d : \alpha_1 + \cdots + \alpha_d \leq k\}| \\ &= |\{B \in \mathbb{N}_0^{d+1} : \beta_1 + \cdots + \beta_{d+1} = k\}| \\ &= \frac{(k+1)\cdots(k+d)}{d!} \\ &\leq \frac{(k+d)^d}{d!} \end{aligned}$$

and given that $\rho_d(n) = (\alpha_1, \ldots, \alpha_d)$, we have $\alpha_1 + \cdots + \alpha_d = k$, and

$$\begin{aligned} k + d &= d * 1 + 1 * \alpha_1 + \cdots + 1 * \alpha_d \\ &= \langle d, 1, \ldots, 1\rangle \bullet \langle 1, \alpha_1, \ldots, \alpha_d\rangle \\ &\leq \|\langle d, 1, \ldots, 1\rangle\| \ \|\langle 1, \alpha_1, \ldots, \alpha_d\rangle\| \qquad \text{(by Cauchy-Schwarz)} \\ &= \sqrt{d^2 + d} \cdot \sqrt{1 + \alpha_1^2 + \cdots + \alpha_d^2}. \end{aligned}$$

Note that the second square root is less than $\langle \rho_d(n)\rangle$. Now we note that for $N \geq 1$, we have

$$0 \leq n \leq N - 1 < N$$

$$\Rightarrow \langle n\rangle < N \leq \frac{(k+d)^d}{d!} \leq \frac{1}{d!}\left(\sqrt{d(d+1)}\langle \rho_d(n)\rangle\right)^d$$

$$\Rightarrow \langle n\rangle \leq c_2 \langle \rho_d(n)\rangle^d$$

so that $c_2 = \frac{\sqrt{d(d+1)}^d}{d!}$ suffices.

□

We are now prepared to proceed with the proof of the theorem; we have

$$L : \mathfrak{s}(\mathbb{N}_0^d) \to \mathfrak{s}(\mathbb{N}_0)$$

and R a map in the reverse direction. For $x \in \mathfrak{s}(\mathbb{N}_0^d)$ and $y \in \mathfrak{s}(\mathbb{N}_0)$, we have

$$L(x) = x \circ \rho_d$$

$$R(y) = y \circ \rho_d^{-1}$$

$$k \geq 0, ||L(x)||_{\alpha,k} = \sup_{n \geq 0} \langle n\rangle^k |x_{\rho_d(n)}|$$

$$\leq c_2^k \sup_{n \geq 0} \langle \rho_d(n)\rangle^{dk} |x_{\rho_d(n)}|$$

where $\sup_{n\geq 0}\langle \rho_d(n)\rangle^{dk}|x_{\rho_d(n)}| = ||x||_{\infty,dk}$. Therefore, L is well-defined and continuous. To determine that R and L are inverse to each other, consider

$$||R(y)||_{\infty,k} = \sup_{\alpha\in\mathbb{N}_0^d} \langle\alpha\rangle^k |y_{\rho_d^{-1}(\alpha)}|$$

$$= \sup_{n\in\mathbb{N}_0} \langle\rho_d(n)\rangle^k |y_n|$$

where $\langle\rho_d(n)\rangle^k \leq \langle n\rangle$. Therefore, $||R(y)||_{\infty,k} \leq ||y||_{\infty,k}$, so that L and R are indeed inverse to each other.

□

2.2.1 Schwartz Space

For $f : \mathbb{R}^d \to \mathbb{K}$ a C^∞ function, $\alpha \in \mathbb{N}_0^d$, and $k \in \mathbb{N}_0$, we define

$$\|f\|_{\alpha,k} \coloneqq \sup_{x\in\mathbb{R}^d} \langle x\rangle^k \ |\partial^\alpha f(x)| \in [0,\infty]$$

The **Schwartz space** $\mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d,\mathbb{K})$ is then defined to be the vector space

$$\mathcal{S}(\mathbb{R}^d,\mathbb{K}) \coloneqq \{f : \mathbb{R}^d \to \mathbb{K} \mid f \text{ is } C^\infty, \|f\|_{\alpha,k} < \infty \text{ for all } \alpha \in \mathbb{N}_0^d, k \in \mathbb{N}_0\}$$

Each $\|\cdot\|_{\infty,k}$ is a seminorm on $\mathcal{S}(\mathbb{R}^d)$. The standard topology on $\mathcal{S}(\mathbb{R}^d)$ is $\mathcal{T}(\{\|\cdot\|_{\alpha,k} \mid \alpha \in \mathbb{N}_0^d, k \in \mathbb{N}_0\})$. Because $\|\cdot\|_{0,0}$ is a norm and the collection of seminorms defining the topology is countable, $\mathcal{S}(\mathbb{R}^d)$ is both Hausdorff and metrizable. We now define the **space of temperate (tempered) distributions on $\mathbb{R}^d$ with values in $\mathbb{K}$**, $\mathcal{S}'(\mathbb{R}^d,\mathbb{K}) = \mathcal{S}'(\mathbb{R}^d)$, to be the (strong) dual of $\mathcal{S}(\mathbb{R}^d)$.

When there are functions of multiple variables at play, we may write $\phi(x)$ or $\mathcal{S}'_x(\mathbb{R}^d)$ to clarify which expressions are evaluated by elements of the Schwartz distribution space and which are treated as constants. For $\phi \in \mathcal{S}'$, we write

$$\phi(f) = \langle\phi, f\rangle = \langle\phi(x), f(x)\rangle_x$$

for the *duality pairing*, i.e., the evaluation of the linear form ϕ at $f \in S$ (thought as a test function). Heuristically, one should think of $\phi(f)$ as

$$\phi(f) = \int_{\mathbb{R}^d} \phi(x)\ f(x)\ d^d(x)$$

for some "function" $\phi(x)$. For $n \in \mathbb{N}_0$, we define the standard nth **Hermite polynomial** $H_n(x)$ by

$$H_n(x) \coloneqq (-1)^n e^{x^2}\ \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

By applying the Faá di Bruno formula, we obtain an explicit formula for $H_n(x)$.

Proposition 2.2.22. *For all $n \in \mathbb{N}_0$,*

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}$$

Proof. By direct application of Faá di Bruno with $g(x) = e^x$ and $f(x) = -x^2$, we obtain

$$\begin{aligned}
(-1)^n H_n(x) &= e^{x^2} \frac{d}{dx^n} g(f(x)) \\
&= e^{x^2} \sum_{k \geq 0} \sum_{n_1, \dots, n_k \geq 1} \mathbb{1}\left\{\sum_{i=1}^k n_i = n\right\} g^{(k)}(f(x)) \frac{f^{(n_1)}(x) \cdots f^{(n_k)}(x)}{k!\, n_1! \cdots n_k!} \\
&= e^{x^2} \sum_{k=0}^{n} \sum_{2 \geq n_1, \dots, n_k \geq 1} \mathbb{1}\left\{\sum_{i=1}^k n_i = n\right\} e^{-x^2} \frac{(-2)^k x^{2-n_1} \cdots x^{2-n_k}}{k!\, n_1! \cdots n_k!} \\
&= \sum_{k=0}^{n} \sum_{2 \geq n_1, \dots, n_k \geq 1} \mathbb{1}\left\{\sum_{i=1}^k n_i = n\right\} \frac{(-1)^k 2^k x^{2k-n}}{k!\, n_1! \cdots n_k!}
\end{aligned}$$

where we used that $f^{(m)}(x) = 0$ for $m > 2$. Note that

$$k \leq n = \sum_{i=1}^k n_k \leq 2k \implies k \geq \frac{n}{2},$$

so the first sum is in fact over $n/2 \leq k \leq n$. Also,

$$\sum_{2 \geq n_1, \dots, n_k \geq 1} \mathbb{1}\left\{\sum_{i=1}^k n_i = n\right\} \binom{k}{n-k} = \frac{k!}{(n-k)!\,(2k-n)!}$$

and

$$n_1! \cdots n_k! = 2^{n-k}$$

Therefore,

$$\begin{aligned}
(-1)^n H_n(x) &= \sum_{\frac{n}{2} \leq k \leq n} \frac{(-1)^k 2^{2k-n} x^{2k-n}}{(n-k)!\,(2k-n)!} \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{n-j} (2x)^{n-2j}}{j!\,(n-2j)!}
\end{aligned}$$

where $j = n - k$. ☐

We now define the nth **Hermite function** $h_n(x)$ by

$$h_n(x) := \pi^{-\frac{1}{4}}\, 2^{-\frac{n}{2}}\, (n!)^{-\frac{1}{2}}\, e^{-\frac{x^2}{2}}$$

Because of the factor of $e^{-\frac{x^2}{2}}$, it is easily seen that $h_n \in \mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ for all $n \geq 0$. Moreover, the collection $\{h_n \mid n \geq 0\}$ forms an orthonormal Schauder basis of $L^2(\mathbb{R})$. More generally, one can define for all $\alpha \in \mathbb{N}_0^d$ the Hermite function h_α on $\mathbb{R}^d$ by

$$h_\alpha(x) \coloneqq h_{\alpha_1}(x) \cdots h_{\alpha_d}(x)$$

Theorem 2.2.23 (Sequence Space Representation)**.** *The map* $\mathcal{S}(\mathbb{R}^d) \to \mathfrak{s}(\mathbb{N}_0^d)$ *given by*

$$f \longmapsto \left(\int_{\mathbb{R}^d} h_\alpha(x)\, f(x)\, d^d x\right)_{\alpha \in \mathbb{N}_0^d}$$

is a well-defined isomorphism of topological vector spaces.

The proof of this result is postponed until Chapter 4.

Corollary 2.2.24. $\mathcal{S}(\mathbb{R}^d) \cong \mathfrak{s}(\mathbb{N}_0^d)$ *as topological vector spaces.*

Corollary 2.2.25. $\mathcal{S}(\mathbb{R}^d)$ *is Fréchet.*

Corollary 2.2.26. *The maps* $\mathcal{S}'(\mathbb{R}^d) \to \mathfrak{s}'(\mathbb{N}_0^d) \to \mathfrak{s}'(\mathbb{N}_0)$ *given by*

$$\phi \longmapsto (\phi(h_\alpha))_{\alpha \in \mathbb{N}_0^d} \longmapsto (\phi(h_{\rho_d^{-1}(\alpha)})_{\rho_d^{-1}(\alpha)}$$

are isomorphisms of topological vector spaces.

2.2.2 The Space $\mathfrak{s}_0$

We will refer the Homework 6 of MATH 7310 much throughout this section. Define the vector space $\mathfrak{s}_0$ by

$$\mathfrak{s}_0 \coloneqq \{x = (x_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}_0} \mid x_n = 0 \text{ for all but finitely many } n\}$$

As a vector space, $\mathfrak{s}_0$ is isomorphic to $\bigoplus_{n\geq 0} \mathbb{K}$ and $\mathbb{K}[x]$ (space of polynomials). Categorically, $\mathfrak{s}_0$ is the "directed colimit" of the directed system

$$\mathbb{K} \hookrightarrow \mathbb{K}^2 \hookrightarrow \mathbb{K}^3 \hookrightarrow \cdots$$

We endow $\mathfrak{s}_0$ with the finest locally convex topology, namely $\mathcal{T}(\mathcal{N}_{all}(\mathfrak{s}_0))$.

Notation. $\mathfrak{s}_0' \coloneqq \mathbb{K}^{\mathbb{N}_0}$ and $\mathfrak{s}_{0,+}' \coloneqq [0,\infty)^{\mathbb{N}_0} \subseteq \mathfrak{s}_0'$

For $x, y \in \mathfrak{s}_0'$, let

$$\langle x, y\rangle \coloneqq \sum_{n=0}^{\infty} x_n y_n$$

if this series converges. If $\omega = (\omega_n)_{n\geq 0} \in \mathfrak{z}'_{0,+}$, then for $x \in \mathfrak{z}_0$ we define

$$\|x\|_\omega \coloneqq \sum_{n=0}^{\infty} \omega_n \, |x_n| < \infty \qquad \text{(finite sum)}$$

$\|\cdot\|_\omega$ is clearly a seminorm on $\mathfrak{z}_0$.

Proposition 2.2.27. *The collection* $\{\|\cdot\|_\omega \mid \omega \in \mathfrak{z}'_{0,+}\}$ *is a defining collection of seminorms for* $\mathfrak{z}_0$*.*

Proof. Let $e_n \coloneqq (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ be the nth standard basis vector of $\mathfrak{z}_0$. Let $\rho \in \mathcal{N}_{\text{all}}(\mathfrak{z}_0)$. If $x \in \mathfrak{z}_0$, we can write $x = \sum_{n=0}^{N} x_n e_n$ for some $N \in \mathbb{N}_0$. But then

$$\rho(x) = \rho\left(\sum_{n=0}^{N} x_n e_n\right) \leq \sum_{n=0}^{N} |x_n| \rho(e_n) = \|x\|_\omega,$$

where $\omega = (\omega_n)_{n\geq 0}$ is defined by $\omega_n \coloneqq \rho(e_n)$. This shows that ρ is continuous with respect to $\{\|\cdot\|_\omega \mid \omega \in \mathfrak{z}'_{0,+}\}$. □

Remark 2.2.28. $\mathfrak{z}_0$ is Hausdorff but **not** metrizable.

Proof. That $\mathfrak{z}_0$ is Hausdorff is trivial. Suppose that $\mathfrak{z}_0$ is metrizable, so that there exists a countable subcollection

$$\mathcal{N} = \{\omega^{(m)} \mid m \in \mathbb{N}\} \subseteq \{\|\cdot\|_\omega \mid \omega \in \mathfrak{z}'_{0,+}\}$$

of seminorms which defines the topology on $\mathfrak{z}_0$. Define $\omega \in \mathfrak{z}'_{0,+}$ by

$$\omega_n = 2^n \left(1 + \max_{i,m\leq n} \omega_i^{(m)}\right)$$

We claim that $\|\cdot\|_\omega$ is not continuous relative to $\mathcal{N}$. If it were, then there would exist $c > 0$ and $m_1, \ldots, m_r \in \mathbb{N}$ such that $\|\cdot\|_\omega \leq c \sum_{i=1}^{r} \|\cdot\|_{\omega^{(m_i)}}$. Put $m \coloneqq \max\{m_1, \ldots, m_r\}$, and choose $N > m$ large enough so that

$$2^N > c \sum_{n=0}^{m} (\omega_n^{(m_1)} + \cdots + \omega_n^{(m_r)})$$

Then for $x \coloneqq (\mathbb{1}\{n \leq N\})_{n\geq 0} \in \mathfrak{z}_0$, it is clear that

$$\begin{aligned}
\|x\|_\omega &= \sum_{n=0}^{N} \omega_n \\
&\geq 2^N \\
&> c \sum_{n=0}^{N} (\omega_n^{(m_1)} + \cdots + \omega_n^{(m_r)}) \\
&= c \left(\|x\|_{\omega_{m_1}} + \cdots + \|x\|_{\omega_{m_r}}\right)
\end{aligned}$$

This is a contradiction. □

For $x \in \mathfrak{s}_0$, define $\operatorname{supp}(x) := \{n \in \mathbb{N}_0 \mid x_N \neq 0\}$. If we endow $\mathbb{N}_0$ with the discrete topology, then $\mathfrak{s}_0$ is precisely the space of compactly supported functions $x : \mathbb{N}_0 \to \mathbb{K}$.

Proposition 2.2.29. *A subset $A \subseteq \mathfrak{s}_0$ is bounded if and only if*

1. *there exists $N \in \mathbb{N}_0$ such that for all $x \in A$, supp$(x) \subseteq \{0, 1, \dots, N\}$, and*
2. *there exists $M > 0$ such that for all $x \in A$, $|x_n| \leq M$ for all $n \geq 0$.*

Proof. See Homework 6 from 7310. □

Proposition 2.2.30. *A sequence $(x^{(m)})_{m \geq 0}$ in $\mathfrak{s}_0$ converges to $x \in \mathfrak{s}_0$ if and only if*

1. *There exists $N \in \mathbb{N}_0$ such that for all $m \geq 0$, supp$(x^{(m)}) \subseteq \{0, 1, \dots, N\}$, and*
2. *For all $n \geq 0$, $\lim_{m \to \infty} x_n^{(m)} = x_n$.*

Proof. See Homework 6 from 7310. □

Proposition 2.2.31 (& Definition)**.** *For $n \geq 0$, define the seminorm ρ_n on $\mathfrak{s}_0'$ by $\rho_n(x) := |x_n|$, and let $\mathcal{T} := \mathcal{T}(\{\rho_n \mid n \geq 0\})$, making $\mathfrak{s}_0'$ a LCTVS. Then $\mathcal{T}$ is the product topology on $\mathfrak{s}_0' = \mathbb{K}^{\mathbb{N}_0}$.*

Proof. Let $\mathcal{T}'$ denote the product topology. Since the projections $x \mapsto x_n$ are certainly continuous with respect to $\mathcal{T}$, we have $\mathcal{T}' \subseteq \mathcal{T}$. Conversely, let $n \geq 0, \varepsilon > 0$, and $x \in \mathfrak{s}_0'$. Then

$$\begin{aligned} B_{n,\varepsilon}(x) &:= \{x' \in \mathfrak{s}_0' \mid \rho_n(x') < \varepsilon\} \\ &= \prod_{k=0}^{n-1} \mathbb{K} \ \times \ \{\lambda \in \mathbb{K} \mid |\lambda - x_n| < \varepsilon\} \ \times \prod_{k>n} \mathbb{K} \\ &\in \mathcal{T}' \end{aligned}$$

The collection $\{B_{n,\varepsilon}(x) \mid n \in \mathbb{N}_0, \varepsilon > 0, x \in \mathfrak{s}_0'\}$ forms a sub-basis for $\mathcal{T}$, so $\tau \subseteq \mathcal{T}'$. □

Theorem 2.2.32. *The map $\mathcal{J} : (\mathscr{S}_0')' \to \mathscr{S}_0$ given by $\mathcal{J}(R) := (R(e_n))_{n \geq 0}$ is a TVS isomorphism. As before, $\mathscr{S}_0'$ denotes the strong dual of $\mathscr{S}_0$, which is $\mathbb{K}^{\mathbb{N}_0}$ with the product topology.*

From this, we obtain an explicit duality given by $R(y) = \langle y, \mathcal{J}(R) \rangle$ for all $R \in (\mathscr{S}_0')'$ and $y \in \mathscr{S}_0'$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\ell^2(\mathbb{N}_0)$.

Proof. First, we show R maps into $\mathscr{S}_0$. Let $R \in (\mathscr{S}_0')'$. Since R is continuous, by our criterion for continuity of linear maps there exists $C > 0$, $N \geq 0$ such that for all $y = (y_n)_{n\geq 0} \in \mathscr{S}_0'$,

$$|R(y)| \leq C(|y_0| + \ldots + |y_N|).$$

It follows that $R(e_n) = 0$ for $n > N$, i.e. $\mathcal{J}(R)$ is a sequence with finite support. Hence, $\mathcal{J}(R) \in \mathscr{S}_0$ for all $R \in (\mathscr{S}_0')'$.

It follows immediately that $\mathcal{J}$ is linear. For instance, for $R, S \in \mathscr{S}_0'$, we have

$$\begin{aligned}\mathcal{J}(R + S) &= ((R + S)(e_n))_{n\geq 0} = (R(e_n) + S(e_n))_{n\geq 0} \\ &= (R(e_n))_{n\geq 0} + (S(e_n))_{n\geq 0} = \mathcal{J}(R) + \mathcal{J}(S).\end{aligned}$$

To show the explicit duality holds, let $y = (y_n)_{n\geq 0} \in \mathscr{S}_0'$ and $R \in (\mathscr{S}_0')'$. We observe that in $\mathscr{S}_0'$, the sequence of finitely supported sequences $(y_0, \ldots, y_n, 0, 0, 0, \ldots)$ converges to y in $\mathscr{S}_0'$ as $n \to \infty$, since $\mathscr{S}_0'$ is equipped with the product topology, i.e. the topology of pointwise convergence, and this sequence evidently converges pointwise to y. By continuity of R, then

$$\sum_{i=0}^{n} y_i R(e_i) = R\left((y_0, \ldots, y_n, 0, 0, 0, \ldots)\right) \to R(y) \text{ as } n \to \infty.$$

Hence, $R(y) = \sum_{n=0}^{\infty} y_n R(e_n) = \langle y, \mathcal{J}(R)\rangle$.

From this, it follows that $\mathcal{J}$ is injective. If $\mathcal{J}(R) = 0$ for some $R \in (\mathscr{S}_0')'$, then $R(y) = \langle y, \mathcal{J}(R)\rangle = 0$ for all $y \in \mathcal{S}_0$, and so $R = 0$.

Next, we show $\mathcal{J}$ is surjective. Let $x \in \mathscr{S}_0$. We define $R : (\mathscr{S}_0')' \to \mathbb{K}$ by $R(y) := \langle y, x\rangle = \sum_{n=0}^{\infty} y_n x_n$. Since x is finitely supported, this is a finite sum and so R is well-defined. The fact that R is linear is clear. And R is continuous; if $N \geq 0$ is such that $\operatorname{supp}(x) \subset \{0, \ldots, N\}$, then $R(y) = \sum_{n=0}^{N} y_n x_n$ for all $y = (y_n)_{n\geq 0} \in \mathscr{S}_0'$, and so

$$|R(y)| \leq \left(\max_{0\leq n\leq N} |x_n|\right) \cdot \sum_{n=0}^{N} |y_n|.$$

Since $\max_{0\leq n\leq N} |x_n|$ is a constant independent of y and the maps $y \mapsto |y_n|$ are continuous seminorms on $\mathscr{S}_0'$, then R is continuous by our continuity criteria. And for all $n \geq 0$, we have $R(e_n) = \langle e_n, x\rangle = x_n$, so $\mathcal{J}(R) = x$. Hence, $\mathcal{J}$ is surjective.

Next, we show $\mathcal{J}$ is continuous. Let $\omega \in \mathscr{S}_{0,+}'$ and $R \in (\mathscr{S}_0')'$. Then $\|\mathcal{J}(R)\|_\omega = \sum_{n=0}^{\infty} \omega_n |R(e_n)| = \sum_{n=0}^{N} \omega_n |R(e_n)|$ for some N depending on R, since $\mathcal{J}(R)$ has finite support. Now, for $0 \leq n \leq N$, let $\lambda_n \in \mathbb{K}$ be of unit magnitude such that $R(e_n) = \lambda_n |R(e_n)|$. Then

$$\|\mathcal{J}(R)\|_\omega = \sum_{n=0}^{N} \omega_n \lambda_n^{-1} R(e_n) = R\left(\sum_{n=0}^{N} \omega_n \lambda_n^{-1} e_n\right).$$

Now let $A := \{y \in \mathscr{S}_0' : |y_n| \leq \omega_n \ \forall n \geq 0\}$. By definition, A is bounded and we have $\sum_{n=0}^{N} \omega_n \lambda_n^{-1} e_n \in A$. Hence, $\|\mathcal{J}(R)\|_\omega \leq \sup_{y \in A} |R(y)| = \|R\|_A$. Hence, $\mathcal{J}$ is continuous, as $\|\cdot\|_A$ is a continuous seminorm on $(\mathscr{S}_0')'$.

Finally, we show that $\mathcal{J}^{-1}$ is continuous, completing the proof. Let A be a bounded set in $\mathscr{S}_0'$. Then for all n, set $\omega_n := \sup_{y \in A} |y_n|$, which is finite by definition of boundedness and the topology on $\mathscr{S}_0'$. Then $\omega := (\omega_n)_{n \geq 0} \in \mathscr{S}_{0,+}'$, and so

$$\begin{aligned} \|R\|_A = \sup_{y \in A} |R(y)| &= \sup_{y \in A} \left| \sum_{n=0}^{N} R(e_n) y_n \right| \\ &\leq \sup_{y \in A} \sum_{n=0}^{N} |R(e_n)| \cdot |y_n| \leq \sum_{n=0}^{N} |R(e_n)| \omega_n = \|\mathcal{J}(R)\|_\omega. \end{aligned}$$

Since $\|\cdot\|_\omega$ is a continuous seminorm on $\mathscr{S}_0$, this proves the claim. □

Corollary 2.2.33. *$\mathscr{S}_0$ and $\mathscr{S}_0'$ are (strongly) reflexive.*

The following concept was used in the preceding proof, and will be useful in the future.

Definition 2.2.34. If $A \subset \mathscr{S}_0' = \mathbb{K}^{\mathbb{N}_0}$, the *envelope* of A is

$$\mathrm{Env}(A) := \left(\sup_{y \in A} |y_n| \right)_{n \geq 0} \in [0, \infty]^{\mathbb{N}_0}.$$

If A is bounded, then $\mathrm{Env}(A) \in \mathscr{S}_{0,+}'$.

2.2.3 The local Schwartz-Bruhat Space.

We give a brief overview of the p-adics, which are constructed as follows. For $p \geq 2$ prime, the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ is defined by $|0|_p := 0$, and for $x \in \mathbb{Q} \setminus \{0\}$, $|x|_p := p^{-a}$, for the unique $a \in \mathbb{Z}$ such that $x = p^a \cdot \frac{r}{s}$, with $r \in \mathbb{Z}$, $s \in \mathbb{Z} \setminus \{0\}$ both relatively prime to p. Thus, integers with large factors of the fixed prime p have small p-adic absolute value and integers with small factors of p have large p-adic absolute value. The field of p-adics, denoted $\mathbb{Q}_p$, is the completion of the metric space $\mathbb{Q}$ with respect to the metric on $\mathbb{Q}$ induced by the absolute value $|\cdot|_p$.

Since $|x + y|_p \leq \max(|x|_p, |y|_p)$ for $x, y \in \mathbb{Q}_p$, it follows that $\mathbb{Q}_p$ is an *ultrametric space*, i.e. $|x - z|_p \leq \max(|x - y|_p, |y - z|_p)$ for $x, y, z \in \mathbb{Q}_p$, a condition stronger than the triangle inequality. To see this, it suffices to show that the above inequality holds for $x, y, z \in \mathbb{Q}$, and extend the result to $\mathbb{Q}_p$ by taking limits.

p-adics have nice decmial-like expansions. Each $x \in \mathbb{Q}_p$ can be written uniquely in the form $x = \sum_{n\in\mathbb{Z}} a_n p^n$ for $a_n \in \{0, 1, \ldots, p-1\}$, where $a_n = 0$ for all but finitely many $n < 0$. Conversely, any such sum is convergent in $\mathbb{Q}_p$. For nonzero x expressed in this form, we have $|x|_p = p^{-N}$, where $N \in \mathbb{Z}$ is minimal such that $a_N \neq 0$.

The closed unit ball of $\mathbb{Q}_p$ is denoted $\mathbb{Z}_p$, since these are the p-adics whose decimal expansions only range over integral powers of p. From the decimal expansion and distance formulas, it follows that $\mathbb{Z}_p$ is homeomorphic to the Cantor set, which is compact. Hence, $\mathbb{Q}_p$ is locally compact, since any point x in an open ball B will contain a scaled and shifted copy of the closed unit ball $\mathbb{Z}_p$, which will be homeomorphic to $\mathbb{Z}_p$ and therefore compact.

The following observations are useful, and show that the geometry of $\mathbb{Q}_p^d$ is quite different from the geometry of $\mathbb{R}^d$.

1. Any point in an open ball $B(x, p^r)$ is its center. If $y \in B(x, p^r)$, then $B(x, p^r) = B(y, p^r)$, for any $x, y \in \mathbb{Q}_p^d$ and $r \in \mathbb{Z}$.
2. Open balls are closed and vice versa.
3. The closed unit ball $\mathbb{Z}_p^d$ contains precisely p^{dr} open balls of radius p^r for $r \geq 0$.

There exists a Lebesgue measure m_p which is translation invariant and such that $m_p(\lambda A) = |\lambda|_p \cdot m_p(A)$ for $\lambda \in \mathbb{Q}_p$ and A measurable, which we normalize to be such $\int_{\mathbb{Z}_p} 1\, dx = 1$. One way to construct m_p is to use the existence of a translation-invariant Haar measure on the topological group $(\mathbb{Q}_p, +)$. To show that this measure's translation invariance implies its scaling property, we observe that $\mathbb{Z}_p$ contains p disjoint copies of $p\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq p^{-1}\}$, namely, $p\mathbb{Z}_p + r$ for $r \in \{0, \ldots, p-1\}$. This, $p \cdot m_p(p \cdot \mathbb{Z}_p) = m_p(\mathbb{Z}_p)$, and so $m_p(p \cdot \mathbb{Z}_p) = |p| \cdot m_p(\mathbb{Z}_p)$ by translation invariance. By extending this argument, it follows translation invariant is sufficient to guarantee scaling.

$\mathbb{Q}_p^d$ is a $\mathbb{Q}_p$-vector space with product measure $d^d x$. We use the max-norm on $\mathbb{Q}_p^d$ given by $|x|_p := \max_{1\leq i\leq d} |x_i|_p$ for $x = (x_1, \ldots, x_d) \in \mathbb{Q}_p^d$. With respect to the metric $(x, y) \mapsto |x - y|_p$, the space $\mathbb{Q}_p^d$ is locally compact since the closed unit ball $\mathbb{Z}_p^d$ is locally compact.

We are ready to define Schwarz-Bruhat space, an analogue of Schwarz-space over p-adics. Note that in the following definition, $\mathbb{K}$ still denotes “either $\mathbb{R}$ or $\mathbb{C}$," not $\mathbb{Q}_p$.

Definition 2.2.35. We say $f : \mathbb{Q}_p^d \to \mathbb{K}$ is *locally constant* if and onnly if for all $x \in \mathbb{Q}_p^d$, there exists $r \in \mathbb{Z}$ such that $f|_{\overline{B}(x,p^r)}$ is constant. For fixed $r \in \mathbb{Z}$, we say f is *locally constant at scale* p^r, or *uniformly locally constant,* if for all $x \in \mathbb{Q}_p^d$, the restriction $f|_{\overline{B}(x,p^r)}$ is constant.

Note that there there exist non-constant, locally constant functions $\mathbb{Q}_p \to \mathbb{K}$ since $\mathbb{Q}_p$ is disconnected. For instance, the function $\mathbb{1}_{\mathbb{Z}_p}$ is locally constant at scale p^0. Since $\mathbb{Q}_p$ is an ultrametric space, it follows that a closed ball of radius 1 about a point x in $\mathbb{Z}_p$ or $\mathbb{Q}_p \setminus \mathbb{Z}_p$ will be contained in $\mathbb{Z}_p$ or $\mathbb{Q}_p \setminus \mathbb{Z}_p$, respectively.

Moreover, every locally constant function f is locally constant at some scale. From the geometry of $\mathbb{Q}_p^d$, it follows that every locally constant function f, there exists $r \in \mathbb{Z}$ such that f is finite linear combination of indicator functions of disjoint translations of $p^r \mathbb{1}_{\mathbb{Z}_p^d}$. Since $\mathbb{Q}_p^d$ is an ultrametric space, it follows that f is locally constant at scale p^r.

Definition 2.2.36. The *Schwarz-Bruhat space* $S(\mathbb{Q}_p^d) = S(\mathbb{Q}_p^d, \mathbb{K})$ is the set of locally constant, compactly supported functions $\mathbb{Q}_p^d \to \mathbb{K}$. We equip $S(\mathbb{Q}_p^d)$ with the finest locally convex topology, i.e. the topology generated by all seminorms on $S(\mathbb{Q}_p^d)$.

This is analogous to Schwarz space, with local constancy taking the role of infinite differentiability, and compact support taking the role of fast decay at infinity.

For $r, s \in \mathbb{Z}$, $r \leq s$, we let $S_{r,s}(\mathbb{Q}_p^d)$ be the set of functions f in $S(\mathbb{Q}_p^d)$ such that f is locally constant at scale p^r and supported in $\overline{B}(0, p^s)$. We have $\dim_{\mathbb{K}} S_{r,s}(\mathbb{Q}_p^d) = p^{d(s-r)}$, since a basis for $S_{r,s}(\mathbb{Q}_p^d)$ is given by

$$\left\{ \mathbb{1}_{z+p^{-r}\mathbb{Z}_p^d} : z = (z_i) \in \mathbb{Q}_p^d \text{ s.t. } z_i = \sum_{n=r+1}^{s} a_n p^{-n} \text{ for } a_n \in \{0, \ldots, p-1\} \right\}.$$

The closed balls $z + p^{-r}\mathbb{Z}_p^d$ form a partition of $\overline{B}(0, p^s)$. By the ultrametric property of p-adics, this is the *unique* partition of $\overline{B}(0, p^s)$ into closed balls of radius p^{-r}, and so the set forms a basis. By definition of $S(\mathbb{Q}_p^d)$, we have the expression $S(\mathbb{Q}_p^d)$ as the increasing union $S(\mathbb{Q}_p^d) = \bigcup_{N \geq 0} S_{-N,N}(\mathbb{Q}_p^d)$.

Theorem 2.2.37. *$S(\mathbb{Q}_p^d)$ and $\mathscr{S}_0$ are TVS-isomorphic.*

Proof. First, we observe that $S(\mathbb{Q}_p^d)$ and $\mathscr{S}_0$ both have countable algebraic bases. We expressed $S(\mathbb{Q}_p^d)$ as the increasing union of finite-dimensional spaces $S_{-N,N}$ (of increasing dimension). Hence, $S(\mathbb{Q}_p^d)$ has a countably infinite algebraic basis. And $\mathscr{S}_0$ has the canonical basis e_k, $k \geq 0$. Hence, there is a linear isomorphism $T : S(\mathbb{Q}_p^d) \to \mathscr{S}_0$. Since $S(\mathbb{Q}_p^d)$ and $\mathscr{S}_0$ are both equipped with their finest locally convex topologies, all linear maps defined on these spaces are continuous. Hence, T and T^{-1} are continuous, and so T is a TVS-isomorphism. □

2.2.4 Duality for $\mathscr{S}$.

We introduce some notation and a definition.

Definition, Notation. 2.2.38. We define $\mathscr{S}'$ to be the space of sequences in $\mathscr{S}_0'$ which grow at most polynomially, or are "of temperate growth." More precisely,

$$\mathscr{S}' := \left\{ y \in \mathscr{S}_0' : \exists C > 0, K \in \mathbb{N}_0 \text{ s.t. } \forall n \geq 0 \ , \ |y_n| \leq C\langle n \rangle^k \right\}.$$

We set $\mathscr{S}_+ := \mathscr{S} \cap [0,\infty)^{\mathbb{N}_0}$ and $\mathscr{S}_+' := \mathscr{S}' \cap [0,\infty)^{\mathbb{N}_0}$.

Definition, Proposition. 2.2.39. For $\omega \in \mathscr{S}_+'$, $\|x\|_\omega := \sum_{n\geq 0} \omega_n |x_n|$ is a continuous seminorm on $\mathscr{S}$. Moreover, $\{\| \cdot \|_\omega : \omega \in \mathscr{S}_+'\}$ is a defining collection of seminorms for $\mathscr{S}$.

Proof. We make use of our criteria for continuity of seminorms. Let $C > 0$ be such that $\omega_n \leq C\langle n \rangle^k$ for all n. Then $\|x\|_\omega \leq C \sum_{n=0}^\infty \langle n \rangle^k |x_n| = C\|x\|_{1,k} < \infty$, and so each $\| \cdot \|_\omega$ is continuous on $\mathscr{S}$. Conversely, $\|x\|_{1,k} = \|x\|_\omega$, for $\omega \in \mathscr{S}_+'$ given by $\omega_n := \langle n \rangle^k$ for all n, and so the definition collection $\{\| \cdot \|_{1,k}\}$ of seminorms is continuous with respect to the topology induced by the seminorms $\| \cdot \|_\omega$ (see Proposition 3.2.14). Hence, the two topologies coincide. □

Theorem 2.2.40. *The map $\mathcal{J} : (\mathscr{S}')' \to \mathscr{S}$, $R \mapsto (R(e_n))_{n\geq 0}$, is a TVS isomorphism and one has the explicit duality: $\forall \in (\mathscr{S}')', \forall\, y \in \mathscr{S}', R(y) =< y, \mathcal{J}(R) >$ $(= \sum_{n=0\infty} y_n x_n$ with $\mathcal{J}(R) = x)$.*

Proof. Since R is continuous, $\exists\, c > 0, \exists\, \nu^{(1)}, ..., \nu^{(p)} \in \mathscr{S}_+, \forall\, y \in \mathscr{S}'$,

$|R(y)| \leq c(||y||_{\nu^{(1)}} + ... + ||y||_{\nu^{(p)}}) = ||y||_\nu$ with $\nu = c(\nu^{(1)} + ... + \nu^{(p)})$. Apply this to $y = e_n = (0, ..., 0, \underset{n}{1}, 0, ..., 0)$.

$|R(e_n)| \leq ||e_n||_\nu = \nu_n \implies (R(e_n))_{n\geq 0} \in \mathscr{S}$. Thus, $\mathcal{J}$ is well-defined. Note that linearity is clear. Now we discuss the explicit duality. Let $y \in \mathscr{S}', y \in \mathscr{S}_+$, $||y - \sum_{n=0}^{N} y_n e_n||_\nu = \sum_{n>N} |y_n| \nu_n \underset{N\to\infty}{\to} 0\ \forall\ \nu$. Thus, $\lim_{N\to\infty} \sum_{n=0}^{N} y_n e_n = y$ in $\mathscr{S}'$ topology.

$$\begin{aligned} R(y) &= \lim_{N\to\infty} R(\sum_{n=0}^{N} y_n e_n) \qquad \text{because } R \text{ is continuous} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} y_n R(e_n) \\ &= \sum_{n=0}^{\infty} y_n R(e_n) \\ &= < y, \mathcal{J}(R) > . \end{aligned}$$

Thus, if $\mathcal{J}(R) = 0$, then $R(y) \equiv 0 \implies R = 0$. Thus, $\mathcal{J}$ is one-one.

Let $x \in \mathscr{S}$; define $R : \mathscr{S}' \to \mathbb{K}$ by $y \mapsto < y, x >$. Let $\nu_n := |x_n| \implies \nu = (\nu_n)_{n\geq 0} \in \mathscr{S}_+$. We have $\sum_{n=0}^{\infty} |x_n y_n| = \sum_{n=0}^{\infty} \nu_n |y_n| = ||y||_\nu < \infty$. Thus, R is well-defined and continuous. $|R(y)| \leq ||y||_\nu$. Thus we have, $\forall\ n, R(e_n) =< e_n, x >= x_n \implies \mathcal{J}(R) = x$. So, surjectivity is proved.

Now we need to show continuity. Let $w \in \mathscr{S}'_+$. $||\mathcal{J}(R)||_w = \sum_{n=0}^{\infty} w_n |R(e_n)|$. $\forall\, n,\ \ \lambda_n \in \mathbb{K}, |\lambda_n| = 1$ such that $R(e_n) = \lambda_n |R(e_n)|$. $||\mathcal{J}(R)||_w = \lim_{N\to\infty} \sum_{n=0}^{N} w_n \lambda_n^{-1} R(e_n) = \lim_{N\to\infty} R(\sum_{n=0}^{N} w_n \lambda_n^{-1} e_n)$.

Let $A := \{y \in \mathscr{S}' \mid \forall\ n \geq 0, |y_n| \leq w_n\}$.

$||\mathcal{J}(R)||_w \leq \sup_{y\in A} |R(y)|$. If $\nu \in \mathscr{S}_+, y \in A$ then,

$||y||_\nu = \sum_{n=0}^{\infty} \nu_n |y_n| \leq \sum_{n=0}^{\infty} \nu_n w_n = ||w||_\nu = ||\nu||_w < \infty$. Thus, for all $\nu, \sup_{y\in A} ||y||_\nu < \infty \implies A$ is bounded. Hence, $\mathcal{J}$ is continuous.

Let A be bounded set in $\mathscr{S}'$; $R \in (\mathscr{S}')'$, so,

$$\begin{aligned} ||R||_A &= \sup_{y\in A} |R(y)| \\ &= \sup_{y\in A} | < y, \mathcal{J}(R) > | \\ &\leq \sup_{y\in A} \sum_{n=0}^{\infty} |y_n||R(e_n)| \\ &\leq \sup_{y\in A} Env(A)_n |R(e_n)| \\ &= ||\mathcal{J}(R)||_w, \quad \text{where } w_n = Env(A)_n \end{aligned}$$

$\implies \exists\ w \in \mathscr{S}'_+,\ \forall\ R, ||R||_A \leq ||J(R)||_w$. □

Corollary 2.2.41. *$\mathscr{S}$‘and $\mathscr{S}'$ are reflexive.*

Corollary 2.2.42. *Via Sequential Representation Theorem, we have $\forall d \geq 1, S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ are reflexive.*

2.2.5 Spaces of Infinite Matrices

Note from now on, we will denote $\mathscr{S}$ as $\mathfrak{s}$.

For $x, y \in \mathbb{K}^{\mathbb{N}_0^2}$, let $< x, y >:= \sum_{(m,n)\in\mathbb{N}_0^2} x_{m,n}y_{m,n}$ if the sum converges absolutely.

$\mathfrak{s}_0\hat{\otimes}\mathfrak{s}$:= space of $x = (x_{m,n})_{0\leq m,n<\infty}$ such that $\forall\ m \geq 0$, row $(x_{m,n})_{n\geq 0} \in \mathfrak{s}$ and identically zero except for finitely many m's.

As a vector space, $\mathfrak{s}_0\hat{\otimes}\mathfrak{s} \simeq \oplus_{m\geq 0}\ \mathfrak{s}$.

$\left(\mathfrak{s}_0\hat{\otimes}\mathfrak{s}\right)_+$:= matrices in $\mathfrak{s}_0\hat{\otimes}\mathfrak{s}$ with non-negative entries.

$\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}'$: space of matrices $y = (y_{m,n,})_{0\leq m,n<\infty} \in \mathbb{K}^{\mathbb{N}_0^2}$ such that $\forall\ m \geq 0$, row $(y_{m,n})_{n\geq 0} \in \mathfrak{s}'$.

$\left(\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}'\right)_+$ = matrices in $\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}'$ with non-negative entries.

$\forall\ x, y \in \mathbb{K}^{\mathbb{N}_0^2}, \nu \in (\mathfrak{s}_0\hat{\otimes}\mathfrak{s})_+, w \in (\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}')_+$,

$$||x||_w = \sum_{(m,n)\in\mathbb{N}_0^2} w_{m,n}|x_{m,n}|,\ ||y||_\nu = \sum_{(m,n)\in\mathbb{K}^{\mathbb{N}_0^2}} \nu_{m,n}|y_{m,n}|.$$

Note that both belong to $[0, \infty]$.

Proposition 2.2.43. *1.* $\mathfrak{s}_0\hat{\otimes}\mathfrak{s} = \{x \in \mathbb{K}^{\mathbb{N}_0^2}\ |\forall w \in (\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}')_+, ||x||_w < \infty\}$.

2. $\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}' = \{y \in \mathbb{K}^{\mathbb{N}_0^2}\ |\ \forall\ \nu \in (\mathfrak{s}_0\hat{\otimes}\mathfrak{s})_+, ||y||_v u < \infty\}$.

3. $||\cdot||_w$ *are semi-norms on* $\mathfrak{s}_0\hat{\otimes}\mathfrak{s}$.

4. $||\cdot||_\nu$ *are semi-norms on* $\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}'$.

Proof. We skip the routine checks of parts 3 and 4. We will prove 1 and 2 can be proved using similarly.

Note that $\mathfrak{s}_0\hat{\otimes}\mathfrak{s} \subset \{x \in \mathbb{K}^{\mathbb{N}_0^2}\ |\forall w \in (\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}')_+, ||x||_w < \infty\}$ clearly.

Now choose an x from RHS and fix an $m \geq 0$. Since the sum exists, we have that for fixed m, $\sum_{n\geq 0} w_{m,n}|x_{m,n}| < \infty$ for all $w \in (\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}')_+$, which shows that $(x_{m,n})_{n\geq 0} \in \mathfrak{s}$. If for infinitely many m, rows $(x_{m,n})_{n\geq 0}$ are non-zero, then choosing a $w \in (\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}')_+$ by inverting absolute value of first non-zero element in rows of x and putting zeros elsewhere in the rows, gives us an infinite sum of 1's which is a contradiction. □

Topology on $\mathfrak{s}_0\hat{\otimes}\mathfrak{s} := \tau\Big(\{||\cdot||_w, w \in (\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}')_+\}\Big)$.

Topology on $\mathfrak{s}_0'\hat{\otimes}\mathfrak{s}' := \tau\Big(\{||\cdot||_\nu, \nu \in (\mathfrak{s}_0\hat{\otimes}\mathfrak{s})_+\}\Big)$.

$(e_{m,n})_{(m,n)\in\mathbb{N}_0^2}$ form the canonical basis, where $e_{m,n}$ is the infinite matrix with 1 at $(m, n)^{th}$ position and 0 elsewhere.

Theorem 2.2.44. *The map* $\mathscr{J} : (\mathfrak{s}_0 \hat{\otimes} \mathfrak{s})' \to \mathfrak{s}_0' \hat{\otimes} \mathfrak{s}', L \mapsto \Big(L(e_{m,n})\Big)_{0 \le m,n < \infty}$ *is a TVS isomorphism. ($\forall\ L, \forall\ x, L(x) = < \mathscr{J}(L), x >$).*

Proof. If $L \in (\mathfrak{s}_0 \hat{\otimes} \mathfrak{s})'$, due to continuity, $\exists C > 0, \exists w^{(1)}, ..., w^{(p)} \in (\mathfrak{s}_0' \hat{\otimes} \mathfrak{s}')_+$ ($w^{(i)} = (w^{(i)}_{m,n})_{0 \le m,n < \infty}$), such that $\forall\ x \in \mathfrak{s}_0 \hat{\otimes} \mathfrak{s}$,

$|L(x)| \le C(\sum_{j=1}^{p} ||x||_{w^{(j)}} \le ||x||_w$ where $w = C(w^{(1)} + ... + w^{(p)})$.

Thus, $\forall\ (m,n)$, $|L(e_{m,n})| \le ||e_{m,n}||_w = w_{m,n}$. Hence, $(L(e_{m,n}))_{0 \le m,n < \infty} \in \mathfrak{s}_0' \hat{\otimes} \mathfrak{s}'$ showing that $\mathscr{J}$ is well-defined.

Linearity is clear.

Explicit Duality: If $x \in \mathfrak{s}_0 \hat{\otimes} \mathfrak{s}, w \in (\mathfrak{s}_0' \hat{\otimes} \mathfrak{s}')_+$, then $\forall\ w$,

$$||x - \sum_{0 \le m \le M, 0 \le n \le N} x_{m,n} e_{m,n}||_w = \sum_{m \ge M, n \ge N} w_{m,n} |x_{m,n}| \underset{M,N \to \infty}{\longrightarrow} 0$$

because $||x||_w < \infty$.

Since, L is continuous, we have

$$\begin{aligned} L(x) &= \lim_{M \to \infty, N \to \infty} L(\sum_{0 \le m \le M, 0 \le n \le N} x_{m,n} e_{m,n}) \\ &= \lim_{M \to \infty, N \to \infty} \sum_{0 \le m \le M, 0 \le n \le N} x_{m,n} L(e_{m,n}) \\ &= \sum_{0 \le m,n < \infty} x_{m,n} L(e_{m,n}) \\ &= < \mathscr{J}(L), x > \end{aligned}$$

Injectivity: $\mathscr{J} = 0 \implies \forall\ x \in \mathfrak{s}_0 \hat{\otimes} \mathfrak{s}, < \mathscr{J}(L), x >= 0 \implies L(x) = 0 \implies L = 0$.

Surjectivity: If $y \in \mathfrak{s}_0' \hat{\otimes} \mathfrak{s}'$, Let $w_{m,n} := |y_{m,n}| \implies w - (w_{m,n})_{(m,n) \in \mathbb{N}_0^2} \in (\mathfrak{s}_0' \hat{\otimes} \mathfrak{s}')_+$. Now for $x \in \mathfrak{s}_0 \hat{\otimes} \mathfrak{s}$, let a linear form L be defined as follows:

$L(x) := < y, x >= \sum_{(m,n) \in \mathbb{N}_0^2} y_{m,n} x_{m,n}$. Hence,

$$\begin{aligned} |L(x)| &= |\sum_{(m,n) \in \mathbb{N}_0^2} y_{m,n} x_{m,n}| \\ &\le \sum_{(m,n) \in \mathbb{N}_0^2} |y_{m,n}||x_{m,n}| \\ &= \sum_{(m,n) \in \mathbb{N}_0^2} w_{m,n} |x_{m,n}| \\ &= ||x||_w \\ &< \infty \end{aligned}$$

Note that $|L(x)| \leq ||x||_w \implies L$ is continuous linear form. Also, $\forall\ (m,n)$, $L(e_{m,n}) =< y, e_{m,n} >= y_{m,n}$. $\mathcal{J}(L) = (L(e_{m,n}))_{0 \leq m,n < \infty} = (y_{m,n})_{0 \leq m,n < \infty} = y \implies \mathcal{J}$ is surjective.

$\mathcal{J}$ is continuous: Let $\nu \in (\mathfrak{s}_0 \hat{\otimes} \mathfrak{s})_+$, $L \in (\mathfrak{s}_0 \hat{\otimes} \mathfrak{s})'$,

$||\mathcal{J}(L)||_\nu = \sum_{(m,n) \in \mathbb{N}_0^2} \nu_{m,n} |L(e_{m,n})|$.

$\forall\ (m,n), \exists \lambda_{m,n} \in \mathbb{K} \setminus \{0\}, |\lambda_{m,n}| = 1$, such that $L(e_{m,n}) = \lambda_{m,n} |L(e_{m,n})|$.

$$\begin{aligned} ||\mathcal{J}(L)||_\nu &= \lim_{M \to \infty, N \to \infty} \sum_{0 \leq m \leq M, 0 \leq n \leq N} {}_{m,n} \lambda_{m,n}^{-1} L(e_{m,n}) \\ &= \lim_{M \to \infty, N \to \infty} L\Big(\sum_{0 \leq m \leq M, 0 \leq n \leq N} {}_{m,n} \lambda_{m,n}^{-1} e_{m,n} \Big) \end{aligned}$$

Let $A := \{ x \in \mathfrak{s}_0 \hat{\otimes} \mathfrak{s} \mid \forall (m,n), |x_{m,n}| \leq \nu_{m,n} \}$.

Note that $0 \leq Env(A) \leq \nu \implies Env(A) \in (\mathfrak{s}_0 \hat{\otimes} \mathfrak{s})_+ \implies A$ is bounded in $\mathfrak{s}_0' \hat{\otimes} \mathfrak{s}$.

$||\mathcal{J}(L)||_\nu \leq \sup_{x \in A} |L(x)| = ||L||_A < \infty$. This shows $\mathcal{J}$ is continuous.

$\mathcal{J}^{-1}$ is continuous: Let A be a bounded set in $\mathfrak{s}_0 \hat{\otimes} \mathfrak{s}, L \in (\mathfrak{s}_0 \hat{\otimes} \mathfrak{s})'$.

$$\begin{aligned} ||L||_A &= \sup_{x \in A} |L(x)| \\ &= \sup_{x \in A} |< \mathcal{J}(L), x >| \\ &\leq \sup_{x \in A} \sum_{(m,n) \in \mathbb{N}_0^2} |L(e_{m,n})||x_{m,n}| \\ &\leq \sum_{(m,n) \in \mathbb{N}_0^2} |L(e_{m,n}| \nu_{m,n} \qquad \text{where } \nu = Env(A) \\ &= ||\mathcal{J}(L)||_\nu \end{aligned}$$

which implies the required continuity. □

Proceeding in the exact same manner, we get the following result:

Theorem 2.2.45. *The map* $\mathcal{J} : (\mathfrak{s}_0' \hat{\otimes} \mathfrak{s}')' \to \mathfrak{s}_0' \hat{\otimes} \mathfrak{s}', R \mapsto (R(e_{m,n}))_{0 \leq m,n < \infty}$ *is a TVS isomorphism.* ($\forall R, \forall y, R(y) =< y, \mathcal{J}(R) >$).

2.2.6 Adelic Schwartz-Bruhat space:

We look at the adeles $\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod'_{p \text{ prime}} \mathbb{Q}_p$. Its elements are of the form $x = (x_\infty, x_2, x_3, x_5, ...)$ such that $x_p \in \mathbb{Z}_p$ for all but finitely many p's.

$f : \mathbb{A}_\mathbb{Q} \to \mathbb{K}$ is called an elementary function if it has the form

$$f(x) = f_\infty(x_\infty) f_2(x_2) f_3(x_3) ...$$

such that

- $f_\infty \in S(\mathbb{R})$
- $\forall p, f_p \in S(\mathbb{Q}_p)$
- $f_p(x) = \mathbb{1}\{x \in \mathbb{Z}_p\}$ except for finitely many p's

Let $S(\mathbb{A}_\mathbb{Q} :=$ finite $\mathbb{K}$-linear combinations of elementary functions.

We quote a result without proof which connects this space to the space in the previous section.

Theorem 2.2.46.

$$S(\mathbb{A}_\mathbb{Q}) \simeq \mathfrak{s}_0 \hat{\otimes} \mathfrak{s}$$

as vector spaces.

2.2.7 The space $\mathscr{D}$ of test function

Let U be a nonempty open set in $\mathbb{R}^d$ $(d \geq 1)$. Define

$$\mathscr{D}(U, \mathbb{K}) := \{f : U \to \mathbb{K} \mid f \in C^\infty \text{ \& } \exists \text{ compact } K \subset U \text{ s.t. } f|_{U \setminus K} \equiv 0\}$$

In other words, if

$$C^\infty(U, K) := \{f \in \mathscr{D}(U) \mid \mathrm{supp}(f) \subset K\}$$

then

$$C^\infty(U) = \bigcup_{\text{compact } K \subset U} C^\infty(U, K).$$

For fixed $k, \alpha \in \mathbb{N}_0^d$ and $f \in \mathscr{D}(U)$, define the norm

$$\|f\|_{K,\alpha} := \sup_{x \in K} |\partial^\alpha f(x)| < \infty$$

The topology on $C^\infty(U, K)$ is the LCTVS structure given by

$$\tau(\{\| \cdot \|_{K,\alpha} \mid \alpha \in \mathbb{N}_0^d\}).$$

Clearly, $C^\infty(U, K)$ is metrizable and $C^\infty(U, \emptyset) = \{0\}$.

Definition 2.2.47. A seminorm ρ on $\mathscr{D}(U)$ is called admissible if $\rho|_{C^\infty(U,K)}$ is continuous with respect to $\tau(\{\| \cdot \|_{K,\alpha}\})$ for any compact $K \subset U$. The standard LCTVS structure on $\mathscr{D}(U)$ is given by

$$\tau(\{\rho \in \mathcal{N}_{all}(\mathscr{D}(U)) \mid \rho \text{ is admissible}\})$$

Theorem 2.2.48 (Valdivia-Vogt 1978)**.**

$$\mathscr{D}(U) \underset{TVS}{\cong} \mathfrak{s}_0 \hat{\otimes} \mathfrak{s}$$

Proof. See the article by Bargetz 2015 (Project 6). □

Definition 2.2.49. Let V be a LCTVS. A Schauder basis in V is a sequence $\{e_n\}_{n\geq 0}$ such that for any $x \in V$ there exists a unique $\{x_n\}_{n\geq 0} \in \mathbb{K}^{\mathbb{N}_0}$ for which

$$\lim_{N\to\infty} \sum_{n=0}^{N} x_n e_n = x$$

and such that for all n the map given by $x \mapsto x_n$ is in V' (the dual of V). For example, $\{e_n = (0, \ldots, 0, 1, 0, \ldots)\}_{n\geq 1}$ is a Schauder basis in $\ell^2(\mathbb{N})$.

Grothendieck's Conjecture:

$$\mathfrak{s} \otimes \mathfrak{s}' \cong \{\text{smooth functions growing at most polynomially}\}$$

was proved by Valdivia in 1980s.

For $x \in \mathbb{K}^{\mathbb{N}_0^2}$ and the norm $\|\cdot\|_\omega = \sum \omega_{m,n}|x_{m,n}|$ where $\alpha_m \geq 0$ and $\beta_n \geq 0$ such that $\omega_{m,n} \leq \alpha_m \beta_n$ for some $\alpha \in \mathfrak{s}'_+$ and $\beta \in \mathfrak{s}_+$.

Horwáth Seminorm: Suppose $\{\theta_\alpha : U \to [0, \infty)\}_{\alpha\in\mathbb{N}_0^d}$ be a family of continuous functions. It is called locally finite family if for any $x \in U$ there exists open subset $x \in V \subset U$ such that

$$\{\alpha \in \mathbb{N}_0^d \mid \exists y \in V,\ \theta_\alpha(y) \neq 0\}$$

is finite.

Theorem 2.2.50. *Let Θ be the set of all locally finite family $\theta = \{\theta_\alpha\}_{\alpha\in\mathbb{N}_0^d}$. For $\theta \in \Theta$ and $f \in \mathscr{D}(U)$, let*

$$\|f\|_\theta := \sup_{\alpha\in\mathbb{N}_0^d} \sup_{x\in U} \theta_\alpha(x)|\partial^\alpha f(x)|$$

Then, $\|\cdot\|_\theta$ is a continuous seminorm on $\mathscr{D}(U)$, called a Horwáth seminorm.

Proof. Well-defined: For any $f \in \mathscr{D}(U)$ there exists nonempty compact subset $K \subset U$ such that $\operatorname{supp}(f) \subset K$. For any open subset $V \subset U$ let

$$F_V := \{\alpha \in \mathbb{N}_0^d \mid \theta_\alpha \not\equiv 0 \text{ on } V\} \qquad \text{and} \qquad G = \left\{V \underset{open}{\subset} U \mid F_V \text{ is finite}\right\}$$

Then θ is locally finite implies $U = \bigcup_{V\in G} V$. So K has a finite cover $\{V_1, \ldots, V_m\} \subset G$. Let $F := \cup_{i=1}^m F_{V_i}$ which is finite, then

$$\|f\|_\theta = \sup_{\alpha\in\mathbb{N}_0^d} \sup_{x\in K} \theta_\alpha(x)|\partial^\alpha f(x)| = \max_{\alpha\in F} \sup_{x\in K} \theta_\alpha(x)|\partial^\alpha f(x)| < \infty$$

Continuous: It's equivalent to show $\|\cdot\|_\theta \leq c(\rho_1 + \ldots + \rho_n)$ for some $c > 0$ and admissible seminorms $\rho_1, \ldots, \rho_n$. In fact we will see that $\rho = \|\cdot\|_\theta$ is admissible. For any $f \in C^\infty(U, K)$ choose F as above, then

$$\|f\|_\theta \leq \left(\max_{\alpha \in F} \sup_{x \in K} \theta_\alpha(x)\right) \max_{\alpha \in F} \|f\|_{K,\alpha} \leq C \sum_{\alpha \in F} \|f\|_{K,\alpha}$$

where C only depends on θ and K. So $\|\cdot\|_\theta\big|_{C^\infty(U,K)}$ is continuous with respect to $\tau(\{\|\cdot\|_{K,\alpha}\})$ for any compact $K \subset U$, i.e. $\|\cdot\|_\theta$ is admissible. $\square$

Theorem 2.2.51. *The Horwáth seminorms $\{\|\cdot\|_\theta : \theta \in \Theta\}$ form a defining collection for $\mathcal{D}(U)$.*

Proof. We have already shown the topology generated by the Horwáth seminorms is contained in the topology of admissible seminorms. It remains to show containment in the other direction holds as well.

Fix an admissible seminorm ρ on $\mathcal{D}(U)$. It suffices to show there exists locally finite family θ such that $\rho \leq \|\cdot\|_\theta$. For U open, there exists an exhausting sequence of compacts in U

$$\emptyset \neq K_1 \subset \mathring{K}_2 \subset \mathring{K}_3 \subset \cdots$$

such that $U = \cup_{N \geq 1} K_N = \cup_{N \geq 1} \mathring{K}_N$. For notation purposes, we'll set $K_0 = \emptyset$ (and $\mathring{K}_0 = \emptyset$).

By the Smooth Urysohn lemma, for each $N \geq 1$, there exists a C^∞ function $g_N : \mathbb{R}^d \to \mathbb{R}$ such that $g_N \equiv 1$ when restricted to K_N and $\operatorname{Supp} g_N \subset \mathring{K}_{N+1}$ with values in $[0,1]$. This implies the increasing sequence of functions

$$0 \leq g_1 \leq g_2 \leq g_3 \leq \cdots \leq 1.$$

For notation purposes, set $g_0 \equiv 0$ and $g_{-1} \equiv 0$.

Observe that $K_{N+1} \backslash \mathring{K}_{N-1}$ is compact and in U for all $N \geq 1$. It follows that ρ admissible implies ρ restricted to $C^\infty(U, K_{N+1} \backslash \mathring{K}_{N-1})$ is a continuous seminorm, i.e., for each $N \geq 1$, there exists $C_N > 0$, $L_N \in \mathbb{N}_0$ such that for all $f \in C^\infty(U, K_{N+1} \backslash \mathring{K}_{N-1})$,

$$\rho(f) \leq C_N \sum_{\alpha, |\alpha| \leq L_N} \|f\|_{K_{N+1} \backslash \mathring{K}_{N-1}, \alpha}.$$

Fix $f \in \mathcal{D}(U)$. Since $\cup_{M \geq 1} \mathring{K}_M$ is an open covering of U, and the support of f is compact in U, there exists $M \geq 1$ such that $\operatorname{Supp} f \subset \mathring{K}_M \subset K_M$. Since g_M is identically 1 on K_M, it follows that $f = f \cdot g_M$ on U. In addition, we can write f as the telescopic sum

$$f = f \cdot \sum_{N=1}^{M} h_N \qquad \text{where} \qquad h_N := g_N - g_{N-1}.$$

Observe that if $x \in K_{N+2} \backslash \mathring{K}_{N+1}$, then $g_{N-1}(x) = g_N(x) = 0$ so $h_N(x) = 0$. In addition, if $x \in \mathring{K}_{N-1}$, $g_N(x) = g_{N-1}(x) = 1$ so $h_N(x) = 0$. Hence the support of h_N is contained in $K_{N+1} \backslash \mathring{K}_{N-1}$. Using that ρ is a seminorm,

$$\rho(f) \leq \sum_{N=1}^{M} \rho(f h_N)$$

and so we can write

$$\begin{aligned}
\rho(f) &\leq \sum_{N=1}^{M} C_N \sum \|f h_N\|_{K_{N+1} \backslash \mathring{K}_{N-1}, \alpha} \\
&= \sum_{N=1}^{M} C_N \sum_{\alpha, |\alpha| \leq L_N} \sup_{x \in K_{N+1} \backslash \mathring{K}_{N-1}} |\partial^\alpha f h_N(x)| \\
&\leq \sum_{N=1}^{M} \binom{L_N + d}{d} C_N \sup_{\alpha, |\alpha| \leq L_N} \sup_{x \in K_{N+1} \backslash \mathring{K}_{N-1}} |\partial^\alpha f h_N(x)|
\end{aligned}$$

where $L_N + d$ choose d is equal to $\sum_{|\alpha| \leq L_N} 1$. By Leibnitz Rule,

$$\partial^\alpha (f h_N) = \sum_{\beta, \gamma \in \mathbb{N}_0^d} \mathbb{1}\{\beta + \gamma = \alpha\} \, \frac{\alpha!}{\beta! \, \gamma!} \, \partial^\beta (f) \cdot \partial^\gamma (h_N)$$

but note that

$$\begin{aligned}
\sum_{\beta, \gamma \in \mathbb{N}_0^d} \mathbb{1}\{\beta + \gamma = \alpha\} \, \frac{\alpha!}{\beta! \, \gamma!} &= \prod_{i=1}^{d} \left(\sum_{\beta_i, \gamma_i \in \mathbb{N}_0^d} \mathbb{1}\{\beta_i + \gamma_i = \alpha_i\} \binom{\alpha_i}{\beta_i} \right) \\
&= \prod_{i=1}^{d} 2^{\alpha_i} = 2^{|\alpha|}.
\end{aligned}$$

Therefore we have the bound

$$\begin{aligned}
\rho(f) \leq \sum_{N=1}^{M} & \binom{L_N + d}{d} C_N 2^{L_N} \times \\
& \times \left(\sup_{|\beta| \leq L_N} \sup_{K_{N+1} \backslash \mathring{K}_{N-1}} |\partial^\beta f(x)| \right) \left(\sup_{|\gamma| \leq L_N} \sup_{K_{N+1} \backslash \mathring{K}_{N-1}} |\partial^\gamma h_N(x)| \right)
\end{aligned}$$

i.e.,

$$\rho(f) \leq \sum_{N=1}^{M} 2^{-N} B_N \times \sup_{|\beta| \leq L_N} \sup_{x \in K_{N+1} \backslash \mathring{K}_{N-1}} |\partial^\beta f(x)|$$

where

$$B_N \coloneqq 2^{N+L_N} C_N \binom{L_N + d}{d} \times \sup_{|\gamma| \le L_N} \sup_{x \in K_{N+1} \setminus \mathring{K}_{N-1}} |\partial^\gamma h_N(x)|$$

(notice B_N doesn't depend on f). This shows that for any $f \in \mathscr{D}(U)$,

$$\rho(f) \le \sup_{N \ge 1} \sup_{|\beta| \le L_N} \sup_{x \in K_{N+1} \setminus \mathring{K}_{N-1}} B_N \, |\partial^\beta f(x)|.$$

For any $\alpha \in \mathbb{N}_0^d$, $x \in U$, let

$$\theta_\alpha(x) \coloneqq \sum_{N \ge 1} B_N \mathbb{1}\{|\alpha| \le L_N\} \times (g_{N+1}(x) - g_{N-2}(x)).$$

We need to show that

1. θ_α is well-defined
2. θ is a locally finite collection of continuous functions on U
3. $\rho \le \| \cdot \|_\theta$

Well-Defined. Recall $U = \cup_{M \ge 1} \mathring{K}_M$. If $x \in \mathring{K}_M \subset K_M$, then for any $N \ge M + 2$, $g_{N-2}(x) = 1 = g_{N+1}(x)$ and so their difference is zero. This means over $\mathring{K}_M$, θ_α is a finite sum of N

$$\theta_\alpha = \sum_{N=1}^{M+1} B_N \mathbb{1}\{|\alpha| \le L_N\}(g_{N+1} - g_{N-2})$$

so it's well-defined. Furthermore, by construction of the g_N's, θ_α is continuous (even C^∞) on U.

Locally Finite Collection. Fix $\alpha \in \mathbb{N}_0^d$, and $x \in \mathring{K}_M$ such that $\theta_\alpha(x) \neq 0$. It follows that there exists $N \in \{1, ..., M+1\}$ such that the set

$$\bigcup_{N=1}^{M+1} \{\beta \in \mathbb{N}_0^d \, : \, |\beta| \le L_N\}$$

is nonempty (contains α) and is finite. So $U = \cup \mathring{K}_M$, each of which is open, and on each $\mathring{K}_M$, θ_α is not identically zero for at most finitely many α's. Hence, θ must be a locally finite collection.

Seminorm bound. Observe that for all $N \ge 1$, if $x \in K_{N+1} \setminus \mathring{K}_{N-1}$, then $g_{N+1}(x) = 0$ and $g_{N-2}(x) = 0$ by support considerations of g_{N-2}. This implies that if $N \ge 1$, $|\alpha| \le L_N$, and $x \in K_{N+1} \setminus \mathring{K}_{N-1}$, then we have $\theta_\alpha(x) \ge B_N$. Therefore

$$\begin{aligned}\rho(f) &\le \sup_{N \ge 1} \sup_{|\alpha| \le L_N} \sup_{x \in K_{N+1} \setminus \mathring{K}_{N-1}} B_N \, |\partial^\alpha f(x)| \\ &\le \sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in U} \theta_\alpha(x) \, \partial^\alpha f(x)|\end{aligned}$$

which is identically $\|f\|_\theta$.

We have found a Horwáth seminorm which controls our given admissible seminorm. Hence we can conclude that the two topologies are equivalent for $\mathscr{D}(U)$. This completes the proof. □

We then get the following corollary.

Corollary 2.2.52. *The pointwise product map on* $\mathscr{D}(U)$

$$\begin{cases} \mathscr{D}(U) \times \mathscr{D}(U) \to \mathscr{D}(U) \\ (f,g) \mapsto fg \end{cases}$$

is a continuous map.

Proposition 2.2.53. *If* $\theta^{(1)}, ..., \theta^{(k)}$ *are locally finite families on* U *and* $c > 0$*, then there exists* θ *such that*

$$c\left(\sum_{i=1}^{k} \|\cdot\|_{\theta^{(i)}}\right) \leq \|\cdot\|_\theta.$$

Proof. For all $\alpha \in \mathbb{N}_0^d$, $\theta_\alpha \coloneqq ck \max_{1\leq i\leq k} \theta^{(i)}$ is continuous on U. To see that it is locally finite, note that if $x \in U$, then for all i there exists a V_i open, such that $x \in V_i \subseteq U$ and only finitely many $\theta_\alpha^{(i)}$ are not identically 0 on V_i. Let $V = \cap_i V_i$ and $F_i = \{\alpha : \ \theta_\alpha^{(i)}|_{V_i} \neq 0\}$. Then each F_i is finite, and so is $F = \bigcup_{i=1}^k F_i$. For $\alpha \notin F$, and for all $1 \leq i \leq k$, $\theta_\alpha^{(i)}|_V = 0$. Hence $\theta_\alpha = ck \max_{1\leq i\leq k} |\theta_\alpha^{(i)} \equiv 0$ on V, and so (θ_α) is locally finite.

Now, we check that the desired inequality holds. Let $f \in \mathscr{D}(U)$. Then

$$\begin{aligned} c\sum_{i=1}^{k} \|f\|_{\theta^{(i)}} &= c\sum_{i=1}^{k} \sup_{\alpha\in\mathbb{N}_0^d} \sup_{x\in U} \theta_\alpha^{(i)}(x)|\partial^\alpha f(x)| \\ &\leq c\sum_{i=1}^{k} \sup_\alpha \sup_x \sup_{1\leq i\leq k} \theta_\alpha^{(i)}(x)|\partial^\alpha f(x)| \\ &\leq \sup_\alpha \sup_x ck \sup_{1\leq i\leq k} \theta_\alpha^{(i)}(x)|\partial^\alpha f(x)| \\ &= \sup_\alpha \sup_x \theta_\alpha(x)|\partial^\alpha f(x)| \\ &= \|f\|_\infty. \end{aligned}$$

□

Corollary 2.2.54. *Let* ρ *be a seminorm on* $\mathscr{D}(U)$*. Then* ρ *is continuous iff* $\exists\ \theta \in \Theta$ *such that* $\rho \leq \|\cdot\|_\theta$*.*

The following theorem shows that pointwise multiplication of test functions on $\mathscr{D}$ is continuous.

Theorem 2.2.55. *The map* $\mathscr{D}(U)\times\mathscr{D}(U)\to\mathscr{D}(U)$ *sending* $(f,g)\mapsto fg$ *is continuous.*

Proof. We want to show that for all continuous seminorms ρ on $\mathscr{D}(U)$ there exists τ_1,τ_2 continous seminorms on $\mathscr{D}(U)$ such that for all $(f,g)\in\mathscr{D}(U)\times\mathscr{D}(U)$,

$$\rho(fg)\le\tau_1(f)\tau_2(g).$$

By Corollary 2.2.54, there exists a θ such that $\rho\le\|\cdot\|_\theta$. We will show that there exists a θ' such that

$$\|fg\|_\theta\le\|f\|_{\theta'}\|g\|_{\theta'},$$

and the result will follow.

If $\alpha\in\mathbb{N}_0^d, x\in U$, then by Leibniz's Rule

$$\partial^\alpha(fg)(x)=\sum_{\beta,\gamma\in\mathbb{N}_0^d}\mathbb{1}\{\beta+\gamma=\alpha\}\frac{\alpha!}{\beta!+\gamma!}\partial^\beta f(x)\partial^\gamma g(x).$$

Since $\mathbb{1}\{\beta+\gamma=\alpha\}$ forces $0\le\beta_i\le\alpha_i$ and $0\le\gamma_i\le\alpha_i$ for all i, we have the following bound:

$$\|\partial^\alpha(fg)(x)|\le 2^{|\alpha|}\left(\max_{\beta\le\alpha}|\partial^\beta f(x)|\right)\left(\max_{\gamma\le\alpha}|\partial^\gamma g(x)|\right).$$

Recall $\|fg\|_\theta=\sup_\alpha\sup_x\theta_\alpha(x)|\partial^\alpha(fg)(x)|$. Let

$$\theta'_\beta(x)=\sup_{\alpha\ge\beta}2^{|\alpha|/2}\sqrt{\theta_\alpha(x)}.$$

for all β. Then for all $\alpha\in\mathbb{N}_0^d$ and all $x\in U$,

$$\begin{aligned}\theta_\alpha(x)|\partial^\alpha(fg)(x)|&\le 2^{|\alpha|}\theta_\alpha(x)\left(\max_{\beta\le\alpha}|\partial^\beta f(x)|\right)\left(\max_{\gamma\le\alpha}|\partial^\gamma g(x)|\right)\\&=\left(\max_{\beta\le\alpha}2^{|\alpha|/2}\sqrt{\theta_\alpha(x)}|\partial^\beta f(x)|\right)\left(\max_{\gamma\le\alpha}2^{|\alpha|/2}\sqrt{\theta_\alpha(x)}|\partial^\gamma g(x)|\right)\\&\le\left(\max_{\beta\le\alpha}\theta'_\beta(x)|\partial^\beta f(x)|\right)\left(\max_{\gamma\le\alpha}\theta'_\gamma(x)|\partial^\gamma g(x)|\right)\\&\le\|f\|_{\theta'}\|g\|_{\theta'}\end{aligned}$$

Hence $\|fg\|_\theta\le\|f\|_{\theta'}\|g\|_{\theta'}$.

Now, we check that $\theta'\in\Theta$, i.e. that it is well-defined, continuous on U, and locally finite. To see that θ' is continuous, note that for all α $2^{|\alpha|/2}\sqrt{\theta_\alpha(x)}$ is continuous since θ_α is continuous and nonnegative. Moreover, since θ is locally finite, for all $x\in U$ there exists an open neighborhood V of x in U and $F\subseteq\mathbb{N}_0^d$ finite so that for all $\alpha\notin F$, $\theta_\alpha|_V\equiv 0$. Then

$$\theta'_\beta 9x)=\sup_{\alpha\in F}2^{|\alpha|/2}\sqrt{\theta_\alpha(x)}=\max_{\alpha\in F}2^{|\alpha|?2}\sqrt{\theta_\alpha(x)}$$

is continuous and well-defined.

To see that θ' is locally finite, let

$$\tilde{F} = \{\beta \in \mathbb{N}_0^d | \exists\ \alpha \in F : \beta \leq \alpha\} = \bigcup_{\alpha \in F} \{\beta | 0 \leq \beta \leq \alpha\}.$$

Then $\tilde{F}$ is finite as a finite union of finite sets. If $\beta \notin \tilde{F}$, then $\theta'_\beta|_V \equiv 0$. Indeed, $\theta'_\beta(x) = \sup_{\alpha \geq \beta} 2^{|\alpha|/2}\sqrt{\theta_\alpha(x)}$, and so if $\beta \notin \tilde{F}$, then all $\alpha \geq \beta$ cannot be in F. Hence $\theta'_\beta(x) = \sup_{\alpha \geq \beta} 0 = 0$. □

We are now ready to prove that pointwise multiplication of test functions is continuous.

Theorem 2.2.56. *The map $S(\mathbb{R}^d) \times S(\mathbb{R})^d) \to S(\mathbb{R}^d)$ given by $(f, g) \mapsto fg$ is continuous as a bilinar map.*

Proof. For all $\alpha \in \mathbb{N}_0^d$ and $k \in \mathbb{N}_0$,

$$\|fg\|_{\alpha,k} = \sup_{x \in \mathbb{R}^d} \langle x \rangle^k |\partial^\alpha (fg)(x)|.$$

If $x \in \mathbb{R}^d$, then

$$\langle x \rangle^k |\partial^\alpha (fg)(x)| \leq \langle x \rangle^k 2^{|\alpha|} \left(\max_{\beta \leq \alpha} |\partial^\beta f(x)| \right) \left(\max_{\gamma \leq \alpha} |\partial^\gamma g(x)| \right).$$

Say $\langle x \rangle^k \leq \langle x \rangle^{2\lceil k/2 \rceil}$. Then

$$\|fg\|_{\alpha,k} \leq 2^{|\alpha|} \left(\max_{\beta \leq \alpha} \|f\|_{\beta,\lceil k/2 \rceil} \right) \left(\max_{\gamma \leq \alpha} \|g\|_{\gamma,\lceil k/2 \rceil} \right).$$

If ρ is a continuous seminorm on $S(\mathbb{R}^d)$, then there exists a finite $F \subseteq \mathbb{N}_0^d \times \mathbb{N}_0$ and $c > 0$ such that

$$\rho \leq c \sum_{(\alpha,k) \in F} \| \cdot \|_{\alpha,k}.$$

Then

$$\begin{aligned} \rho(fg) &\leq c \sum_{(\alpha,k) \in F} 2^{|\alpha|} \left(\max_{\beta \leq \alpha} \|f\|_{\beta,\lceil k/2 \rceil} \right) \left(\max_{\gamma \leq \alpha} \|g\|_{\gamma,\lceil k/2 \rceil} \right) \\ &\leq c|F| \max_{(\alpha,k) \in F} 2^{|\alpha|} \left(\max_{\beta \leq \alpha} \|f\|_{\beta,\lceil k/2 \rceil} \right) \left(\max_{\gamma \leq \alpha} \|g\|_{\gamma,\lceil k/2 \rceil} \right) \\ &\leq \tau(f)\tau(g). \end{aligned}$$

where $\tau = \sqrt{c|F|} \sum_{(\alpha,k) \in F} \sum_{\beta \leq \alpha} 2^{|\alpha|/2} \| \cdot \|_{\beta,\lceil k/2 \rceil}$. Since τ is a finite sum of defining seminorms, τ is continuous. □

Definition 2.2.57. The **space of distributions on** U is $\mathscr{D}'(U)$, i.e. the strong dual of $\mathscr{D}(U)$.

Proposition 2.2.58. *$A \subseteq \mathscr{D}(U)$ is bounded iff*

1. *there exists $K \subseteq U$ compact such that for all $f \in A$, $supp(f) \subset K$, and*
2. *for all $\alpha \in \mathbb{N}_0^d$, $\sup\limits_{f\in A}\sup\limits_{x\in U} |\partial^\alpha f(x)| < \infty$.*

In other words, A is bounded iff the elements of A are uniformly bounded and have common compact support.

Proof. ($\Rightarrow$) We need to show that for all $\theta \in \Theta$, $\sup_{f\in A} \|f\|_\theta < \infty$. Let θ be given. Since θ is locally finite, for every $x \in K$, we can find a neighborhood V_x of x so that only finitely many θ_α are nonzero on V_x. Since K is compact, we can then find a finite $F \subseteq \mathbb{N}_0^d$ such that $\theta_\alpha|_K \equiv 0$ for all $\alpha \notin F$. Since (1) $\Rightarrow |\partial^\alpha f(x)| = 0$ for $x \notin K$ and since θ_α is continuous for all α, we have for any $f \in A$,

$$\begin{aligned}\|f\|_\theta &= \sup_{\alpha\in\mathbb{N}_0^d}\sup_{x\in U}\theta_\alpha(x)|\partial^\alpha f(x)| \\ &= \sup_{\alpha\in F}\sup_{x\in U}\theta_\alpha(x)|\partial^\alpha f(x)| \\ &\le \left(\max_{\alpha\in F}\max_{x\in K}\theta_\alpha(x)\right)\left(\sup_{\alpha\in F}\sup_{x\in K}|\partial^\alpha f(x)|\right),\end{aligned}$$

where $\max\limits_{\alpha\in F}\max\limits_{x\in K}\theta_\alpha(x)$ is constant and $\sup\limits_{\alpha\in F}\sup\limits_{x\in K}|\partial^\alpha f(x)|$ is bounded by (2). □

Definition 2.2.59. A set of functions D has **common compact support** if there exists a compact K s.t. for every function $f \in D$, the support of f is contained in K.

Theorem 2.2.60. *$A \subset \mathfrak{D}(U)$ is bounded iff both*

1. *A has common compact support*
2. *$\forall\alpha \in \mathbb{N}_0^d$, $\sup\limits_{f\in A}\sup\limits_{x\in U} |\partial^\alpha f(x)| < \infty$.*

Proof. We prove this theorem in three parts:

i "(1) + (2) $\implies$ A bounded"

We wish to show that $\forall\theta \in \Theta, \sup_{f\in A}\|f\|_\theta < \infty$, so let $\theta \in \Theta$. Because of (1), $\exists K, \forall f \in A, \mathrm{Supp}(f) \subset K$. Because θ is locally finite and K

is compact, only a finite number of θ_α's are nonzero on K. Let F be that finite set of α's. We want A to be bounded, so we compute

$$\begin{aligned}
\sup_{f\in A} \|f\|_\theta &:= \sup_{f\in A} \sup_{\alpha\in\mathbb{N}_0^d} \sup_{x\in U} \theta_\alpha(x) \left|\partial^\alpha f(x)\right| \\
&= \sup_{f\in A} \sup_{\alpha\in F} \sup_{x\in K} \theta_\alpha(x) \left|\partial^\alpha f(x)\right| \\
&= \sup_{f\in A} \max_{\alpha\in F} \sup_{x\in K} \theta_\alpha(x) \left|\partial^\alpha f(x)\right| \\
&\leq \left(\sup_{f\in A} \max_{\alpha\in F} \sup_{x\in K} \theta_\alpha(x)\right) \left(\sup_{f\in A} \max_{\alpha\in F} \sup_{x\in K} \left|\partial^\alpha f(x)\right|\right) \\
&= \left(\sup_{f\in A} \max_{\alpha\in F} \sup_{x\in K} \theta_\alpha(x)\right) \left(\max_{\alpha\in F} \sup_{f\in A} \sup_{x\in K} \left|\partial^\alpha f(x)\right|\right)
\end{aligned}$$

where the right group is finite by (2), but the left group needs a bit more attention. Because the support of θ_α is contained in K compact and θ_α is continuous, then the image $\theta_\alpha(K)$ is compact also and thus bounded. So $\sup_{x\in K} \theta_\alpha(x)$ is finite, so $\max_{\alpha\in F} \sup_{x\in K} \theta_\alpha(x)$ is finite. Moreover, θ_α does not depend on f, so taking the supremum over all $f \in A$ changes nothing. So

$$\forall\theta \in \Theta,\ \sup_{f\in A} \|f\|_\theta < \infty$$

as desired, so that A is bounded.

ii "A bounded $\implies$ (2)"

blar

iii "A bounded $\implies$ (1)"

We shall prove this by the contrapositive, so we prove "Not (1) $\implies$ A not bounded".

Choose an increasing sequence of compact sets $K_1 \subset K_2 \subset\subset\subset$ in U such that

$$U = \bigcup_{N\geq 1} K_N.$$

$$\begin{aligned}
&\text{"Not (1)"} \\
\implies& \forall K_N, \exists f_N \in A, \mathrm{Supp}(f) \not\subset K_N \\
\implies& \forall K_N, \exists f_N \in A, \exists x_N \notin K_N, f(x_N) \neq 0 \\
\implies& \forall K_N, \exists f_N \in A, \exists x_N \notin K_N, \exists \varepsilon_N, B(x_N, \varepsilon_N/2) \cap K_N = \varnothing
\end{aligned}$$

and f_N is nonzero on that ball.

Let θ be defined by

$$\theta_\alpha(y) := \mathbb{1}\{\alpha = 0\} \sum_{N \geq 1} \frac{N}{|f_N(x_N)|} \psi\Big(\frac{1}{\varepsilon_N}(y - x_N)\Big).$$

so that $\psi\Big(\frac{1}{\varepsilon_N}(\cdot - x_N)\Big)$ is 0 outside of $B(x_N, \varepsilon_N/2)$. In particular, $\psi(\cdot)$ is 0 on K_N. To get our contradiction, we wish to show that $\|\cdot\|_\theta$ blows up on A (this would make A unbounded). But before we proceed, we must verify that $\theta \in \Theta$.

Lemma 2.2.61. $\theta \in \Theta$

Proof.

- **well-defined**:

  Let $y \in U$ be arbitrary. Then the exhaustive sequence of compact sets guarantees $\exists M$ s.t. $y \in K_M$. For all $P > M$, $\psi\Big(\frac{1}{\varepsilon_P}(y - x_P)\Big) = 0$ because $\psi(\cdot)$ is 0 on $K_P \supset K_M \ni y$.

  The information about y's location allows us to write θ_0 as a finite sum:
  $$\theta_0(y) = \sum_{1 \leq N < M} \frac{N}{|f_N(x_N)|} \psi\Big(\frac{1}{\varepsilon_N}(y - x_N)\Big)$$

  Any finite sum converges, so θ_0 is defined.

- **nonnegativity**:

  Each θ_α is either 0 or the sum of products of N, $\frac{1}{|f_N(x_N)|}$, and $\psi(\cdot)$, each of which are nonnegative, so every θ_α has nonnegative range.

- **continuity**:

  (Is more justification needed? Continuous wrt what?) Each θ_α is the sum and product of continuous functions, hence continuous itself (in fact C^∞).

- **locally finite**:

  θ_0 is the only θ_α that is nonzero.

□

We continue with the main proof, trying to show that $\|\cdot\|_\theta$ blows up on A. For any f_N,

$$\begin{aligned}
\|f_N\|_\theta &= \sup_{\alpha\in\mathbb{N}_0^d}\sup_{y\in U}\theta_\alpha(y)\,|\partial^\alpha f_N(y)| \\
&\geq \theta_0(x_N)\left|\partial^0 f_N(x_N)\right| \\
&= \theta_0(x_N)\,|f_N(x_N)| \\
&= |f_N(x_N)|\,\theta_0(x_N) \\
&= |f_N(x_N)|\sum_{M\geq 1}\frac{M}{|f_M(x_M)|}\,\psi\left(\frac{1}{\varepsilon_M}(x_M-x_N)\right) \\
&\geq |f_N(x_N)|\,\frac{N}{|f_N(x_N)|}\,\psi(0) \qquad \text{(only the } M=N \text{ summand)} \\
&= N.
\end{aligned}$$

Finally, we can show

$$\sup_{f\in A}\|f\|_\theta \geq \sup_{N\geq 1}\|f_N\|_\theta \geq \sup_{N\geq 1} N = \infty$$

which implies that A is unbounded. This contradicts the hypothesis that A is bounded! ↯

□

2.2.8 Recap + Outlook

Metrizability

The following spaces are metrizable:

$$\mathscr{s}_0' \cong \mathcal{S}(\mathbb{Q}_p^d), \mathscr{s} \cong \mathcal{S}(\mathbb{R}^d)$$

The following spaces are not metrizable:

$$\mathscr{s}_0 \cong \mathcal{S}(\mathbb{Q}_p^d), \mathscr{s}' \cong \mathcal{S}'(\mathbb{R}^d), \mathscr{s}_0\hat{\otimes}\mathscr{s} \cong \mathcal{S}(\mathbb{A}_\mathbb{Q}) \cong \mathscr{D}(U), \mathscr{s}_0'\hat{\otimes}\mathscr{s}' \cong \mathcal{S}'(\mathbb{A}_\mathbb{Q}) \cong \mathscr{D}'(U)$$

(Note: We have not yet proven that $\mathcal{S}(\mathbb{R}^d) \cong \mathscr{s}$, but we will!)

Other generalizations of distributions

- $(\mathscr{M}, g)$ is a compact Riemmanian manifold, $\mathscr{M} \cong (\mathbb{R}/\mathbb{Z})^d$, and g is a section. $T * \mathscr{M} \otimes T * \mathscr{M}$ is the symmetric part. $\mathcal{S}(\mathbb{R}^d) = \mathscr{D}(\mathscr{M})$ is metrizable.

- Currents (de Rhan):

  k is a differentiable form on $\mathbb{R}^d$. We have $\sum_{1\le i_1\le\le\le i_k\le d} f_{i_1,,,i_k} dx_{i_1} \wedge \wedge \wedge dx_{i_k}$. To generalize this theory, let the f's be distributions instead of currents. $z \to \delta_z^d$. $U \to \mathscr{D}'(U)$.

- Distributions on Abelian locally compact groups (Bruhat):

  $\mathscr{S}'(\mathbb{Q}_p^d)$ generalizes to $\mathscr{S}'(\mathbb{A}_\mathbb{Q})$.

- Lie groups that are noncompact (see "Langlands program" or "Tate's thesis") (Harish-Chandra):

  $GL_n(\mathbb{R})$ generalizes to $\mathscr{S}'(GL_n(\mathbb{A}_\mathbb{Q}))$.

2.3 Basic properties of distributions

2.3.1 The mother of all distributions

The mother of all distributions is the constant function 1, also known as the Lebesque measure!

Proposition 2.3.1. *If $U \neq \varnothing$ is an open set in $\mathbb{R}^d$, then the map*

$$\begin{cases} \mathscr{D}(U) & \to \mathbb{K} \\ f & \mapsto \int_U f(x)\, d^d x \end{cases}$$

is in $\mathscr{D}'(U)$.

Proof. Here we prove that the seminorm $\rho(f) := \left|\int_U f(x)\,\mathrm{d}^d x\right|$ is continuous with respect to Θ. (There may be something more to show in order to prove the proposition).

It is sufficient to find a $\theta \in \Theta$ s.t. ρ is continuous with respect to $\|\cdot\|_\theta$. Define $\|\cdot\|_\theta$ by choosing $\theta_\alpha(x) := \mathbb{1}\{\alpha = 0\}\,\langle x\rangle^{d+1}$. The θ_α's we just chose are nonnegative, continuous, and locally finite, so $\theta \in \Theta$ indeed. For all $f \in \mathscr{D}(U)$,

$$\begin{aligned} \rho(f) &= \left|\int_U f(x)\,\mathrm{d}^d x\right| \\ &\le \int_U \langle x\rangle^{-d-1}\langle x\rangle^{d+1}\,|f(x)|\,\mathrm{d}^d x \\ &\le \left(\int_{\mathbb{R}^d} \langle x\rangle^{-d-1}\,\mathrm{d}^d x\right) \sup_{x\in U}\,\langle x\rangle^{d+1}\,|f(x)| \\ &=: \left(\int_{\mathbb{R}^d} \langle x\rangle^{-d-1}\,\mathrm{d}^d x\right) \|f\|_\theta\,. \end{aligned}$$

The left parenthesized group is finite, so ρ is continuous with respect to $\|\cdot\|_\theta$ as desired. ☐

Proposition 2.3.2. *There is a canonical continuous injective map*

$$\begin{aligned}\mathcal{D}(U) &\hookrightarrow \mathcal{D}'(U)\\ \varphi &\mapsto \left(f \mapsto \int_U \varphi(x) f(x)\, \mathrm{d}^d x\right) =: L_\varphi.\end{aligned}$$

(This allows us to identify $\mathcal{D}(U)$ with a linear subspace of $\mathcal{D}'(U)$.)

Proof. (It looks like we show continuity in this proof, but not injectivity. Injectivity still needs to be shown.) Consider

$$\begin{array}{ccccc}\mathcal{D}(U) & \to & \mathcal{D}(U) & \to & \mathbb{K}\\ f & \mapsto & \varphi \circ f & \mapsto & \int_U \varphi \circ f\end{array}$$

where L_φ is the composition of both arrows.

Let $A \in \mathcal{D}(U)$ be bounded.

$$\begin{aligned}\|L_\varphi\|_A &= \sup_{f\in A}\left|\int \varphi\circ f\right|\\ &\le \sup_{f\in A}\left(\int_{\mathbb{R}^d}\langle x\rangle^{-d-1}\,\mathrm{d}^d x\right)\|\varphi\circ f\|_\theta\\ &= \left(\int_{\mathbb{R}^d}\langle x\rangle^{-d-1}\,\mathrm{d}^d x\right)\sup_{f\in A}\|\varphi\circ f\|_\theta\,,\end{aligned}$$

where the second line is the result of reusing the inequality we had for f in the previous proof on $\varphi \circ f$. We know there exists a $\theta' \in \Theta$ such that $\|\varphi\circ f\|_\theta \le \|f\|_{\theta'}\|\varphi\|_{\theta'}$, so

$$\begin{aligned}\|L_\varphi\|_A &\le \left(\int_{\mathbb{R}^d}\langle x\rangle^{-d-1}\,\mathrm{d}^d x\right)\sup_{f\in A}\left(\|f\|_{\theta'}\|\varphi\|_{\theta'}\right)\\ &\le \left(\int_{\mathbb{R}^d}\langle x\rangle^{-d-1}\,\mathrm{d}^d x\right)\left(\sup_{f\in A}\|f\|_{\theta'}\right)\left(\sup_{f\in A}\|\varphi\|_{\theta'}\right)\\ &= \left(\int_{\mathbb{R}^d}\langle x\rangle^{-d-1}\,\mathrm{d}^d x\right)\left(\sup_{f\in A}\|f\|_{\theta'}\right)\|\varphi\|_{\theta'}\,.\end{aligned}$$

We know that $\left(\int_{\mathbb{R}^d}\langle x\rangle^{-d-1}\,\mathrm{d}^d x\right)$ is finite as shown in the previous proof, and A bounded implies common compact support of the f's implies $\sup_{f\in A}\|f\|_{\theta'}$ is finite. We have shown $\|L_\varphi\|_A \le c\|\varphi\|_{\theta'}$ for some finite number c, so for any $A \in \mathcal{D}(U)$ bounded, $\|\cdot\|_A$ is continuous with respect to $\|\cdot\|_{\theta'}$, and hence with respect to Θ. □

We have already shown that the following map is both injective and continuous:

$$\mathcal{D}(U) \hookrightarrow \mathcal{D}(U)'$$

$$\phi \mapsto (f \mapsto \int_U \phi(x) f(x) d^d x) := L_\phi.$$

This map has a dense image (which we take for granted). This density justifies the intuition that any distribution can be seen as

$$f \mapsto \int \phi f$$

even if ϕ doesn't exist as an honest function "$\phi(x)$: " i.e., for all $\phi \in \mathscr{D}'$, (DIFFERENT ϕ THAN ABOVE) $\exists (\phi_N)_{N\geq 1}$ in $\mathscr{D}$ such that $L_{\phi_N} \to \phi$ in topology of $\mathscr{D}'$, so for all A bounded in $\mathscr{D}$,

$$||L_{\phi_N} - \phi||_A \to 0$$

or

$$\sup_{f\in A} |\int \phi_N f - \phi(f)| \to 0.$$

Theorem 2.3.3. *There is a canonical continuous injective map*

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$$

$$\phi \mapsto (f \mapsto \int_{\mathbb{R}^d} \phi f)$$

which allows identification of $\mathcal{S}$ as a subset of $\mathcal{S}'$.

Remark: we've established the following chain of inclusions:

$$\mathscr{D}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathscr{D}'(\mathbb{R}^d)$$

Generic example: Recall a Borel measure $\mu \geq 0$ on $U \subset \mathbb{R}^d$ is called Radon if it's finite on compact sets. A family $(\mu_i)_{i\in I}$ of Radon measures is called locally finite if and only if for all compact subsets K of U, $\mu_i(K) = 0$ except for at most finitely many i's.

Theorem 2.3.4. *Let $(\mu_\alpha^+)_{\alpha\in\mathbb{N}_0^d}$ and $(\mu_\alpha^-)_{\alpha\in\mathbb{N}_0^d}$ be two locally finite families of Radon measures on U. Then the map*

$$\phi : \mathscr{D}(U, \mathbb{R}) \to \mathbb{R}$$

$$f \mapsto \phi(f) = \sum_{\alpha\in\mathbb{N}_0^d} \Big(\int_U \partial^\alpha f \, d\mu_\alpha^+ - \int_U \partial^\alpha f \, d\mu_\alpha^- \Big)$$

is well-defined and $\phi \in \mathscr{D}'(U, \mathbb{R})$.

Proof.

- **well-defined**:

  Let $f \in \mathscr{D}(U, \mathbb{R})$ be arbitrary. Then $f \in C^\infty(K, \mathbb{R})$ for some compact K. Then

$$\exists F \text{ finite } \subset \mathbb{N}_0^d \text{ such that } \alpha \notin F \Rightarrow \mu_\alpha^\pm(K) = 0$$

$$\Rightarrow \phi(f) = \sum_{\alpha \in F} \Big(\int_K \partial^\alpha f d\mu_\alpha^+ - \int_K \partial^\alpha f d\mu_\alpha^- \Big)$$

  where ϕ is well-defined and linear.

- **continuous**:

  By theorem 3.1.21, we need only find a seminorm $|\cdot|$ on $\mathbb{R}$ and a seminorm $\|\cdot\|$ on U s.t. $|\phi(f)| \leq c\|f\|$ for all f.

  If $f \in C^\infty(K, \mathbb{R})$, then

$$|\cdot|\,\phi(f) \leq \sum_{\alpha \in F} \Big(\mu_\alpha^+(K) + \mu_\alpha^-(K)\Big)||f||_{k,\alpha}$$

  where $||f||_{k,\alpha} := \sup_{x \in K} |\delta^\alpha f(x)|$.

□

Now we recall the Dirac δ functions: for $z \in U$, $\delta_z^d \in \mathscr{D}(U)'$ by definition,

$$\delta_z^d(f) := \langle \delta_z^d, f \rangle = \langle \delta_z^d(x), f(x) \rangle = \int_U \delta^d(z - x) f(x) d^d x$$

which is a spike at $f(z)$. For our particular case:

$$\mu_\alpha^- \equiv 0, \forall \alpha$$

$$\mu_\alpha^+ \equiv 0, \alpha \neq 0$$

$$\mu_0^+ = \text{ unit point mass at } z$$

$$\forall B \text{ Borel } \subset U, \mu_0^+(B) = \mathbb{1}\{z \in B\}$$

defines a map

$$U \hookrightarrow \mathscr{D}'(U)$$

$$z \mapsto \delta_z^d.$$

This map is continuous. Furthermore consider

$$\frac{d}{dt}\delta_{z+t}^d|_{t=0} := \lim_{t \to 0^+} \frac{1}{t}(\delta_{z+t}^d - \delta_z^d) = \psi - \partial_i f(z)$$

in the topology of $\mathscr{D}'$.

Definition 2.3.5. A Borel measure $f : U \to \mathbb{K}$ is called locally integrable if and only if for all $x \in U$, there exists $V \subset U$ a neighborhood of x such that $\int_V |f| < \infty$. We call $L^{1,\text{loc}}(U)$ the space of locally integrable functions on U modulo $f \sim g$ if and only if $f = g$ Lebesgue almost everywhere.

Note that

$$L^{1,\text{loc}}(U) \hookrightarrow \mathscr{D}'(U)$$
$$f \mapsto (f \mapsto \int_U \phi f)$$

all $\mu_\alpha^\pm \equiv 0$, $d\mu_0^\pm(x) = \phi_\pm(x)d^d x$.

Proposition 2.3.6. *For a sequence* $(f_n)_{n\geq 1}$, $f \in \mathscr{D}$, *we have* $f_n \to f$ *in* $\mathscr{D}(U)$ *if and only if we satisfy the following:*

1. *The set* $\{f\} \cup \{f_n \mid n \geq 1\}$ *has common compact support.*
2. $\forall \alpha, \partial^\alpha f_n \to \partial^\alpha f$ *uniformly in* U.

Proof. Sketch: if $f_n \to f$, then $A := \{f\} \cup \{f_n\}_{n\geq 1}$ is bounded by the single semi-norm criterion and the definition of convergence of sequences for ρ implies $|\rho(f_n) - \rho(f)| \to 0$ which implies boundedness and therefore common compact support. □

Proposition 2.3.7. *If* $\phi : \mathscr{D}(U) \to \mathbb{K}$ *is a linear form, then* ϕ *is continuous, i.e.* $\phi \in \mathscr{D}'$ *if and only if for all compact* $K \subset U$, $\forall f$ *and* $(f_n)_{n\geq 1}$ *with support in* K *such that* $\forall \alpha$, $\partial^\alpha f_n \to \partial^\alpha f$ *uniformly, we have* $\phi(f_n) \to \phi(f)$.

Proof. Sketch: the forward implication is immediate. For the other direction, we consider that $\rho(f) = \phi(f)|$ is a semi-norm. We can verify that ρ is continuous. Then we can verify that $\rho|_{C^\infty(K,\mathbb{R})}$ is continuous using the statement of the proposition and the fact that U is metrizable. Then we can apply the sequential characterization of the continuity of linear forms to obtain our needed result.

□

2.3.2 Multiplication of Test Functions and Distributions

Remark. Everything that follows has analogues for $\mathscr{S}$ and $\mathscr{S}'$.

Suppose that $\varphi(x)$ is a distribution and $f(x)$ is a test function. We would like to define the product distribution $f\varphi$, but it is not immediately clear how this should be defined because φ is not a function, per se. Assuming for the moment that φ is a function, we get

$$\langle f\varphi, g\rangle = \int_U (f(x)\varphi(x))g(x)\ d^d x = \int_U \varphi(x)(f(x)g(x)) = \langle \phi, fg\rangle$$

The term on the right-hand side makes sense for any distribution φ, and we take it to define $f\varphi$.

Definition 2.3.8. Let $f \in \mathscr{D}$, $\varphi \in \mathscr{D}'$. We define the **pointwise product** $f\varphi$ to be the distribution given by

$$\langle f\varphi, g\rangle := \langle \varphi, fg\rangle$$

Note that $f\varphi : \mathscr{D} \to \mathbb{K}$ is continuous because it can be written as a composite of continuous functions

$$\mathscr{D} \overset{m_f}{\leftrightarrow} \mathscr{D} \overset{\varphi}{\leftrightarrow} \mathbb{K}$$

where m_f is multiplication by f.

Remarks.

(i) The multiplication above makes $\mathscr{D}'$ into a $\mathscr{D}$-module.

(ii) The bilinear map $\mathscr{D}\times\mathscr{D}' \to \mathscr{D}'$ given by $(f, \varphi) \mapsto f\varphi$ is **not** continuous. It is, however, **hypocontinuous**, i.e., continuous on bounded sets.

(iii) The product $(f, \varphi) \mapsto f\varphi$ extends the pointwise product of functions: if $\varphi \in \mathscr{D} \hookrightarrow \mathscr{D}'$, then the distribution determined by the pointwise product $f\varphi \in \mathscr{D}$ is precisely the distribution $f\varphi \in \mathscr{D}'$ that we have just defined.

2.3.3 Derivatives

For $\varphi \in \mathscr{D}'$, we again wish to define a notion of for φ. Following the same line of reasoning as before, we assume for the moment that φ is a function:

$$\begin{aligned} \langle \partial_i \varphi, f\rangle &= \int \partial_i \varphi(x) f(x)\ d^d x \\ &= \int_U \partial_i(\varphi f)(x)\ d^d x\ - \int_U \varphi(x)\partial_i f(x)\ d^d x \\ &= -\int_U \varphi(x)\partial_i f(x)\ d^d x \\ &= \langle \varphi, -\partial_i f\rangle \end{aligned} \tag{$*$}$$

Thus we define $\langle \partial_i\varphi, f\rangle := \langle \varphi, \partial_i f\rangle$. In the above computation, we used that $\int_U \partial_i(\varphi f)(x)\ d^d x = 0$ when $\varphi, f \in \mathscr{D}$. This will be justified shortly.

Definition 2.3.9 (& Proposition)**.** For $\varphi \in \mathscr{D}'$, the derivative $\partial_i\varphi \in \mathscr{D}'$ is defined to be the linear form $f \mapsto \langle \varphi, -\partial_i f\rangle$.

Proof. We need to show that $\partial_i\varphi : \mathscr{D} \to \mathscr{D}$ is continuous. Let $\theta \in \Theta$ be a Horwath seminorm. We have

$$\|\partial_i f\|_\theta = \sup_\alpha \sup_x \theta_\alpha(x)|\partial^{e_i+\alpha} f| = \|f\|_{\theta'}$$

where

$$\theta'_\beta := \begin{cases} \theta_{\beta-e_i}, & \beta_i \geq 1 \\ 0, & \beta_i = 0 \end{cases}$$

One sees immediately that θ' is locally finite (because θ is) and each θ'_β is continuous (because the corresponding θ_α is), so we are done. □

Remark. By induction on $|\alpha| \geq 0$, we see that for each $\alpha \in \mathbb{N}_0^d$, the map $\partial^\alpha : \mathscr{D} \to \mathbb{K}$ given by

$$\langle \partial^\alpha \varphi, f\rangle := (-1)^{|\alpha|}\langle \varphi, \partial^\alpha f\rangle$$

is in $\mathscr{D}'$.

Example 2.3.10. The Heaviside function $\varphi(x) = \mathbb{1}x \geq 0$ defines the map

$$f \longmapsto \langle \varphi, f\rangle = \int_{\mathbb{R}} \varphi(x) f(x)\ dx = \int_0^\infty f(x)\ dx$$

Notice that φ is **not** differentiable as an ordinary function, but it **is** differentiable as a distribution:

$$\begin{aligned} \langle \varphi', f\rangle = -\langle \varphi, f'\rangle &= -\int_0^\infty f'(x)\ dx \\ &= f(x) \\ &= \langle \delta_0, f\rangle \end{aligned}$$

That is, $\varphi' = \delta_0$.

Proposition 2.3.11. *$\partial_i\varphi$ generalizes the classical derivative of ordinary functions via the inclusion $\mathscr{D} \hookrightarrow \mathscr{D}'$.*

Proof. Suppose $\varphi \in \mathscr{D}$, and let $L_\varphi \in \mathscr{D}'$ denote the associated distribution. We need to show $\partial_i L_\varphi = L_{\partial_i\varphi}$. By the computation $(*)$ at the beginning of this subsection, this is reduced to showing that $\int_U \partial_i g(x)\ d^dx = 0$ where $g(x) = \varphi(x)f(x)$. Let $\widetilde{g} \in \mathscr{D}(\mathbb{R}^d)$ be the extension of g to $\mathbb{R}^d$ by zero. Then, applying Fubini's theorem, we have

$$\begin{aligned} \int_U \partial_i g(x)\ d^dx &= \int_{\mathbb{R}^d} \widetilde{g}(x)\ d^dx \\ &= \int_{\mathbb{R}^{d-1}} \prod_{j\neq i} dx_j \int_{\mathbb{R}} \partial_i\widetilde{g}(x)\ dx_i \\ &= 0 \end{aligned}$$

because $\widetilde{g}$ has compact support. □

2.3.4 Composition With Diffeomorphisms

Let $F : U \to V$ be a C^∞ diffeomorphism and $\varphi \in \mathscr{D}'(V)$. We would like to define the "composition" $\varphi \circ F \in \mathscr{D}'(U)$. We apply the same reasoning as before:

$$\begin{aligned} \langle \varphi \circ F, f\rangle &= \int_U \varphi(F(x))f(x)\ d^d x \\ &= \int_V \varphi(y)f(F^{-1}(y))\ |J_y(F^{-1})|\ d^d y \end{aligned}$$

by change of variables.

Definition 2.3.12 (& Proposition)**.** For $\varphi \in \mathscr{D}'(V)$, we define the composition $\varphi \circ F \in \mathscr{D}'(U)$ by

$$\langle \varphi \circ F, f\rangle := \langle \varphi, (f \circ F^{-1})|J(F^{-1})|\rangle$$

Proof. We need to check that the map $\mathscr{D}(U) \to \mathscr{D}(V)$ given by

$$f \mapsto ((\varphi \circ F)(f) = \varphi((f \circ F^{-1})|J(F^{-1})|)$$

is continuous. By Proposition 2.3.7, it suffices to show that if $f_n, f \in \mathscr{D}(U)$ all have support contained in some compact $K \subseteq U$ and $\partial^\alpha f_n \to \partial^\alpha f$ uniformly for all $\alpha \in \mathbb{N}_0^d$, then $(\varphi \circ F)(f_n) \to (\varphi \circ F)(f)$. That is, we need to show that $\varphi(g_n) \to \varphi(g)$, where $G = F^{-1}$ and

$$g_n(y) = |J_y G|(f_n \circ G), \qquad g(y) = |J_y G|(f \circ G)$$

Now φ is assumed to be continuous, so again by Proposition 2.3.7, it suffices to show that g_n, g have a common compact support in V and $\partial^\alpha g_n \to \partial^\alpha g$ uniformly for all $\alpha \in \mathbb{N}_0^d$. The common compact support is clear, since G is a differomorphism. Choose $K \subseteq V$ compact such that $\mathrm{Supp}(g_n), \mathrm{Supp}(g) \subseteq K$. Now $|J_y G|$ and any finite number of its derivatives are uniformly bounded on K, so in view of the product rule, we can assume $g_n = f_n \circ G$ and $g = f \circ G$. Write $G = (G_1, \ldots, G_d)$ for $G_1, \ldots, G_d$ smooth functions on V. A more general version of the Faá di Bruno formula says that

$$\partial_y^\alpha g(y) = \sum_{(\sigma,\beta)\in B} C_{\sigma,\beta}\ \partial_x^\sigma f(G(y))\ \prod_{i=1}^d \left(\partial_y^{\beta^i} G_i(y)\right)^{k_{\sigma,\beta,i}}$$

and

$$\partial_y^\alpha g_n(y) = \sum_{(\sigma,\beta)\in B} C_{\sigma,\beta}\ \partial_x^\sigma f_n(G(y))\ \prod_{i=1}^d \left(\partial_y^{\beta^i} G_i(y)\right)^{k_{\sigma,\beta,i}}$$

for some finite subset B of $\mathbb{N}_0^d \times (\mathbb{N}_0^d)^d$, $k_{\sigma,\beta,i} \in \mathbb{N}_0$, and $C_{\sigma,\beta} \in \mathbb{K}$. Now $\prod_{i=1}^d \left(\partial_y^{\beta^i} G_i(y)\right)^{k_{\sigma,\beta,i}}$ is uniformly bounded on K, and by assumption

$$\partial_x^\sigma f_n(G(y)) \longrightarrow \partial_x^\sigma f(G(y))$$

uniformly for each σ. It follows that $\partial^\alpha g_n \to \partial^\alpha g$ uniformly. □

3

The Fourier Transform

For $f : \mathbb{R}^d \to \mathbb{C}$, we define its **Fourier transform** $\mathcal{F}[f] = \hat{f}$ by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi x} f(x)\ d^d x$$

whenever this integral converges. There is more than one way to interpret $\hat{f}$.

First, $\widetilde{F}$ defines a linear map $\widetilde{F} : L^1(\mathbb{R}^d) \to C(\mathbb{R}^d)$, where $C(\mathbb{R}^d)$ is the set of continuous functions on $\mathbb{R}^d$. Indeed, if $f \in L^1(\mathbb{R}^d)$, then $\hat{f}$ is a well defined function of $\xi \in \mathbb{R}^d$. By the dominated convergence theorem, $\hat{f}(\xi_n) \to \hat{f}(\xi)$ whenever $\xi_n \to \xi$, so $\hat{f}$ is continuous.

More classically, $\mathcal{F}$ defines a bijective linear map $\mathcal{S}(\mathbb{R}^d, \mathbb{C}) \to \mathcal{S}(\mathbb{R}^d, \mathbb{C})$ with inverse given by the **inverse Fourier transform** $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[f](x) = \check{f}(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi x} \hat{f}(\xi)\ d\xi$$

To show this, first, we show our formula $\hat{f} : \xi \mapsto \int_{\mathbb{R}^d} e^{-\xi x} f(x)\, d^d x$ defines a function into $S(\mathbb{R}^d, \mathbb{C})$. First, we verify that the formula which defines $\hat{f}$ makes sense. For each $\xi \in \mathbb{R}^d$, we have

$$|e^{-i\xi x} f(x)| = \langle x \rangle^{-(d+1)} \left(\langle x \rangle^{d+1} |f(x)| \right) \leq \|f\|_{0,d+1} \cdot \langle x \rangle^{-(d+1)}.$$

Since $\|f\|_{0,d+1}$ is finite $x \mapsto \langle x^{-(d+1)}$ is integrable over $\mathbb{R}^d$, and so $x \mapsto e^{-i\xi x} f(x)$ is an absolutely integrable function on $\mathbb{R}^d$.

Next, we show that $\hat{f}$ is C^∞, that is, $\partial^\alpha \hat{f}$ exists for all $\alpha \in \mathbb{N}_0^d$. By induction on $|\alpha|$, we will prove that $\partial^\alpha \hat{f}$ exists and is given by the following formula:

$$\partial^\alpha \hat{f}(\xi) = \int_{\mathbb{R}^d} (-ix)^\alpha e^{-i\xi x} f(x)\, d^d x.$$

(Recall that $x^\alpha = x_1^{\alpha_1}\dots x_d^{\alpha_d}$ for $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ and $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}_0^d$). This holds by definition for $\alpha=0$. At each inductive step, this follows from an application of "differentiation under the integral sign," e.g. [Folland, Theorem 2.27.] To apply this principle, we need to show that $\partial^\alpha \hat{f}$ is dominated by an absolutely integrable function. Since $|\cdot|\le\langle x\rangle$, we have

$$\begin{aligned}|(-ix)^\alpha e^{-i\xi x}f(x)| &= |x^\alpha|\cdot|f(x)|\\ &= |x_1|^{\alpha_1}\dots|x_d|^{\alpha_d}\langle x\rangle^{|\alpha|}\\ &\le \langle x\rangle^{|\alpha|}|f(x)|\\ &= \langle x\rangle^{-(d+1)}\left(\langle x\rangle^{|\alpha|+d+1}f(x)\right)\\ &\le \|f\|_{0,|\alpha|+d+1}\cdot\langle x\rangle^{-(d+1)}.\end{aligned}$$

Since $\|f\|_{0,|\alpha|+d+1}$ is a finite constant and $x\mapsto\langle x\rangle^{-(d+1)}$ is absolutely integrable, this proves the claim.

Next, we show that each $\hat{f}$ has fast decay. For $\alpha,\beta\in\mathbb{N}_0^d$, by Fubini's Theorem we have

$$\begin{aligned}\xi^\beta\partial^\alpha\hat{f}(\xi) &= \int_{\mathbb{R}^d} e^{-i\xi x}(-ix)^\alpha\xi^\beta f(x)\,d^dx\\ &= i^{|\beta|-|\alpha|}\int_{\mathbb{R}^d} e^{-i\xi x}(-i\xi)^\beta\cdot x^\alpha f(x)\,d^dx\\ &= i^{|\beta|-|\alpha|}\int_{\mathbb{R}^d}\partial_x^\beta\left(e^{-i\xi x}\right)\cdot\left(x^\alpha f(x)\right)\,d^dx\\ &= i^{|\beta|-|\alpha}\prod_{i=1}^d\int_{\mathbb{R}}\partial_{x_i}^{\beta_i}\left(e^{-i\xi_i x_i}\right)x_i^{\alpha_i}f(x)\,d^dx.\end{aligned}$$

We observe that any term $x^{\gamma_i}f(x)$ for $\gamma_i\in\mathbb{N}_0$ is such that $x^{\gamma_i}f(x)\big|_0^\infty = \lim_{x\to\infty}x^{\gamma_i}f(x)-0=0$ by the rapid decay of f. Hence, by repeated application of integration by parts and Fubini's Theorem, we have

$$\begin{aligned}\xi^\beta\partial^\alpha\hat{f}(\xi) &= i^{|\beta|-|\alpha|}\prod_{i=1}^d(-1)^{\beta_i}\int_{\mathbb{R}} e^{-i\xi_i x_i}\partial_{x_i}^\beta\left(x_i^{\alpha_i}f(x)\right)\,d^dx\\ &= i^{|\beta|-|\alpha|}(-1)^{|\beta|}\int_{\mathbb{R}^d} e^{-i\xi x}\partial_x^\beta(x^\alpha f(x))\,d^dx.\end{aligned}$$

By Leibniz' formula, we have

$$\xi^\beta\partial^\alpha\hat{f}(\xi) = (-i)^{|\beta|+|\alpha|}\int_{\mathbb{R}^d} e^{-i\xi x}\sum_{0\le\gamma\le\beta}\frac{\beta!}{\gamma!(\beta-\gamma)!}(\partial_x^\gamma x^\alpha)\cdot(\partial_x^{\beta-\gamma}f(x))\,d^dx.$$

By repeated differentiation and Fubini's Theorem, we see that

$$\partial_x^\gamma x^\alpha = \mathbb{1}\{\gamma\ge\alpha\}\cdot\frac{\alpha!}{(\alpha-\gamma)!}x^{\alpha-\gamma}.$$

Hence,

$$\begin{aligned}|\xi^\beta \partial^\alpha \hat{f}(\xi)| &\leq \sum_{\gamma \in \mathbb{N}_0^d} \mathbb{1}\{\gamma \leq \alpha, \beta\} \frac{\alpha!\beta!}{\gamma!(\alpha-\gamma)!(\beta-\gamma)!} \int_{\mathbb{R}^d} \langle x \rangle^{|\alpha|-|\gamma|} \left| \partial^{\beta-\gamma} f(x) \right| \, d^d x \\ &\leq \sum_{\gamma \in \mathbb{N}_0^d} \mathbb{1}\{\gamma \leq \alpha, \beta\} \frac{\alpha!\beta!}{\gamma!(\alpha-\beta)!(\beta-\gamma)!} \cdot \|f\|_{\beta-\gamma, |\alpha|-|\gamma|+d+1} \cdot \int_{\mathbb{R}^d} \langle x \rangle^{-(d+1)} \, d^d x,\end{aligned}$$

which is finite as the sum over γ is finite. This proves that $\sup_{\xi \in \mathbb{R}^d} |\xi^\beta \partial^\alpha \hat{f}(\xi)| < \infty$ for all $\alpha, \beta \in \mathbb{N}_0$, i.e. $\hat{f} \in S$. We also see that $|\xi^\beta \hat{f}(\xi)| \leq C\|f\|_{\nu,\ell}$ for some $\nu \in \mathbb{N}_0^d$ and $\ell \geq 0$.

Finally, we show $\mathcal{F}$ is continuous. For $k \geq 0$ and $\alpha \in \mathbb{N}_0^d$, we set $m := \lceil \frac{k}{2} \rceil$. Since $\langle \cdot \rangle \geq 1$ and $k \leq 2m$, then

$$\|\hat{f}\|_{\alpha,k} = \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^k |\partial^\alpha \hat{f}(\xi)| \leq \|\hat{f}\|_{\alpha,2m}.$$

We have

$$\langle \xi \rangle^{2m} \partial^\alpha \hat{f}(\xi) = \left(1 + \xi_1^2 + \ldots + \xi_d^2\right) \partial^\alpha \hat{f}(\xi).$$

We observe that this is a finite linear combination of expressions $\xi^\beta \partial^\alpha \hat{f}(\xi)$, for $\beta = 0, (2, 0, \ldots, 0), \ldots, (0, \ldots, 0, 2)$. We showed that each of these are expressions bounded above by a constant multiple of a seminorm $C\|f\|_{\nu,\ell}$. Hence, $\|\hat{f}\|_{\alpha,k} \leq \sum_{i=0}^M C_i \|f\|_{\nu_i,\ell_i}$ for some $M \geq 0$, $\nu_i \in \mathbb{N}_0^d$, $\ell_i \geq 0$, $C_i > 0$. And so $\mathcal{F}$ is continuous.

Remark 3.0.1. In the course of this proof, we showed the following relations between derivatives of Fourier transforms and multiplications by monomials x^α:

1. $\partial^\alpha \hat{f}(\xi) = \mathcal{F}\left[x \mapsto (-ix)^\alpha f(x)\right](\xi)$.

2. By integration by parts,

$$\begin{aligned}\widehat{\partial^\alpha f}(\xi) &= \int_{\mathbb{R}^d} e^{-i\xi x} \partial^\alpha f(x) \, d^d x \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^d} (-i\xi)^\alpha e^{-i\xi x} f(x) \, d^d x = (i\xi)^\alpha \hat{f}(\xi)\end{aligned}$$

Example 3.0.2. Let $0 < \alpha < d$.

"$\displaystyle\int_{\mathbb{R}^d} e^{-i\xi x} \frac{1}{|x|^\alpha} \, d^d x = \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{d}{2})} 2^{d-\alpha} \pi^{\frac{d}{2}} \frac{1}{|\xi|^{d-\alpha}}$"

Proof. Note that L.H.S. does not make sense as Lebesgue integral as $\int_{\mathbb{R}^d} |x|^{-\alpha} d^d x = \infty$. But we will work in the language of distributions. Define

$$\phi(x) = \begin{cases} \frac{1}{|x|^\alpha} & x \neq 0, \\ 0 & x = 0 \end{cases}$$

Define the distribution associated to ϕ, say T, i.e. given $f \in S'$, we have $< T, f > = \int_{\mathbb{R}^d} \phi(x) f(x) d^d x$.

$$\int |\phi f| = \int \frac{1}{|x|^\alpha} \frac{< x >^{d+1}}{< x >^{d+1}} |f(x)| d^d x \leq ||f||_{0,d+1} \int_{\mathbb{R}^d} \frac{d^d x}{|x|^\alpha < x >^{d+1}} < \infty.$$

Thus T is well-defined and continuous. So, $T \in S'$.

$\forall f \in S'$,

$$\begin{aligned} < \hat{T}, f > := < T, \hat{f} > &= \int_{\mathbb{R}^d \setminus \{0\}} d^d x \frac{1}{|x|^\alpha} \hat{f}(x) \\ &= \int_{x \neq 0} d^d x \Big(\frac{1}{\Gamma(\frac{\alpha}{2}} \int_0^\infty \frac{dt}{t} t^{\frac{\alpha}{2}} e^{-t|x|^2} \Big) \hat{f}(x) \\ &\overset{Fubini}{=} \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \frac{dt}{t} t^{\frac{\alpha}{2}} \int_{\mathbb{R}^d \setminus \{0\}} d^d x \ \overline{e^{-t|x|^2}} \hat{f}(x) \quad (*) \end{aligned}$$

For $a > 0$,

$$\begin{aligned} \mathscr{F}[\xi \to e^{-a\xi^2}](x) &= \int_{\mathbb{R}^d} e^{-ix\xi} e^{-a\xi^2} d^d \xi \\ &= (2a)^{-\frac{d}{2}} \int e^{-\frac{\eta^2}{2} - i \frac{x}{\sqrt{2a}\eta}} d^d \eta, \quad \eta = \sqrt{2a}\xi \\ &= (\frac{\pi}{a})^{\frac{d}{2}} e^{-\frac{x^2}{4a}} \end{aligned}$$

Take $\frac{1}{4a} = t$; $e^{-t|x|^2} = \mathscr{F}[\xi \to (4\pi t)^{-\frac{d}{2}} e^{-\frac{\xi^2}{4t}}](x)$

Substitute into (*) and use Plancherel,

$$\begin{aligned} < T, \hat{f} > &= \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty \frac{dt}{t} t^{\frac{\alpha}{2}} (2\pi)^d \int_{\mathbb{R}^d \setminus \{0\}} d^d \xi \ (4\pi)^{-\frac{d}{2}} e^{-\frac{\xi^2}{4t}} f(\xi) \\ &\overset{Fubini}{=} \frac{1}{\Gamma(\frac{\alpha}{2})} \int_{\xi \neq 0} d^d \xi \ f(\xi) \pi^{\frac{d}{2}} \int_0^\infty \frac{dt}{t} t^{\frac{\alpha - d}{2}} e^{-\frac{\xi^2}{4t}} \\ &= \frac{1}{\Gamma(\frac{\alpha}{2})} \int_{\xi \neq 0} d^d \xi \ f(\xi) \pi^{\frac{d}{2}} (\frac{\xi^2}{4})^{\frac{\alpha - d}{2}} \int_0^\infty \frac{ds}{s} s^{\frac{d-\alpha}{2}} e^{-s}, \quad s = \frac{\xi^2}{4t} \end{aligned}$$

Thus, $< \hat{T}, f > = \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} 2^{d-\alpha} (\pi)^{\frac{d}{2}} \int_{\xi \neq 0\} d^d \xi \ \frac{1}{|\xi|^{d-\alpha}} f(\xi)}$. The last part does define an element in S' as $d - \alpha < d \iff \alpha > 0$.

□

Remark: This was a particular case of Fourier transform for homogeneous distributions $\phi(x) \in S'(\mathbb{R}^d)$— it is homogeneous of degree $\gamma \in \mathbb{R}$ if and only if $\forall \lambda > 0$, "$\phi(\lambda x) = \lambda^\gamma \phi(x)''$.

Remark: $S' \subset \mathcal{D}'$. Let $\phi \in S'(\mathbb{R}^d,\ f--C^\infty$ diffeomorphism of $\mathbb{R}^d; in general \phi \circ f \in \mathcal{D}'$ but not in S'. However, if we have $f(x) = \lambda x$, then $\phi \circ f$ does lie in S'.

We quote a theorem without proof.

Theorem 3.0.3. *If ϕ is homogeneous of degree γ, then $\mathcal{F}[\phi]$ is homogeneous of degree $-\gamma - d$.*

Thus, what we proved earlier is a particular case of this.

3.0.1 Convolution

Definition 3.0.4. Let $f, g \in S(\mathbb{R}^d, \mathbb{C})$. Their convolution is defined as follows:

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)d^d y.$$

f is bounded and $g \in L_1$, so this definition makes sense. In fact, $f*g \in \mathcal{S}$.

Theorem 3.0.5.

1 . *Convolution is a continuous bilinear map from $S(\mathbb{R}^d, \mathbb{C}) \times S(\mathbb{R}^d, \mathbb{C}) \to S(\mathbb{R}^d, \mathbb{C})$.*

2 . *$*$ is symmetric: $f * g = g * f$.*

3 . *$\forall\ \alpha \in \mathbb{N}_0^d$, $\partial^\alpha(f * g) = (\partial^\alpha f) * g + f * (\partial^\alpha g)$.*

4 . *$\forall\ f, g,\ \mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.*

Proof. We will prove the first half of (3) first.

$\partial^\alpha f \in L^\infty, g \in L^1$.

$|\partial_x^\alpha[f(x-y)g(y)]| \leq$constant $\times |g(y)|$. Thus derivative can be taken inside the integral as we have seen in Math 7310. Thus we have $f * g$ is C^∞ and $\partial^\alpha(f * g) = \int \partial^\alpha(f(x-y)g(y))\ dy$.

Now we show part 2, which when combined with what we just saw, will give us 3.

$f * g(x) = \displaystyle\int f(x-y)g(y)d^d y = \int f(z)g(x-z)d^d z = g * f(x)$ using $z = x - y$ and the translation invariance of Lebesgue measure.

$$||f*g||_{\alpha,k} = \sup_x < x >^k |\partial^\alpha(f*g)(x)| \leq \sup_x \int < x >^k |\partial^\alpha f(x-y)||g(y)|d^d y$$

from 3.

We previously had shown that $\forall\, u, v \in \mathbb{R}^d, < u+v > \leq \sqrt{2} < u >< v >$. Applying this in the previous calculation and continuing we have,

$$\leq 2^{\frac{k}{2}} \sup_x \int < x-y >^k |\partial^\alpha f(x-y)| < y >^k |g(y)| d^d y$$
$$\leq 2^{\frac{k}{2}} \Big(\int_{\mathbb{R}^d} < y >^{-d-1} d^d y \Big) \times ||f||_{\alpha,k} ||g||_{0,k+d+1} \qquad < \infty$$

and hence we have continuity.

Finally to see 4, we have

$$\begin{aligned} \mathcal{F}(f*g) &= \int e^{-i\xi x} \Big(\int f(x-y) g(y) d^d y \Big) d^d x \\ &\overset{Fubini}{=} \int \int e^{-i\xi(x-y)} f(x-y) e^{-i\xi y} g(y) d^d y d^d x \\ &= \int g(y) e^{-i\xi y} \Big(e^{-i\xi(x-y)} f(x-y) d^d x \Big) d^d y \\ &= \hat{f}(\xi) \hat{g}(\xi) \end{aligned}$$

□

3.0.2 Poisson Equation Revisited

Let $\rho \in S$. The question was to find out a ϕ such that $-\Delta\phi = \rho$. We saw that in $d = 3$, we solved $\phi(x) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \dfrac{1}{|x-y|} \rho(y) d^3 x$.

Let's start a heuristic discussion to handle this case using the tools we have developed.

"

$\mathcal{F}[-\Delta\phi] = -\displaystyle\sum_{j=1}^{3} \widehat{\partial_j^2 \phi}(\xi) = -\sum_{j=1}^{3} (i\xi_j)^2 \hat{\phi}(\xi) = |\xi|^2 \hat{\phi}(\xi)$. Thus we should have from the given Poisson Equation $|\xi|^2 \hat{\phi}(\xi) = \hat{\rho}(\xi)$ which should give us $\phi = \mathcal{F}^{-1}\Big[\dfrac{1}{|\xi|^2} \hat{\rho}(\xi)\Big]$. Now if we have $\dfrac{1}{|\xi|^2} = \hat{h}()\xi$ for some h, then we can write $\phi = \mathcal{F}^{-1}[\mathcal{F}[h*\rho]] = h*\rho$ i.e. $\phi(x) = \displaystyle\int_{\mathbb{R}^3} h(x-y)\rho(y) d^3 y$.

Now the question is whether $h(x) = \dfrac{1}{4\pi|x|}$ works or not.

$$
\begin{aligned}
\hat{h}(\xi) &= \frac{1}{4\pi}\int_{\mathbb{R}^d} e^{-i\xi x}\frac{1}{|x|^\alpha}d^dx \\
&= \frac{1}{4\pi}\frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}2^{d-\alpha}\pi^{\frac{d}{2}}\frac{1}{|\xi|^{d-\alpha}} \text{ (from what we have seen)} \\
&= \frac{1}{4\pi}\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}2^2\pi^{\frac{3}{2}}\frac{1}{|\xi|^2}, \text{ (as } d=3, \alpha=1) \\
&= \frac{1}{|\xi|^2}
\end{aligned}
$$

”

4

Sequence Space Representation

4.1 Completeness of $\mathcal{S}$

Proposition 4.1.1. *Let U be an open subset in $\mathbb{R}^d$ and $\{f_n\}_{n\geq 1} \subset C^1(U)$. Suppose there is a function $f : U \to \mathbb{R}$ such that $f_n \to f$ pointwise and for any $1 \leq i \leq d$, $\partial_i f_n$ converge locally uniformly to some function g_i. Then $f \in C^1(U)$ and $g_i = \partial_i f$ for all i.*

Proof. Since g_i is locally uniform limit of continuous functions, it is also continuous and for any $x \in U$ there exists $\epsilon > 0$ and $B(x, \epsilon) \subset U$ such that

$$\sup_{y \in B(x,\epsilon)} |\partial_i f_n(y) - g_i(y)| \to 0 \quad \text{as } n \to \infty.$$

Now for $|t| < \epsilon$, by FTC

$$f_n(x + te_i) - f_n(x) = t \int_0^1 \partial_i f_n(x + ste_i) ds$$

By taking $n \to \infty$ (note that $\partial_i f_n \to g_i$ uniformly on $B(x, \epsilon)$)

$$f(x + te_i) - f(x) = t \int_0^1 g_i(x + ste_i) ds$$

Then by uniform continuity

$$\begin{aligned} \left| \frac{f(x + te_i) - f(x)}{t} - g_i(x) \right| &= \left| \int_0^1 [g_i(x + ste_i) - g_i(x)] ds \right| \\ &\leq \sup_{s \in [0,1]} |g_i(x + ste_i) - g_i(x)| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

So all partial derivatives of f exist and $\partial_i f = g_i$ which are continuous, and $f \in C^1(U)$. □

Theorem 4.1.2. *Let $\{f_n\}_{n\geq 1} \subset \mathscr{S}(\mathbb{R}^d)$ such that for any $(\alpha, k) \in \mathbb{N}_0^d \times \mathbb{N}_0$, $\{f_n\}$ is Cauchy in the norm $\|\cdot\|_{\alpha,k}$ (i.e. $\forall \epsilon > 0$, $\exists N \geq 0$ s.t. $\forall m, n \geq N$, $\|f_m - f_n\|_{\alpha,k} \leq \epsilon$). Then there exists $f \in \mathscr{S}(\mathbb{R}^d)$ such that $f_n \to f$ in the topology of $\mathscr{S}(\mathbb{R}^d)$.*

Proof. It is known that $\{f_n\}$ is Cauchy in $\|\cdot\|_{\alpha,k}$, in particular for $(\alpha, k) = (0,0)$ or $(1,0)$ this implies $\{f_n\}$ and $\{\partial_i f_n\}$ converge uniformly and by Proposition 4.1.1,

$$f_n \to f \in C^1(\mathbb{R}^d), \qquad \partial_i f_n \to \partial_i f$$

One can iterate the above procedure and show that $f \in C^\infty(\mathbb{R}^d)$ and $\partial^\alpha f_n \to \partial^\alpha f$ uniformly for all α. In what follows we will show $f_n \to f$ in $\mathscr{S}(\mathbb{R}^d)$: By the hypothesis, with α, k fixed, for any $\epsilon > 0$ there exists N such that for $m, n \geq N$ and $x \in \mathbb{R}^d$

$$\langle x\rangle^k \left|\partial^\alpha f_m(x) - \partial^\alpha f_n(x)\right| \leq \epsilon$$

For fixed ϵ, N, m, x, by taking $n \to \infty$ in the above inequality we get

$$\langle x\rangle^k \left|\partial^\alpha f_m(x) - \partial^\alpha f(x)\right| \leq \epsilon$$

Since it holds for all x we proved that for any $\epsilon > 0$ there is N such that $\|f_m - f\|_{\alpha,k} \leq \epsilon$ when $m \geq N$, and since α, k are arbitrary this means $f_n - f \to 0$ in $\mathscr{S}(\mathbb{R}^d)$. Finally, $f \in \mathscr{S}(\mathbb{R}^d)$ because for $\epsilon = 1$ there is N_0 such that

$$\|f\|_{\alpha,k} \leq \|f_{N_0}\|_{\alpha,k} + \|f_{N_0} - f\|_{\alpha,k} \leq \|f_{N_0}\|_{\alpha,k} + 1 < \infty$$

□

Corollary 4.1.3. *$\mathscr{S}(\mathbb{R}^d)$ is a Fréchet space.*

Proof. Let $\rho : \mathbb{N} \to \mathbb{N}_0^d \times \mathbb{N}_0$ by $n \mapsto (\alpha_n, k_n) =: \rho(n)$ be a bijection. Then the sequence of seminorms $\|\cdot\|_{\rho(n)}$ defining the metric

$$d(f, g) := \sum_{n\geq 1} 2^{-n} \min\{1, \|f - g\|_{\rho(n)}\}$$

satisfies the requirement of being Fréchet: Clearly it is a distance (translation invariant) in $\mathscr{S}(\mathbb{R}^d)$ which defines a topology $\tau(d)$ of $\mathscr{S}(\mathbb{R}^d)$. And clearly this is a complete metric space. We only need to show $\tau(d) = \tau(\mathscr{S}(\mathbb{R}^d))$.

($\subseteq$): For any $B_\epsilon(f)$ in $\tau(d)$, the multi-ball

$$\left\{g \mid \|f - g\|_{\rho(1)} < \epsilon/2, \ldots, \|f - g\|_{\rho(n)} < \epsilon/2\right\}$$

with $2^{-n} < \epsilon/2$ is contained in $B_\epsilon(f)$.

($\supseteq$): On the other hand, note that

$$B_{2^{-n}\epsilon}(f) \subset \left\{g \mid \|f - g\|_{\rho(1)} < \epsilon, \ldots, \|f - g\|_{\rho(n)} < \epsilon\right\}$$

□

4.1.1 Old and New density results in $L^2(\mathbb{R}^d, \mathbb{R})$

Recall that

$$C^0(\mathbb{R}^d, \mathbb{K}) := \{f : \mathbb{R}^d \to \mathbb{K} \text{ which are continuous}\}$$
$$C_c^0(\mathbb{R}^d, \mathbb{K}) := \{f : \mathbb{R}^d \to \mathbb{K} \text{ which are continuous and compactly supported}\}$$

and also we have inclusion

$$\mathscr{D}(\mathbb{R}^d) \subset C_c^0(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$$
$$\mathscr{D}(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$$

Proposition 4.1.4. *$C_c^0(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$.*

Proof. Recall in Math 7310 – Hwk8 we showed that $C^0(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. For any $f \in L^2$ take $g \in C^0(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $\|f - g\|_{L^2} < \epsilon$. Now let

$$g_N(x) = g(x)\Phi(x/N), \qquad N \geq 1$$

where $\Phi(x) = \varphi_3(|x|)$ which is the bump function defined in section 1.2. Then $g_N \in C_c^0(\mathbb{R}^d)$ and by DCT

$$\|g - g_N\|_{L^2}^2 = \int_{\mathbb{R}^d \setminus B(0,N/2)} g(x)^2 \left(1 - \Phi(x/N)\right)^2 dx \to 0 \quad \text{ as } n \to \infty$$

so we can take large N such that $\|f - g_N\|_{L^2} < 2\epsilon$. □

Proposition 4.1.5. *$\mathscr{D}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. (This implies $\mathscr{S}(\mathbb{R}^d)$ is also dense in $L^2(\mathbb{R}^d)$.)*

Proof. By Proposition 4.1.4 it suffices to show $\overline{\mathscr{D}(\mathbb{R}^d)}$ is dense in $C_c^0(\mathbb{R}^d)$ under L^2 norm. Let $f \in C_c^0(\mathbb{R}^d)$ and $K = \overline{\operatorname{supp}(f)}$ which is compact. Let $R > 0$ be such that $K \subseteq \overline{B(0, R)}$. Then by Theorem 1.3.13 there exists $\{f_N\} \subset \mathscr{D}(\mathbb{R}^d)$ such that $\operatorname{supp}(f_N) \subset \overline{B(0, R)}$ and

$$\|f - f_N\|_\infty \to 0$$

Then

$$\|f - f_N\|_{L^2}^2 = \int_{\overline{B(0,R)}} |f(x) - f_N(x)|^2 dx \leq \mathbf{Vol}(B(0, R))\|f - f_N\|_\infty^2 \to 0$$

□

Theorem 4.1.6. *Let V be the $\mathbb{C}$-linear span of $\left\{e^{-\frac{x^2}{2} + i\xi x} \mid \xi \in \mathbb{R}^d\right\}$. Then V is dense in $L^2(\mathbb{R}^d, \mathbb{C})$.*

Proof. It suffices to show V is dense in $\mathscr{D}(\mathbb{R}^d)$ under L^2 norm. Let $f \in \mathscr{D}(\mathbb{R}^d, \mathbb{C})$. Then $g(x) = e^{\frac{x^2}{2}} f(x) \in \mathscr{D} \subset \mathscr{S}$. Since Fourier transform is invertible on $\mathscr{S}$,

$$e^{\frac{x^2}{2}} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi x} \hat{g}(\xi) d\xi$$

so

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\frac{x^2}{2} + i\xi x} \hat{g}(\xi) d\xi$$

Let

$$f_M(x) = \frac{1}{(2\pi)^d} \int_{[-M,M)^d} e^{-\frac{x^2}{2} + i\xi x} \hat{g}(\xi) d\xi$$

Then

$$|f(x) - f_M(x)| \leq \frac{e^{-\frac{x^2}{2}}}{(2\pi)^d} \int_{\mathbb{R}^d \setminus [-M,M)^d} |\hat{g}(\xi)| d\xi$$

and by DCT

$$\|f(x) - f_M(x)\|_{L^2}^2 \leq 2^{-2d} (\pi)^{-\frac{3d}{2}} \int_{\mathbb{R}^d \setminus [-M,M)^d} |\hat{g}(\xi)| d\xi \to 0 \quad \text{as } M \to \infty.$$

Next for $N \geq 1$, divide $[-M, M)^d$ into $(2MN)^d$ cubes of side length N^{-1} and write

$$f_M(x) = \frac{1}{(2\pi)^d} \sum_{\xi \in [-M,M)^d \cap (N^{-1}\mathbb{Z})^d} \int_{\xi + [0,N^{-1})^d} e^{-\frac{x^2}{2} + i\eta x} \hat{g}(\eta) d\eta$$

Define

$$f_{M,N}(x) = \frac{1}{(2\pi)^d} \sum_{\xi \in [-M,M)^d \cap (N^{-1}\mathbb{Z})^d} e^{-\frac{x^2}{2} + i\xi x} \int_{\xi + [0,N^{-1})^d} \hat{g}(\eta) d\eta$$

By definition $f_{M,N} \in V$ and

$$|f_M(x) - f_{M,N}(x)| \leq \frac{e^{-\frac{x^2}{2}}}{(2\pi)^d} \sum_{\xi \in [-M,M)^d \cap (N^{-1}\mathbb{Z})^d} \int_{\xi + [0,N^{-1})^d} \left| e^{i\eta x} - e^{i\xi x} \right| |\hat{g}(\eta)| d\eta$$

For $u, v \in \mathbb{R}$,

$$|e^{iv} - e^{iu}| = \left| \int_0^1 i(v-u) e^{i(u + t(v-u))} dt \right| \leq |u - v|$$

For $\eta \in \xi + [0, N^{-1})^d$,

$$|\eta x - \xi x| \le |x||\eta - \xi| \le |x|\frac{\sqrt{d}}{N}$$

So

$$|f_M(x) - f_{M,N}(x)| \le \frac{|x|e^{-\frac{x^2}{2}}}{(2\pi)^d}\frac{\sqrt{d}}{N}\int_{[-M,M)^d} |\hat{g}(\eta)| d\eta$$

Then

$$\|f_M(x) - f_{M,N}(x)\|_{L^2}^2 \le \frac{d}{N^2(2\pi)^d}\|\hat{g}\|_{L^1}^2 \int_{\mathbb{R}} x^2 e^{-x^2} dx \to 0 \quad \text{as } N \to \infty.$$

$\square$

Theorem 4.1.7. *Let $W_{\mathbb{K}} := \{e^{-x^2/2}p(x) \;:\; p(x) \in \mathbb{K}[x_1, ..., x_d]\}$ (here x^2 understood in sense $x_1^2 + \cdots + x_d^2$), Then $W_{\mathbb{K}}$ is a dense subspace of $L^2(\mathbb{R}^d, \mathbb{K})$.*

Remark 1. The claim of theorem is well posed for $W_{\mathbb{K}}$ is clearly contained in Schwartz space $\mathcal{S}$ and thus $L^2(\mathbb{R}^d, \mathbb{K})$.

Remark 2. Notes from Math 7305 have a proof for the $d = 1$ case (it is "very ad hoc" method) and it's possible to extend the proof to d dimensions. We'll present an alternative proof which uses methods discussed.

Proof. We'll prove two cases.

($\mathbb{K} = \mathbb{C}$). Fix function $g(x) = e^{-\frac{x^2}{2} + i\xi x}$. To approximate g in $W_{\mathbb{C}}$, we'll expand $e^{i\xi x}$ for $N \in \mathbb{N}_0$. Define

$$g_N(x) = e^{-\frac{x^2}{2}} \sum_{i=0}^{N} \frac{(i\xi x)^n}{n!} \in W_{\mathbb{C}}.$$

We easily get the following bound on the difference of g and g_N:

$$|g(x) - g_N(x)| \le e^{-\frac{x^2}{2}} \sum_{n \ge N+1} \frac{|i\xi x|^n}{n!} \le e^{-x^2/2} \sum_{n \ge N+1} \frac{|\xi|^n |x|^n}{n!}$$

hence

$$|g(x) - g_N(x)|^2 \le e^{-x^2} \sum_{m,n \ge N+1} \frac{|\xi|^{m+n}|x|^{m+n}}{m!\, n!}.$$

It follows that

$$\|g(x) - g_N(x)\|_{L^2}^2 \le \sum_{m,n \ge N+1} \frac{|\xi|^{m+n}}{m!\, n!} \int_{\mathbb{R}^d} e^{-x^2} |x|^{m+n} \, d^d x$$

which by Cauchy-Schwartz is bounded by

$$\sum_{m,n\geq N+1} \frac{|\xi|^{m+n}}{m!\,n!} \left(\int_{\mathbb{R}^d} e^{-x^2}|x|^{2m} d^d x\right)^{1/2} \left(\int_{\mathbb{R}^d} e^{-x^2}|x|^{2n} d^d x\right)^{1/2}.$$

The right side of the inequality above can be denoted as ${R_N}^2$ where

$$R_N := \sum_{n\geq N+1} \frac{|\xi|^n}{n!} \left(\int_{\mathbb{R}^d} e^{-x^2}|x|^{2n}\, d^d x\right)^{1/2}. \tag{4.1}$$

It remains to show that R_N converges. By use of spherical coordinates, notice

$$\begin{aligned}\int_{\mathbb{R}^d} e^{-x^2}|x|^{2n}\, d^d x &= \mathbf{Vol}_{d-1}(\mathbb{S}^{d-1}) \times \int_0^\infty e^{-r^2}\, r^{2n+d-1} dr \\ &= \mathbf{Vol}_{d-1}(\mathbb{S}^{d-1}) \times \frac{1}{2}\int_0^\infty e^{-s}\, s^{n+\frac{d}{2}}\, \frac{ds}{s}\end{aligned}$$

where last equality is result of change of variables $s = r^2$. Notice the integral term is identical to the Gamma function evaluated at $n+d/2$. Putting things together, we now have

$$\int_{\mathbb{R}^d} e^{-x^2}|x|^{2n}\, d^d x = \mathbf{Vol}_{d-1}(\mathbb{S}^{d-1}) \times \frac{1}{2}\Gamma\left(n + \frac{d}{2}\right)$$

It follows that we can approximate line (4.1) by the power series $\sum_{n=0}^{\infty} c\, a_n |\xi|^n$ where

$$a_n = \frac{\sqrt{\Gamma\left(n+\frac{d}{2}\right)}}{n!} \quad \text{and} \quad c = \sqrt{\frac{1}{2}\mathbf{Vol}_{d-1}(\mathbb{S}^{d-1})}$$

Making use of the gamma function property $\Gamma(z+1) = z\Gamma(z)$, we see the series has an infinite radius of convergence

$$\left|\frac{a_n}{a_{n+1}}\right| = (n+1)\sqrt{\frac{\Gamma\left(n+\frac{d}{2}\right)}{\Gamma\left(n+1+\frac{d}{2}\right)}} = \frac{n+1}{\sqrt{n+\frac{d}{2}}} \xrightarrow{\text{as } n\to\infty} \infty$$

and so all ξ, R_N converges.

($\mathbb{K} = \mathbb{R}$). Fix a function $f \in L^2(\mathbb{R}^2, \mathbb{R}) \subset L^2(\mathbb{R}^2, \mathbb{C})$. By our first case, there exists a complex polynomial $P_N \in \mathbb{C}[x_1, ..., x_d]$ such that

$$f_N(x) := e^{-\frac{x^2}{2}} P_N(x) \to f$$

in L^2 where $f_N \in W_{\mathbb{C}}$. We may write the polynomial P_N in the form

$$P_N(x) = \sum_{\text{finite } \alpha} C_{N,\alpha}\, x^\alpha$$

where $C_N, \alpha \in \mathbb{C}$ and $x^\alpha \in \mathbb{R}$. Define $Q_N(x)$ to be the real part of polynomial P_N. Then

$$Q_N(x) = \sum_\alpha \mathrm{Re}(C_{N,\alpha})\ x^\alpha \in \mathbb{R}[x_1, ..., x_d]$$

and

$$\mathrm{Re}(f_N)(x) = e^{-\frac{x^2}{2}}\, Q_N(x) \in W_\mathbb{R}.$$

From here, the result follows by observing

$$\|f - \mathrm{Re}(f_N)\|^2_{L^2(\mathbb{R}^d,\mathbb{R})} = \|\mathrm{Re}(f - f_N)\|^2_{L^2(\mathbb{R}^d,\mathbb{R})} \le \|f - f_N\|^2_{L^2(\mathbb{R}^d,\mathbb{C})}$$

which goes to zero as N goes to infinity. □

4.2 Hermite Polynomials

We'll only define Hermite polynomials in one dimension setting before moving on to the multi-dimension definition. See Math 7305 notes for details on the one dimension Hermite polynomials.

Definition 4.2.1. For $n \in \mathbb{N}_0$, define the (n^{th}) standard Hermite Polynomial[1] as

$$H_n(x) := (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}. \tag{4.2}$$

Here is a list of the first six Hermite polynomials

$$H_0(x) = 1 \qquad H_3(x) = 8x^3 - 12x$$

$$H_1(x) = 2x \qquad H_4(x) = 16x^4 - 38x^2 + 12$$

$$H_2(x) = 4x^2 - 2 \qquad H_5(x) = 32x^5 - 160x^3 + 120x$$

From the following proposition (proven in Math 7305 summer class), $H_n(x)$ is a polynomial of degree n with leading coefficient 2^n.

Proposition 4.2.2. *For any $n \in \mathbb{N}_0$, $H_n(x) = \displaystyle\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}$.*

Since we have exactly one Hermite polynomial of degree n, it's possible to represent any polynomial over $\mathbb{K}$ as a linear sum of Hermite polynomials.

Proposition 4.2.3. $\{H_n(x)\}_{n\ge 0}$ *forms a basis for* $\mathbb{K}[x]$.

Furthermore, Proposition 4.2.2 also shows that the Hermite polynomials form an orthogonal system in $L^2(\mathbb{R}, \mathbb{K})$ with respect to weight e^{-x^2}.

Proposition 4.2.4. *Let $n, m \in \mathbb{N}_0$. Then*

$$\int_\mathbb{R} H_n(x) H_m(x) e^{-x^2}\, dx = 2^n n! \sqrt{\pi}\ \delta_{n,m}.$$

Proof. Repeated use of integration-by-parts. □

[1] There are many different equivalent formulations of Hermite polynomials.

Multi-Dimensional Hermite Polynomials

The results for Hermite polynomials in one-dimension can be extended to *multi-dimensional* Hermite polynomials.

Definition 4.2.5. Let $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$. Then the d-dimensional α-Hermite polynomial $H_\alpha(x)$ is the product of the one-dimensional Hermite polynomials on the components α_i. That is,

$$H_\alpha(x) := H_{\alpha_1}(x_1)H_{\alpha_2}(x_2)\cdots H_{\alpha_d}(x_d). \tag{4.3}$$

By construction, $H_\alpha(x)$ is separable, i.e., a product of different variable functions. Therefore orthogonality translates to the system of multi-dimensional Hermite polynomials

$$\int_{\mathbb{R}^d} H_\alpha(x)H_\beta(x)e^{-x^2}\,dx^d = 2^{|\alpha|}\,\alpha!\,\pi^{d/2}\ \mathbb{1}\{\alpha=\beta\}.$$

In addition, the d-dimensional Hermite polynomials form a basis on the polynomial ring $\mathbb{K}[x_1, \cdots, x_d]$. However, since $H_\alpha(x)$ are strictly polynomials for all $\alpha \in \mathbb{N}_0^d$,, they are not in $L^2(\mathbb{R}^d)$. The goal is to manipulate the Hermite polynomials enough such that not only their manipulated counterparts are in $L^2(\mathbb{R})$, but they also form an orthonormal basis. We will call this maniuplated function a *Hermite function*.

4.2.1 Hermite Functions

We'll first define these *Hermite functions* in one-dimension.

Definition 4.2.6. Let $d = 1$ and $n \in \mathbb{N}_0$. The n^{th}-Hermite function $h_n(x)$ is defined as a re-scaling of the n^{th} Hermite polynomial multiplied by an exponential

$$h_n(x) = \pi^{-1/4}\ 2^{-n/2}\ (n!)^{-1/2}\ e^{-x^2/2}\ H_n(x). \tag{4.4}$$

By construction, the Hermite functions form an *orthonormal* system in $L^2(\mathbb{R})$.

$$\begin{aligned}\int_{\mathbb{R}} h_n(x)h_m(x)\,dx &= \pi^{-1/2}\,2^{-n}(n!)^{-1}\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}\,dx \\ &= \pi^{-1/2}\,2^{-n}(n!)^{-1}\,(2^n n!\sqrt{\pi}\ \delta_{m,n}) \\ &= \delta_{m,n}\end{aligned}$$

Moreover the Hermite functions form an orthonormal *basis* in $L^2(\mathbb{R})$. Indeed by construction, the Hermite polynomials span the space $W_{\mathbb{K}} = \{e^{-x^2/2}p(x) : p(x) \in \mathbb{K}[x]\}$. We showed in Theorem 4.1.7 that $W_{\mathbb{K}}$ is a dense subspace of $L^2(\mathbb{R})$ and so $(h_n)_{n\geq 0}$ form a basis in $L^2(\mathbb{R})$.

This can be extended to multi-dimensions in an analogous way to Hermite polynomials. Namely if $\alpha \in \mathbb{N}_0^d$ where $d \geq 2$, define

$$\begin{aligned} h_\alpha(x) &\coloneqq h_{\alpha_1}(x_1)h_{\alpha_2}(x_2)\cdots h_{\alpha_d}(x_d) \\ &= \pi^{-1/4}\, 2^{-|\alpha|/2}\, (\alpha!)^{-1/2}\, (-1)^{|\alpha|}\, e^{x^2/2}\, \partial^\alpha e^{-x^2} \end{aligned} \tag{4.5}$$

As shown in dimension one, the multi-dimension Hermite functions form an orthonormal system in $L^2(\mathbb{R}^d)$.

Proposition 4.2.7. *For any* $\alpha, \beta \in \mathbb{N}_0^d$, $\langle h_\alpha\,,\, h_\beta\rangle = \mathbb{1}\{\alpha = \beta\}$.

Proof. Observe that

$$\langle h_\alpha\,,\, h_\beta\rangle = \prod_{i=1}^{d}\langle h_{\alpha_i}\,,\, h_{\beta_i}\rangle$$

which reduces to dimension 1 result proven above. □

Moreover, the multi-dimensional Hermite functions form an orthonormal basis in $L^2(\mathbb{R}^d)$. Before we present the proof, we need to define the following.

Definition 4.2.8. Let I be a countably infinite set, V a TVS, and $(v_i)_{i\in I}$ a family of elements in V. Then the sum $\sum_{i\in I} v_i$ is **unconditionally convergent** to $v \in V$ if and only if for any exhausting sequence of finite sets $\{\Lambda_N\}_{N\geq 1}$, $\lim_{N\to\infty}\sum_{i\in\Lambda_i} v_i = v$ in the sense of the topology on V.

Note that by "exhausting sequence", we mean

$$\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \bigcup_{N\geq 1}\Lambda_N = I.$$

Theorem 4.2.9. $(h_\alpha)\alpha \in \mathbb{N}_0^d$ *is a Hilbertian orthonormal basis of* $L^2(\mathbb{R}^d, \mathbb{K})$. *In particular,*

$$\sum_{\alpha\in\mathbb{N}_0^d}\langle h_\alpha\,,\, f\rangle_{L^2} h_\alpha = f \tag{4.6}$$

unconditionally for any $f \in L^2(\mathbb{R}^d, \mathbb{K})$.

Proof. Let Λ_N be an exhausting sequence and V_N be the linear span of h_α for $\alpha \in \Lambda_N$.Since there are finitely many h_α in V_N, it follows that V_N is a finite dimensional closed subspace of $L^2(\mathbb{R}, \mathbb{K})$. Define P_N to be the orthogonal projection on V_N, namely

$$P_N(f) = \sum_{\alpha\in\Lambda_N}\langle h_\alpha\,,\, f\rangle h_\alpha.$$

In L^2 norm of f, $\|f - P_N(f)\|_{L^2} \to 0$ as $N \to \infty$ and since $\cup V_n = W_{\mathbb{K}}$ which we know is dense in L^2, the desired sum follows. □

Theorem 4.2.10. *The map*

$$\Phi : L^2(\mathbb{R}^d,\, \mathbb{K}) \to \ell^2(\mathbb{N}_0^d, \mathbb{K})$$

where

$$f \mapsto (\langle h_\alpha\,,\, f\rangle)_{\alpha\in\mathbb{N}_0^d}$$

is an isomorphism of Hilbert spaces.

Proof. We'll prove the map is well=defined and surjective. The preservation of inner product is left as an exercise (difference between $\mathbb{R}$ and $\mathbb{C}$ cases).

Well-defined. Assume $f \in L^2(\mathbb{R}^d, \mathbb{K})$. Let P_N denote the projection operator on the subspaces V_N as defined previously. By application of Parseval,

$$\|f\|_{L^2}^2 = \lim_{N\to\infty} \|P_N(f)\|_{L^2}^2 = \lim_{N\to\infty} \sum_{\alpha\in\Lambda_N} |\langle h_\alpha\,,\, f\rangle|^2 = \sum_{\alpha\in\mathbb{N}_0^d} |\langle h_\alpha\,,\, f\rangle|^2$$

where the RHS is the definition of the (squared) $\ell^2(\mathbb{N}_0^d, \mathbb{K})$. Thus $f \in L^2$ insures $(\langle h_\alpha\,,\, f\rangle)_{\alpha\in\mathbb{N}_0^d} \in \ell^2$ and so the map Φ is well-defined. Furthermore, this also shows the map Φ is a linear isometry.

Surjective. Assume $(z_\alpha) \in \ell^2(\mathbb{N}_0^d)$. Pick ${}_N = \{\alpha : |\alpha| \leq N\}$ to be the set of multi-indices below level N. Let $f_N = \sum_{\alpha\in\Lambda_N} z_\alpha h_\alpha$ be finite z_α combination of Hermite functions. By construction, $f_N \in L^2(\mathbb{R}^d)$. If $M \leq N$, then

$$\|f_M - f_N\|_{L^2} = \|\sum_{\alpha\in\Lambda_N\setminus\Lambda_M} z_\alpha h_\alpha\| = \sum_{\alpha\in\Lambda_N\setminus\Lambda_M} |z_\alpha|^2$$

where the last equality is by orthonormality. Since $(z_\alpha) \in \ell^2$, this approaches 0 ans $N, M \to \infty$. Hence (f_N) is a Cauchy seqeunce in L^2 and L^2 complete immediately gives us (f_N) is a convergent sequence in L^2. Let $f \coloneqq \lim_{N\to\infty} f_N \in L^2$. It suffices to show $\langle h_\alpha\,,\, f\rangle = z_\alpha$ for all $\alpha \in \mathbb{N}_0^d$ but this follows from construction of f.

$$\langle h_\alpha\,,\, f\rangle_{L^2} = \lim_{N\to\infty} \langle h_\alpha\,,\, f_N\rangle_{L^2} = \lim_{N\to\infty} \mathbb{1}\{\alpha \in \Lambda_N\} z_\alpha = z_\alpha$$

Hence $\Phi(f) = (z_\alpha)_{\alpha\in\mathbb{N}_0^d}$ and so Φ is surjective. □

4.3 Proof of Sequence Space Representation

Recall from section 2.2.3 that the map

$$\begin{aligned} \Gamma_d \colon \mathcal{S}(\mathbb{R}^d, \mathbb{K}) &\to (\mathbb{N}_0^d, \mathbb{K}) \\ f &\mapsto \left(\int_{\mathbb{R}^d} h_\alpha(x) f(x)\, \mathrm{d}^d x\right)_{\alpha\in\mathbb{N}_0^d} \end{aligned}$$

is a TVS isomorphism. Since h_α is real-valued and $f \in \mathcal{S}$ is seen as a distribution because of the embedding $\mathcal{S} \hookrightarrow \mathcal{S}'$, we have

$$\int_{\mathbb{R}^d} h_\alpha(x) f(x)\, \mathrm{d}^d x = \langle h_\alpha | f \rangle_{L^2} = \langle f | h_\alpha \rangle.$$

Recall that $\mathscr{L}_1(\mathcal{S}, \mathcal{S})$ is the $\mathbb{K}$-VS of continuous linear maps from $\mathcal{S}$ to itself, where multiplication is composition and $[A, B] \coloneqq AB - BA$ is the commutator.

Recall that for all $i \in \mathbb{N}$, $\partial_i \in \mathscr{L}_1(\mathcal{S}, \mathcal{S})$, and so for any $\alpha \in \mathbb{N}_0^d$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \in \mathscr{L}_1(\mathcal{S}, \mathcal{S})$. Likewise, x^α are elements of $\mathscr{L}_1(\mathcal{S}, \mathcal{S})$, where x^α is viewed as the operator which is left multiplication by the monomial x^α. To see that this multiplication operator is continuous, let $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}_0^d$. Then $x^\alpha f \in C^\infty$, and using Leibnitz rule, we deduce

$$\begin{aligned}
\|x^\alpha f\|_{\beta,k} &\coloneqq \sup_{x \in U} \langle x \rangle^k \left| \partial^\beta x^\alpha f(x) \right| \\
&\leq \sup_{x \in U} \langle x \rangle^k \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} |\partial^\gamma x^\alpha| \left| \partial^{\beta-\gamma} f(x) \right| \\
&= \sup_{x \in U} \langle x \rangle^k \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \left| \mathbb{1}\{\gamma \leq \alpha\} \frac{\alpha!}{(\alpha-\gamma)!} x^{\alpha-\gamma} \right| \left| \partial^{\beta-\gamma} f(x) \right| \\
&\leq \sup_{x \in U} \langle x \rangle^k \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \mathbb{1}\{\gamma \leq \alpha\} \frac{\alpha!}{(\alpha-\gamma)!} \langle x \rangle^{|\alpha|-|\gamma|} \left| \partial^{\beta-\gamma} f(x) \right| \\
&\leq \sum_{\gamma \in \mathbb{N}_0^d} \mathbb{1}\{\gamma \leq \alpha, \beta\} \frac{\alpha! \beta!}{\gamma!(\alpha-\gamma)!(\beta-\gamma)!} \|f\|_{\beta-\gamma, |\alpha|-|\gamma|+k}
\end{aligned}$$

where the $\|f\|_{\beta-\gamma,|\alpha|-|\gamma|+k}$'s are continuous seminorms, hence we have continuity.

——————————— Next we will give some useful commutation relations. Since multiplication by monomials is commutative, we have for all $i, j \in \{1, ..., d\}$,

$$[x_i, x_j] = 0$$

By Schwarz's Theorem (or Clairaut's theorem on equality of mixed partials) we have $\forall i, j \in \{1, \ldots, d\}$,

$$[\partial_i, \partial_j] = 0$$

. Finally, for all $i, j \in \{1, \ldots, d\}$,

$$[\partial_i, x_j] = \delta_{ij} I,$$

where I is the identity operator on $\mathcal{S}$. Indeed, for any $f \in \mathcal{S}$, we have $[\partial_i, x_j] f(x) = (\partial_i x_j - x_j \partial_i) f(x)$, were, by Leibnitz we have that

$$\partial_i(x_j f(x)) = (\partial_i x_j) f(x) + x_j (\partial_i f)(x).$$

For all i, define

$$
\begin{aligned}
a_i &\coloneqq \frac{1}{\sqrt{2}}(x_i + \partial_i) && \text{(the \underline{annihilation separator})}\\
a_i^* &\coloneqq \frac{1}{\sqrt{2}}(x_i - \partial_i) && \text{(the \underline{creation separator})}\\
\theta_i &\coloneqq \frac{1}{2}(x_i^2 - \partial_i^2) && \text{(the \underline{harmonic oscillator})}\\
1 &= [\partial_i, x_i] && \text{(why??)}
\end{aligned}
$$

The terminology comes from quantum mechanics and a system corresponding to a harmonic oscillator.

Note that we have

$$\theta_i = a_i^* a_i + \frac{1}{2}$$

Indeed,

$$
\begin{aligned}
2a_i^* a_i &= 2\frac{1}{\sqrt{2}}(x_i - \partial_i)\frac{1}{\sqrt{2}}(x_i + \partial_i)\\
&= (x_i - \partial_i)(x_i + \partial_i)\\
&= x_i^2 - \partial_i x_i + x_i \partial_i - \partial_i^2\\
&= x_i^2 + -[\partial_i, x_i] - \partial_i^2\\
&= 2\theta_i - 1
\end{aligned}
$$

From this, we get the following commutation relations for all $i, j \in \{1, ..., d\}$.

- $[a_i, a_j] = 0$
- $[a_i^*, a_j^*] = 0$
- $[a_i, a_j^*] = \delta_{ij} I$

  proof: For this, we need only to check when $i = j$.

$$
\begin{aligned}
[a_i, a_i^*] &= \frac{1}{2}[x_i + \partial_i, x_i - \partial_i]\\
&= \frac{1}{2}\left([\partial_i, x_i] - [x_i, \partial_i]\right)\\
&= \frac{1}{2}\left(I - (-I)\right)\\
&= I
\end{aligned}
$$

- $\forall n \geq 1, [a_i, (a_i^*)^n] = n(a_i^*)^{n-1}$

proof: Proceed by induction, where the case $n = 1$ was shown above. For $n \geq 1$,

$$\begin{aligned}
[a_i, (a_i^*)^{n+1}] &= a_i(a_i^*)^{n+1} - (a_i^*)^{n+1}a_i \\
&= a_i(a_i^*)^n a_i^* - (a_i^*)^n a_i^* a_i \\
&= ([a_i, (a_i^*)^n] + (a_i^*)^n a_i)\, a_i^* - (a_i^*)^n a_i^* a_i \\
&= [a_i, (a_i^*)^n]a_i^* + (a_i^*)^n a_i a_i^* - (a_i^*)^n a_i^* a_i \\
&= [a_i, (a_i^*)^n]a_i^* + (a_i^*)^n [a_i, a_i^*] \\
&= n(a_i^*)^{n-1} a_i^* + (a_i^*)^n \qquad \text{(by the I.H.)} \\
&= (n+1)(a_i^*)^n
\end{aligned}$$

Lemma 4.3.1. *The following are true:*

1. $a_i^* h_\alpha = \sqrt{\alpha_i + 1}\; h_{\alpha+e_i}$
2. $a_i h_\alpha = \begin{cases} \sqrt{\alpha_i}\; h_{\alpha-e_i} & \text{if } \alpha_i \geq 1 \\ 0 & \text{if } \alpha_i = 0 \end{cases}$
3. $\theta_i h_\alpha = (\alpha_i + \frac{1}{2})h_\alpha$, *i.e. the* h_α *are joint eigenvectors for the* θ_i.
4. $h_\alpha = \alpha!^{-\frac{1}{2}}(a_1^*)^{\alpha_1} \cdots (a_d^*)^{\alpha_1 d} h_0$.

Proof. Recall that

$$\partial^\alpha(e^{-x_1^2} \cdots e^{-x_d^2}) = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}(e^{-x_1^2} \cdots e^{-x_d^2}) = (\partial_1^{\alpha_1} e^{-x_1^2}) \cdots (\partial_d^{\alpha_d} e^{-x_d^2}).$$

Then

$$h_\alpha(x) = \pi^{-\frac{d}{4}} 2^{-\frac{|\alpha|}{2}} \alpha!^{-\frac{1}{2}} (-1)^{|\alpha|} e^{\frac{x^2}{2}} \partial^\alpha e^{-x^2}.$$

Applying a_i^* gives

$$\begin{aligned}
\partial_i h_\alpha(x) &= \left(\pi^{-\frac{d}{4}} 2^{-\frac{|\alpha|}{2}} \alpha!^{-\frac{1}{2}} (-1)^{|\alpha|}\right) \partial_i e^{\frac{x^2}{2}} \partial^\alpha e^{-x^2} \\
&= \pi^{-\frac{d}{4}} 2^{-\frac{|\alpha|}{2}} \alpha!^{-\frac{1}{2}} (-1)^{|\alpha|} \partial_i e^{\frac{x^2}{2}} \left(x_i e^{\frac{x^2}{2}} \partial^\alpha e^{-x^2} + e^{\frac{x^2}{2}} \partial^{\alpha+e_i} e^{-x^2}\right)
\end{aligned}$$

and so

$$\begin{aligned}
a_i^* h_\alpha(x) &= \frac{1}{\sqrt{2}} \left(\pi^{-\frac{d}{4}} 2^{-\frac{|\alpha|}{2}} \alpha!^{-\frac{1}{2}} (-1)^{|\alpha|}\right) (-1) e^{\frac{x^2}{2}} \partial^{\alpha+e_i} e^{-x^2} \\
&= \sqrt{\alpha_i + 1}\; h_{\alpha+e_i}(x),
\end{aligned}$$

as desired. Now, for any $\alpha = \alpha_1 e_1 + \cdots + \alpha_d e_d$, where e_i are the standard basis vectors, we get

$$h_\alpha = \alpha!^{-\frac{1}{2}} (a_1^*)^{\alpha_1} \cdots (a_d^*)^{\alpha_d} h_0$$

by starting at $0 = (0, ..., 0)$ and iterating the identity $a_i^* h_\alpha = \sqrt{\alpha_i + 1}$.
For (2), first note that since

$$h_0 = \pi^{-\frac{d}{4}} e^{-\frac{x^2}{2}}$$

the computation

$$\partial_i h_0(x) = \pi^{-\frac{d}{4}} (-x_i) e^{-\frac{x^2}{2}} = -x_i h_0(x)$$

shows that $a_i h_0 = 0$. [2]

Now, for $\alpha \neq 0$, we have for each i,

$$\begin{aligned} a_i h_\alpha &= \alpha!^{-\frac{1}{2}} a_i \prod_{j=1}^{d} (a_j^*)^{\alpha_j} h_o \\ &= \alpha!^{-\frac{1}{2}} a_i \left(\prod_{j \neq i} (a_j^*)^{\alpha_j} \right) a_i (a_i^*)^{\alpha_i} h_o \\ &= \alpha!^{-\frac{1}{2}} a_i \left(\prod_{j \neq i} (a_j^*)^{\alpha_j} \right) \left([a_i, (a_i^*)^{\alpha_i}] + (a_i^*)^{\alpha_i} a_i \right) h_o \\ &= \alpha!^{-\frac{1}{2}} a_i \left(\prod_{j \neq i} (a_j^*)^{\alpha_j} \right) \left(\alpha_i (a_i^*)^{\alpha_i - 1} + (a_i^*)^{\alpha_i} a_i \right) h_o \\ &= \alpha!^{-\frac{1}{2}} a_i \left(\prod_{j \neq i} (a_j^*)^{\alpha_j} \right) (\alpha_i (a_i^*)^{\alpha_i - 1}) h_o. \qquad \text{(since } a_i h_0 = 0\text{)} \end{aligned}$$

Now, if $\alpha_i = 0$, then we have $a_i h_\alpha = 0$. On the other hand, if $\alpha_i \geq 1$, then the above is equal to

$$\begin{aligned} a_i h_\alpha &= \alpha_i \alpha!^{-\frac{1}{2}} a_i \left(\prod_{j \neq i} (a_j^*)^{\alpha_j} \right) \left((a_i^*)^{\alpha_i - 1} \right) h_o \\ &= \sqrt{\alpha_i} h_{\alpha - e_i}. \end{aligned}$$

[2]It jumps by "quanta". Quanta are discrete, and this is where the term "quantum mechanics" comes from!

To show (3), we simply compute

$$\begin{aligned}
\theta_i h_\alpha &= \left(a_i^* a_i + \frac{1}{2}\right) h_\alpha \\
&= a_i^* a_i h_\alpha + \frac{1}{2} h_\alpha \\
&= a_i^* \sqrt{\alpha_i}\; h_{\alpha - e_i} + \frac{1}{2} h_\alpha && \text{(using above fact)} \\
&= \sqrt{\alpha_i}\sqrt{(\alpha_i - 1) + 1}\; h_{(\alpha - e_i) + e_i} + \frac{1}{2} h_\alpha && \text{(using corresponding * fact)} \\
&= \sqrt{\alpha_i}\sqrt{\alpha_i}\; h_\alpha + \frac{1}{2} h_\alpha \\
&= \left(\alpha_i + \frac{1}{2}\right) h_\alpha,
\end{aligned}$$

which concludes the proof of the lemma. □

Proposition 4.3.2. *Γ_d is well-defined and continuous.*

Proof. $f \in \mathcal{S}$ and $k \in \mathbb{N}_0$.

$$\begin{aligned}
&\sum_{\alpha \in \mathbb{N}_0^d} \prod_{i=1}^d \left(\alpha_i + \frac{1}{2}\right)^{2k} |\langle h_\alpha, f\rangle_{L^2}|^2 \\
&\leq \sum_{\alpha \in \mathbb{N}_0^d} \left| \left\langle \prod_{i=1}^d \left(\alpha_i + \frac{1}{2}\right)^k h_\alpha, f \right\rangle_{L^2} \right|^2 \\
&= \sum_{\alpha \in \mathbb{N}_0^d} \left| \left\langle \theta_1^k \cdots \theta_d^k h_\alpha, f \right\rangle_{L^2} \right|^2 \\
&= \sum_{\alpha \in \mathbb{N}_0^d} \left| \left\langle h_\alpha, \theta_1^k \cdots \theta_d^k f \right\rangle_{L^2} \right|^2 && \text{(by integration by parts)} \\
&= \left\| \theta_1^k \cdots \theta_d^k f \right\|_{L^2} && \text{(by Parseval)} \\
&< \infty.
\end{aligned}$$

Note that this weighted L^2 seminorms composed with $\sqrt{\cdot}$ is a continuous seminorm. So, Γ_d is continuous on $\mathcal{S}$. □

Note that we can use the separating family $\|\cdot\|_{p,k}$ of L^2 seminorms instead of $\|\cdot\|_{\infty,k}$.

We've seen already that for k ranging through $\mathbb{N}_0$ and all $p \in [1, \infty)$, the following is a defining collection of seminorms for $\mathfrak{s}(\mathbb{N}_0)$:

$$\|z\|_{p,k} = \Big(\sum_{\alpha \in \mathbb{N}_0^d} \langle \alpha \rangle^k |z_\alpha|^p \Big)^{1/p}.$$

We consider $p = 2$:

$$\|\Gamma_d(f)\|_{2,k}^2 = \sum_{\alpha \in \mathbb{N}_0^d} \langle\alpha\rangle^k |z_\alpha|^2$$

$$\langle\alpha\rangle^2 = 1 + \alpha_1^2 + \cdots + \alpha_d^2 \leq \prod_{i=1}^{d}(1 + \alpha_i^2)$$

Note that

$$\langle\alpha\rangle \leq 2^d \prod_{i=1}^{d}(\alpha_i + \frac{1}{2})$$

and

$$\|\Gamma_d(f)\|_{2,k}^2 \leq 2^{kd} \sum_{\alpha} \prod_{i} (\alpha_i + \frac{1}{2})^k |\langle h_\alpha, f\rangle_{L^2}|^2$$
$$\leq 2^{2k} ||\theta_1^k \dots \theta_d^k f||_{L^2}^2$$

where $\theta_1^k \dots \theta_d^k$ is a continuous operator from $\mathcal{S}$ to $\mathcal{S}$. Furthermore, $||.||_{L^2}$ is a continuous seminorm on $\mathcal{S}(\mathbb{R}^d)$. We have that for $g \in \mathcal{S}$,

$$||g||_{L^2}^2 = \int \langle x\rangle^{-(d+1)} \langle x\rangle^{d+1} |g(x)|^2 d^d x$$

$$\leq ||g||_{\theta, \lceil \frac{d+1}{2} \rceil}^2 \int_{\mathbb{R}^d} \langle x\rangle^{-(d+1)} d^d x.$$

We've proven that the following map is continuous:

$$\Gamma : \mathcal{S}(\mathbb{R}^d) \to \mathfrak{s}(\mathbb{R}^d).$$

Now, define:

$$\Xi_d : \mathfrak{s}(\mathbb{N}_0^d) \to \mathcal{S}(\mathbb{R}^d)$$
$$Z = (z_\alpha)_{\alpha \in \mathbb{N}_0^d} \mapsto \lim_{N \to \infty} \sum_{|\alpha| \leq N} z_\alpha h_\alpha.$$

We'll show Ξ_d is well-defined, continuous, and inverse to Γ_d.

Lemma 4.3.3. *For all $\beta \in \mathbb{N}_0^d$ and for all $k \in \mathbb{N}_0$, there exists $c < 0$ and exists $m \in \mathbb{N}_0$ such that for all $\alpha \in \mathbb{N}_0^d$,*

$$\|h_\alpha\|_{\beta,k} \leq c\langle\alpha\rangle^m.$$

We have that c, m are independent from α but can depend on β, k.

Proof. Now we complete the proof of the theorem (not the lemma above!). For z fixed, we consider:

$$f_N = \sum_{|\alpha| \leq N} z_\alpha k_\alpha \in \mathcal{S}$$

$$M \leq N \Rightarrow \|f_N - f_M\|_{\beta,k} = \|\sum_{M\leq|\alpha|\leq N} z_\alpha h_\alpha\|_{\beta,k}$$
$$\leq \sum_{M\leq|\alpha|\leq N} |z_\alpha|\|h_\alpha\|_{\beta,k}$$

because

$$\sum_{\alpha\in\mathbb{N}_0^d} |z_\alpha|\|h_\alpha\|_{\beta,k} < \infty$$

and indeed by lemma 1, for $z \in \mathfrak{s}(\mathbb{N}_0^d)$,

$$\sum_{\alpha\in\mathbb{N}_0^d} |z_\alpha|\|h_\alpha\|_{\beta,k} \leq \sum_\alpha |z_\alpha| C\langle\alpha\rangle^m < \infty$$
$$\Rightarrow (f_N) \text{ is Cauchy for all } \|.\|_{\beta,k}$$
$$\Rightarrow \text{we have a convergent sequence, } f = \lim_{N\to\infty} f_N$$
$$\Rightarrow \Xi_d \text{ is well-defined, linear, and continuous.}$$
$$\|\Xi_d(x)\|_{\beta,k} = \|f\|_{\beta,k} \leq c\|z\|_{\beta,k}$$

where we're employing a continuous semi-norm on $\mathfrak{s}(\mathbb{N}_0^d)$. Now we consider:

$$z \in \mathfrak{s}, \Gamma_d \circ \Xi_d(z) = \Gamma_d\left(\lim_{N\to\infty}\sum_{|\alpha|\leq N} z_\alpha h_\alpha\right)$$
$$= \lim_{N\to\infty}\Gamma_d\left(\sum_{|\alpha|\leq N} z_\alpha h_\alpha\right)$$
$$= \lim_{N\to\infty}\sum_{|\alpha|\leq N} z_\alpha \Gamma_d(h_\alpha).$$

we have that

$$\Gamma_d(h_\alpha) = \Big(\langle h_\beta, h_\alpha\rangle_{L^2}\Big)_{\beta\in\mathbb{N}_0^d}$$

where $\langle h_\beta, h_\alpha\rangle_{L^2} = \mathbb{1}\{\beta = \alpha\}$. Therefore, we have the following:

$$\Gamma_d \circ \Xi_d(z) = \lim_{N\to\infty}(\mathbb{1}\{|\alpha| \leq N\} z_\alpha)_{\alpha\in\mathbb{N}_0^d} \to z$$
$$\Rightarrow \Gamma_d \circ \Xi_d = \mathrm{Id}_{\mathfrak{s}}$$
$$\Xi_d \circ \Gamma_d(f) = \lim_{N\to\infty}\sum_{|\alpha|\leq N}\langle h_\alpha, f\rangle h_\alpha =: g \in \mathcal{S}$$
$$g \to f \text{ in } L^2$$
$$f = g \text{ in } L^2$$
$$f, g \text{ are the same functions in } \mathcal{S}$$
$$\Rightarrow \Xi_d \circ \Gamma_d = Id_{\mathfrak{s}}$$

□

Proof of Lemma. Call a seminorm $\|\cdot\|$ on $\mathcal{S}(\mathbb{R}^d)$ **good** if for all $c > 0$ there is $m \in \mathbb{N}_0$ such that for all $\alpha \in \mathbb{N}_0^d$,

$$\|h_\alpha\| \leq c\langle\alpha\rangle^m$$

We need to show that for all $\beta \in \mathbb{N}_0^d$ and $k \geq 0$, $\|\cdot\|_{\beta,k}$ is good. First, we claim that the set $\mathcal{N}$ of of good seminorms has the property that if ρ is a seminorm continuous relative to $\mathcal{N}$, then $\rho \in \mathcal{N}$.

Indeed, for such ρ, there exist $\tau_1, \dots, \tau_n \in \mathcal{N}$ and $c > 0$ such that

$$\rho \leq c(\tau_1 + \cdots + \tau_n)$$

By hypothesis, there exist $A_i \geq 0$ and $m_i \in \mathbb{N}_0$, $1 \leq i \leq n$, such that for all $\alpha \in \mathbb{N}_0^d$,

$$\tau_i(h_\alpha) \leq A_i\langle\alpha\rangle^{m_i} \leq A\langle\alpha\rangle^m$$

where $A = \max\{A_1, \dots, A_n\}$ and $m = \max\{m_1, \dots, m_n\}$. That is, for all $\alpha \in \mathbb{N}_0^d$,

$$\rho(h_\alpha) \leq cnA\langle\alpha\rangle^m$$

We now show that each $\|\cdot\|_{\alpha,k}$ is continuous relative to $\mathcal{N}$. Recall that if $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$\|f\|_{\alpha,k} := \sup_x \langle x\rangle^k |\partial^\alpha f(x)|$$

Expanding the right-hand side of the inequality

$$\langle x\rangle^k \leq (1 + x^2)^k,$$

we see that $\|\cdot\|_{\alpha,k}$ is continuous relative to the seminorms of the form

$$\|f\|'_{\alpha,\beta} := \sup_x |x^\beta \partial^\alpha f(x)|, \qquad \beta \in \mathbb{N}_0^d$$

We now invoke the Fourier inversion formula:

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi x} \hat{f}(\xi)\ d^d\xi.$$

As in the proof that $\mathcal{F}(\mathcal{S}) \subseteq \mathcal{S}$, integration by parts and Fubini's theorem give

$$x^\beta \partial^\alpha f(x) = \frac{i^{|\alpha|+|\beta|}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi} \partial_\xi^\beta \left(\xi^\alpha\ \hat{f}(\xi)\right)\ d^d\xi$$

Thus, by Cauchy-Schwarz,

$$\begin{aligned} |x^\beta \partial^\alpha f(x)| &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \langle\xi\rangle^{-(d+1)} \langle\xi\rangle^{d+1} \left|\partial_\xi^\beta \left(\xi^\alpha\ \hat{f}(\xi)\right)\right|\ d^d\xi \\ &\leq \frac{1}{(2\pi)^d} \|\langle\xi\rangle^{-(d+1)}\|_{L^2}\ \|(1+\xi^2)^{d+1} \partial_\xi^\beta (\xi^\alpha\ \hat{f}(\xi))\|_{L^2} \end{aligned}$$

After expanding the last expression above using the Leibnitz rule and the triangle inequality, we see that $\|f\|'_{\alpha,\beta}$ is continuous relative to seminorms of the form $\|f\|''_{\gamma,\delta} := \|\xi^\gamma \partial^\delta \hat{f}\|_{L^2}$.

Recall that $\partial^\delta \hat{f}(\xi) = \mathscr{F}[(-ix)^\delta f(x)](\xi)$; that is,

$$\partial^\delta \circ \mathscr{F} = \mathscr{F} \circ (-i)^{|\delta|} x^\delta$$

Integration by parts gives us

$$\begin{aligned}
\xi^\gamma \hat{f}(\xi) &= \int_{\mathbb{R}^d} (i\partial_x)^\gamma \left(e^{-ix\xi}\right) f(x)\, d^d x \\
&= \int_{\mathbb{R}^d} e^{-ix\xi} \, (-i\partial_x)^\gamma (f(x))\, d^d x \\
&= \mathscr{F}[(-i\partial_x)^\gamma f(x)](\xi)
\end{aligned}$$

That is,

$$\xi^\gamma \circ \mathscr{F} = F \circ (-i)^{|\gamma|} \partial^\gamma$$

Putting these two observations together yields

$$\xi^\gamma \circ \partial^\delta \circ \mathscr{F} = (-i)^{|\delta|+|\gamma|} \mathscr{F} \circ \partial^\gamma \circ x^\delta$$

so that

$$\begin{aligned}
\|f\|''_{\gamma,\delta} &= \|\mathscr{F}[\partial_x^\gamma (x^\delta \, f(x))]\|_{L^2} \\
&= (2\pi)^{d/2} \|\partial^\gamma (x^\delta \, f(x))\|_{L^2} && \text{(Plancherel)}
\end{aligned}$$

By again expanding via the Leibnitz rule and triangle inequality, we find that $\|f\|''_{\gamma,\delta}$ is continuous relative to seminorms of the form

$$\|f\|'''_{\alpha,\beta} := \|x^\alpha \, \partial^\beta f(x)\|_{L^2}$$

Because the seminorms $\|\cdot\|_{\beta,k}$, which define the topology on $\mathcal{S}(\mathbb{R}^d)$, are continuous relative to the continuous seminorms $\|\cdot\|'''_{\alpha,\beta}$ above, the latter form a defining collection of seminorms for the topology on $\mathcal{S}(\mathbb{R}^d)$. Let a_i, a_i^* $(1 \leq i \leq d)$ be the annihilation and creation operators, respectively, which were defined in section 4.3. For $B = (b_1, b_2, \ldots, b_n)$ a finite sequence in $\{a_1, \ldots, a_d, a_1^*, \ldots, a_d^*\} \subseteq \mathscr{L}_1(\mathcal{S}, \mathcal{S})$, define yet another seminorm $\|\cdot\|_B^{(4)}$ by

$$\|f\|_B^{(4)} := \|b_1 b_2 \cdots b_n f\|_{L^2}$$

Then $\|\cdot\|'''_{\alpha,\beta}$ is continuous relative to $\|\cdot\|_B^{(4)}$ because

$$x_i = \frac{a_i + a_i^*}{\sqrt{2}} \quad \text{and} \quad \partial_i = \frac{a_i - a_i^*}{\sqrt{2}}$$

The last step in the proof is to show that each $\|\cdot\|_B^{(4)}$ is good. Recall that

$$a_i^* h_\alpha = \sqrt{\alpha_i + 1} h_{\alpha+e_i} \quad \text{and} \quad a_i h_\alpha = \begin{cases} \sqrt{\alpha_i}\, h_{\alpha_i - e_i}, & \alpha_i \geq 1 \\ 0, & \alpha_i = 0 \end{cases}$$

Thus, $b_{1n} h_\alpha = \sqrt{C}\, h_\beta$ for some $\beta \in \mathbb{N}_0^d$ and constant C which is a product of numbers bounded above by $|\alpha| + n + 1$. Therefore, for any $c > 0$ and B fixed, we can choose $m \in \mathbb{N}_0$ large enough so that

$$\|h_\alpha\|_B^{(4)} \leq (|\alpha| + n + 1)^n \|h_\alpha\|_{L^2} = (|\alpha| + n + 1)^n \leq c\langle\alpha\rangle^m,$$

showing that $\|\cdot\|_B^{(4)}$ is good. □

We introduce a general lemma about topological vector spaces.

Lemma 4.3.4. *Assume X and Y are topological vector spaces and that $T : X \to Y$ is continuous and linear. If $A \subset X$ is bounded, then $T(A) \subset Y$ is bounded.*

Proof. We apply the general definition of "bounded" for subsets of TVS's. Let $U \subset Y$ be an open neighborhood of the origin. Then $T^{-1}(U)$ is an open neighborhood of the origin in X. By definition, then, there exists $\lambda > 0$ such that $\lambda A \subset T^{-1}(U)$. By linearity of T,

$$\lambda T(A) = T(\lambda A) \subset T(T^{-1}(U)) \subset U,$$

and so $T(A)$ is bounded. □

The strong dual construction for topological vector spaces can be viewed as a *contravariant functor* from the category of TVS's to itself. This functor sends a TVS V to its strong dual V' and sends a linear map (morphism) $T : X \to Y$ to its *transpose* $T' : Y' \to X'$, the map such that $T'(L) = L \circ T$ for all $L \in Y'$. The adjective "contravariant" refers to the fact that the functor reverses morphisms' arrows. By definition, strong duals are always locally convex. So, this functor maps into the *full subcategory* of locally convex TVS's, a subcategory of the category of TVS'S. The adjective "full" means if X, Y are members of the subcategory, then $T : X \to Y$ is a morphism (continuous linear map) regardless of whether it is viewed as a member of the original category or the subcategory. This is indeed the case, since linearity is an intrinsic property of T.

We include a quick verification that the transpose is indeed a functor of TVS's. That is, it sends continuous linear maps to continuous linear maps. The fact that it preserves linearity is evident. And if A is bounded in X, then

$$\|T'(L)\|_A = \sup_{x\in A} |T'(L)(x)| = \sup_{x\in A} |L(T(x))| = \sup_{y\in T(A)} |L(y)| = \|L\|_{T(A)}.$$

Since the collection of seminorms $\|\cdot\|_A$ for A bounded defines the strong operator topology on Y' and each $T(A)$ is bounded by the previous Lemma, then T' is continuous by Theorem 2.1.21.

The following relation between transposes and composition is also evident:

$$(T_1T_2)' = T_2'T_1'$$

for any linear maps $T_1 : Y \to Z$, $T_2 : X \to Y$ and TVS's X, Y, Z.

In sections 4.4 and 2.2.2, we showed that we have the following chain of TVS-isomorphisms (we are renaming the map "t" as A_d).

$$S(\mathbb{R}^d) \xrightarrow{\Gamma_d} \mathscr{S}(\mathbb{N}_0^d) \xrightarrow{A_d} \mathscr{S}(\mathbb{N}_0) = \mathscr{S}.$$

Applying the transpose functor yields TVS-isomorphisms between strong duals:

$$\mathscr{S}(\mathbb{N}_0)' = (\mathscr{S})' \xrightarrow{A_d'} \mathscr{S}(\mathbb{N}_0^d)' \xrightarrow{\Gamma_d'} S'(\mathbb{R}^d).$$

In Section 2.2.6, we showed there is a TVS-isomorphism $\mathscr{I}$ between the abstract strong dual $(\mathscr{S})'$ and the sequence space $\mathscr{S}' \subset \mathbb{K}^{\mathbb{N}_0}$. Hence, viewing $S'(\mathbb{R}^d)$ as TVS-isomorphic to $(\mathscr{S})'$, we have a non-explicit isomorphism between $S'(\mathbb{R}^d)$ and a sequence space. But it is useful to produce such an isomorphism explicitly.

First, we introduce some generalizations of previous notation:

1. $\mathscr{S}_0'(\mathbb{N}_0^d) := \mathbb{K}^{\mathbb{N}_0}$ denotes the space of all multisequences.
2. $\mathscr{S}_{0,+}'(\mathbb{N}_0^d) := [0,\infty)^{\mathbb{N}_0}$ denotes the space of all non-negative multisequences.
3. For $\omega \in \mathscr{S}_{0,+}'(\mathbb{N}_0^d)$ and $x \in \mathscr{S}_0'(\mathbb{N}_0^d)$, let $\|x\|_\omega := \sum_{\alpha\in\mathbb{N}_0^d} \omega_\alpha |x_\alpha|$.
4. $\mathscr{S}'(\mathbb{N}_0^d) := \{x \in \mathscr{S}_0'(\mathbb{N}_0^d) : \exists C > 0\, K \in \mathbb{N}_0\, \forall \alpha \in \mathbb{N}_0^d\ , |x_\alpha| \le C\langle\alpha\rangle^k$. This is the space of multisequences with at most polynomial growth.
5. $\mathscr{S}_+(\mathbb{N}_0^d) := \mathscr{S}(\mathbb{N}_0^d) \cap \mathscr{S}_{0,+}'(\mathbb{N}_0^d)$. This is the space of rapidly decaying non-negative multisequences.
6. If $x, y \in \mathscr{S}_0'(\mathbb{N}_0^d)$, we set $\langle x, y\rangle := \sum_{\alpha\in\mathbb{N}_0^d} x_\alpha y_\alpha$, if this series is absolutely convergent.

Let C denote the counting measure on $\mathbb{N}_0^d$ and let $x \in \mathscr{S}_0'(\mathbb{N}_0^d)$. Then it is convenient to write

$$\|x\|_\omega = \int_{\alpha \in \mathbb{N}_0^d} w_\alpha |x_\alpha| \, dC(\alpha).$$

Recall (Sect. 2.2.6) that for all $d \geq 2$ there is a bijection $\rho_d : \mathbb{N}_0 \to \mathbb{N}_0^d$ such that for some constants $C_1, C_2 > 0$, for all $n \geq 0$, we have

$$\langle \rho_d(n) \rangle \leq C_1 \langle n \rangle$$

and

$$\langle n \rangle \leq C_2 \langle \rho_d(n) \rangle^d.$$

From this bijection, we obtain a map $A_d : \mathscr{S}_0'(\mathbb{N}_0^d) \to \mathscr{S}_0'(\mathbb{N}_0)$ given by $A_d(x) := x \circ \rho_d = (x_{\rho_d(n)})_{n \geq 0}$ for $x = (x_\alpha)_{\alpha \in \mathbb{N}_0^d}$. This is $\mathbb{K}$-vector space isomorphism. Since $\mathscr{S}(\mathbb{N}_0^d) \subset \mathscr{S}_0'(\mathbb{N}_0^d)$ and $\mathscr{S}(\mathbb{N}_0) \subset \mathscr{S}_0'(\mathbb{N}_0)$, we also let A_d denote the restricted map $A_d : \mathscr{S}(\mathbb{N}_0^d) \to \mathscr{S}(\mathbb{N}_0)$, which we showed is a well-defined TVS-isomorphism.

Hence, for $x = (x_\alpha) \in \mathscr{S}_0'(\mathbb{N}_0^d)$ and $\omega = (\omega_\alpha) \in \mathscr{S}_{0,+}'$, we have

$$\begin{aligned} \|x\|_\omega &= \int_{\alpha \in \mathbb{N}_0^d} \omega_\alpha |x_\alpha| \, dC(\alpha) = \int_{\alpha \in \mathbb{N}_0} \omega_{\rho_d(n)} |x_{\rho_d(n)}| \, d(\rho_d^* C)(\alpha) \\ &= \int_{\alpha \in \mathbb{N}_0} \omega_{\rho_d(n)} |x_{\rho_d(n)}| \, dC(\alpha) = \|(x_{\rho_d(n)})_{n \geq 0}\|_{(\omega_{\rho_d(n)})_{n \geq 0}} < \infty, \end{aligned}$$

since $\rho_d^* C = C$.

From this computation and the growth property of ρ_d, it follows that $\|\cdot\|_\omega$ defines a seminorm on $\mathscr{S}'(\mathbb{N}_0^d)$ for $\omega \in \mathscr{S}_+(\mathbb{N}_0^d)$. We define the topology on $\mathscr{S}'(\mathbb{N}_0^d)$ to be the locally convex topology generated by these seminorms $\|\cdot\|_\omega$.

Theorem 4.3.5. *The map $\mathscr{I}_d : (\mathscr{S}(\mathbb{N}_0^d))' \to \mathscr{S}'(\mathbb{N}_0^d)$ given by $L \mapsto (L(e_\alpha))_{\alpha \in \mathbb{N}_0^d}$ is a TVS-isomorphism. As before, for all $x \in \mathscr{S}(\mathbb{N}_0^d)$ and $L \in \mathscr{S}(\mathbb{N}_0^d)'$ we have*

$$L(x) = \langle \mathscr{I}_d(L), x \rangle = \sum_{\alpha \in \mathbb{N}_0^d} L(e_\alpha) x_\alpha.$$

Proof. The case $d = 1$ was proven in Section 2.2.6. So we assume $d \geq 2$. First, we observe that $A_d : \mathscr{S}'(\mathbb{N}_0^d) \to \mathscr{S}'(\mathbb{N}_0)$ is a TVS-isomorphism. A_d is well-defined; if $x = (x_\alpha)_{\alpha \in \mathbb{N}_0^d}$, then there exists some constants $C > 0$, $k \geq 1$ such that $|x_\alpha| \leq C \langle \alpha \rangle^k$ for all α. Hence, for some constant $C' > 0$, for all $n \geq 0$ we have

$$|A_d(x)_n| = |x_{\rho_d(n)}| \leq C \langle \rho_d(n) \rangle^k \leq CC' \langle n \rangle^k,$$

i.e. $A_d(x) \in \mathscr{S}'(\mathbb{N}_0)$. A_d is evidently bijective since ρ_d is a bijection. Linearity is also immediate.

We observe that if $\omega = (\omega_n)_{n\geq 0} \in \mathscr{S}_+(\mathbb{N}_0^d)$, then since the bijection ρ_d grows at most polynomially, $\omega \circ \rho_d^{-1} = (\omega_{\rho_d^{-1}(\alpha)})_{\alpha\in\mathbb{N}_0^d} \in \mathscr{S}_+(\mathbb{N}_0)$. To see this, let $C, k > 0$ be such that $|w_\alpha| \leq C\langle\alpha\rangle^k$ for all α. Then for all $n \geq 0$,

$$|\omega_{\rho_d^{-1}(\alpha)}| \leq C\langle\rho_d^{-1}(n)\rangle^k \leq CC'\langle n\rangle^k$$

for some constant $C' > 0$, since ρ_d grows at most polynomially. Similarly, if $\nu = (\nu_n)_{n\geq 0} \in \mathscr{S}_+$, then $\nu \circ \rho_d \in \mathscr{S}_+(\mathbb{N}_0)$.

Continuity follows from the observations that

$$\|A_d(x)\|_\nu = \|x\|_{\nu\circ\rho_d^{-1}}$$

and

$$\|A_d^{-1}(y)\|_\omega = \|y\|_{\omega\circ\rho_d}$$

for $x \in \mathscr{S}'(\mathbb{N}_0^d)$, $y \in \mathscr{S}'(\mathbb{N}_0)$, $\omega \in \mathscr{S}_+(\mathbb{N}_0^d)$, and $\nu \in \mathscr{S}_+(\mathbb{N}_0)$. As observed, $\omega \circ \rho_d \in \mathscr{S}_+(\mathbb{N}_0)$, and $\nu \circ \rho_d^{-1} \in \mathscr{S}_+(\mathbb{N}_0^d)$. Hence, A_d and A_d^{-1} are continuous by our criteria for continuity of linear operators on locally convex TVS's.

Next, we observe that $\mathcal{J}_d$ can be written as the following composition of TVS-isomorphisms:

$$(\mathscr{S}(\mathbb{N}_0^d)' \xrightarrow{(A_d')^{-1}} \mathscr{S}(\mathbb{N}_0)' \xrightarrow{\ \mathcal{J}\ } \mathscr{S}'(\mathbb{N}_0) \xrightarrow{A_d^{-1}} \mathscr{S}'(\mathbb{N}_0^d)$$

To see this, let $\alpha \in \mathbb{N}_0^d$ and $L \in (\mathscr{S}(\mathbb{N}_0^d)'$. By applying the definitions outlined above, we have

$$\begin{aligned}(A_d^{-1}\mathcal{J}(A_d')^{-1})(L)_\alpha &= \mathcal{J}(A_d')^{-1}(L)_{e_{\rho_d^{-1}(\alpha)}} \\ &= (A_d')^{-1}(L)(e_{\rho_d^{-1}(\alpha)}) = LA_d^{-1}(e_{\rho_d^{-1}(\alpha)}) = L(e_\alpha),\end{aligned}$$

from which it follows that $\mathcal{J}_d$ is a TVS-isomorphism such that $L(x) = \langle \mathcal{J}_d(L), x\rangle$ for all x. □

This theorem gives us a clearer picture of the strong dual $(\mathscr{S}(\mathbb{N}_0^d)'$ by realizing it as a concrete space of sequences. We apply it immediately to give a similar realization for the space of distributions.

Theorem 4.3.6. *The map* $\widetilde{\Gamma_d} : S'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{N}_0^d)$ *such that* $\widetilde{\Gamma_d}(\phi) = (\phi(h_\alpha))_{\alpha\in\mathbb{N}_0^d}$ *for all distributions* ϕ *is a TVS-isomorphism. (Recall the notation* h_α *for Hermite functions). This is such that*

$$\phi(f) = \langle\widetilde{\Gamma_d}(\phi), \Gamma_d(f)\rangle = \sum_{\alpha\in\mathbb{N}_0^d} \phi(h_\alpha)\langle h_\alpha, f\rangle_{L^2}$$

for all distributions ϕ and test functions. The inverse function is given by the formula

$$\widetilde{\Gamma_d}^{-1} : (z_\alpha)_{\alpha\in\mathbb{N}_0^d} \mapsto \left(\sum_{\alpha\in\mathbb{N}_0^d} z_\alpha \langle h_\alpha, \cdot\rangle_{L^2} \right).$$

Proof. $\widetilde{\Gamma_d}$ is equal to $\mathscr{I}_d \circ (\Gamma_d^{-1})'$, since for any $\phi \in S'(\mathbb{R}^d)$, we have

$$(\mathscr{I}_d(\Gamma_d^{-1})')(\phi) = \mathscr{I}_d(\phi\Gamma_d^{-1}) = (\phi\Gamma_d^{-1}(e_\alpha)_{\alpha\in\mathbb{N}_0^d} = (\phi(h_\alpha))_{\alpha\in\mathbb{N}_0^d}.$$

As shown, each of these functions are TVS-isomorphisms. □

4.3.1 Dual Characterization of Polynomial Growth

To conclude this chapter, we prove that certain sequences spaces we have introduced can be used to characterize other sequence spaces' polynomial growth.

Theorem 4.3.7.

$$\mathscr{S}'(\mathbb{N}_0^d) = \{x \in \mathscr{S}_0'(\mathbb{N}_0^d) : \forall\omega \in \mathscr{S}_+(\mathbb{N}_0^d)\ ,\ \|x\|_\omega < \infty\}.$$

Recall our previous notation: $\|x\|_\omega$ denotes the sum $\sum_{\alpha\in\mathbb{N}_0^d} w_\alpha |x_\alpha|$

Likewise, for $\mathscr{S}(\mathbb{N}_0^d)$, we have

$$\mathscr{S}'(\mathbb{N}_0^d) = \{x \in \mathscr{S}_0'(\mathbb{N}_0^d) : \forall\omega \in \mathscr{S}_+'(\mathbb{N}_0^d)\ ,\ \|x\|_\omega < \infty\}.$$

Proof. This can be reduced immediately to the case $d = 1$ by use of the maps ρ_d and A_d. Recall that ρ_d is a bijection between $\mathscr{S}_+(\mathbb{N}_0^d)$ and $\mathscr{S}_+(\mathbb{N}_0)$ which grows at most polynomially. The inclusion of the left-hand side in the right-hand side is also immediate from the proof of Theorem 4.3.5. Thus, we need to show that if a non-negative sequence $\nu \in \mathscr{S}_{0,+}'$ is such that $\sum_{n\geq 0} \omega_n \nu_n < \infty$ for all $\omega \in \mathscr{S}_+$, then $\nu \in \mathscr{S}'$.

Suppose to the contrary that $\nu \notin \mathscr{S}'$. We proceed by a diagonal-type argument to derive a contradiction. By induction, we construct an increasing sequence

$$0 \leq n_0 < n_1 < n_2 < \dots$$

such that $\nu_{n_k} > \langle n_k\rangle^k$ for all k. We do this as follows. Since $\nu \notin \mathscr{S}'$, then in particular ν is not bounded. So there exists n_0 such that $\nu_{n_0} \geq 1 = \langle n_0\rangle^0$. Now assume inductively that we have chosen $0 \leq n_< \dots < n_k$. If there did not exist n_{k+1} such that $\nu_k < \langle n\rangle^k$ for all $n > n_k$, then we would have $\nu \in \mathscr{S}'$, a contradiction. So we choose a suitable n_{k+1}.

Now, we define

$$\omega_n := \begin{cases} \langle n_k \rangle^{-k} & \text{if } n = n_k \\ 0 & \text{otherwise} \end{cases},$$

then $\omega = (\omega_n)_{n\geq 0} \in \mathscr{S}'_{0,+}$. We claim that $\omega \in \mathscr{S}_+$. Indeed, for any $\ell \geq 0$, we have

$$\sup_{n\geq 0} \langle n \rangle^{\ell} \omega_n = \sup_{k\geq 0} \langle n_k \rangle^{\ell} \langle n_k^{-k} \leq 1 < \infty.$$

Now, by definition of ω, we have

$$\sum_{n\geq 0} \omega_n \nu_n = \sum_{k\geq 0} \omega_{n_k} \nu_{n_k} \geq \sum_{k\geq 0} 1 = \infty,$$

since this last equality holds termwise. This is a contradiction, and so the desired set-equality holds. □

5

Infinite Dimensional Multilinear Algebra

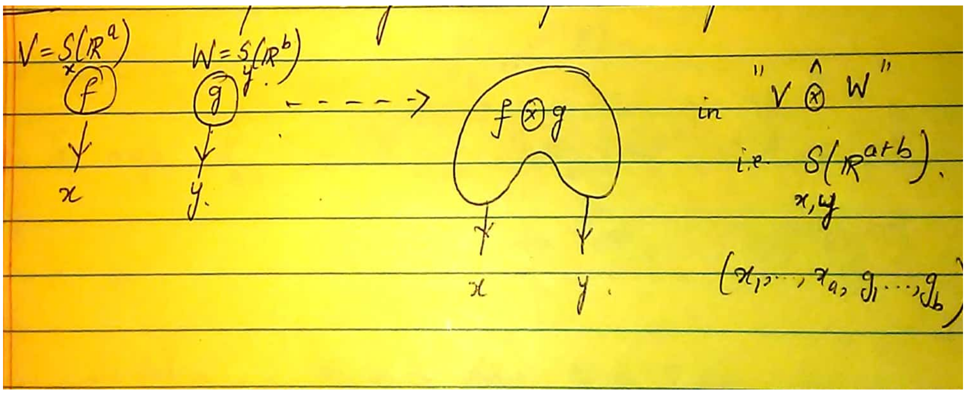
Apart from what the name suggests, one could also say that this lays down the foundations for the Game of Centipedes for infinite dimensional vector spaces.

The only reference which can be mentioned here are Volumes 3,4 of the book by Schwartz (Distributions Vectorielles)

This is currently being further investigated by Anderson-Kashnev who are working in the Topoligical Quantum Field Theory and recent works from 2014 are of interest.

5.1 Tensor Product of Test Function Spaces

Recall the following diagram which we had talked about earlier. We are going to discuss the notion of tensor product of functions as indicated in the diagram.



Theorem 5.1.1. *The map* $S(\mathbb{R}^a) \times S(\mathbb{R}^b) \to S(\mathbb{R}^{a+b})$, $(f, g) \mapsto f \otimes g$, *where*

$(f \otimes g)(x,y) := f(x)g(y)$, is well-defined, bilinear, continuous and the span of its image is dense.

Proof. First let $< x, y >$ denote the bracketed norm where the x consists of the first a coordinates and y, the last b coordinates. Clearly we have $1 + |x|^2 + |y|^2 \leq (1 + |x|^2)(1 + |y|^2)$ and hence we have the inequality
$< x, y > \leq < x >< y >$.

• $f \otimes g$ is C^∞.

We have for $\alpha \in \mathbb{N}_0^a, \beta \in \mathbb{N}_0^b$,
$< x, y >^k \quad |\partial_x^\alpha \partial_y^\beta [f(x)g(y)]| \leq < x >^k < y >^k |\partial_x^\alpha f(x)||\partial_y^\beta g(y)|$. Taking supremum over x, y, we have
$||f \otimes g||_{(\alpha,\beta),k} \leq ||f||_{\alpha,k}||g||_{\beta,k} < \infty$. This shows well-definedness as well as continuity.

• Density

We can use the following key property and density of the span of Hermite functions to immediately conclude the required density.
$h_{\alpha,\beta}(x,y) = h_\alpha(x)h_\beta(y)$ and by definition $h_{\alpha,\beta} = h_\alpha \otimes h_\beta$. □

5.2 Nuclear Theorem 1 (NT 1)

Version 1:

Theorem 5.2.1. *The map $K_{a,b} : S'(\mathbb{R}^{a+b}) \to \mathscr{L}_2(S(\mathbb{R}^a), S(\mathbb{R}^b); \mathbb{K})$ given by $\phi \mapsto \Big((f,g) \mapsto \phi(f \otimes g)\Big)$ is well defined and an isomorphism of $\mathbb{K}$-vector spaces. Here $\mathscr{L}_2(X, Y; \mathbb{K})$ is the set of all bilinear forms from $X \times Y \to \mathbb{K}$.*

Proof. We can reduce to sequence spaces, prove the theorem and then conclude about the isomorphism in the discussion. Consider the following diagram.

$$S'(\mathbb{N}_0^{a+b}) \overset{(\tilde{\Gamma}_{a+b})^{-1}}{\to} S'(\mathbb{R}^{a+b}) \overset{K_{a,b}}{\to} \mathscr{L}_2(S(\mathbb{R}^a, S(\mathbb{R}^b); \mathbb{K}) \overset{\tilde{\Gamma}_a,\tilde{\Gamma}_b}{\to} \mathscr{L}_2(S(\mathbb{N}_0^a, S(\mathbb{N}_0^b); \mathbb{K})$$

Let the composition be denoted as $k_{a,b}$. Let us explicitly look at what this map does.

Let $z = (z_{\alpha,\beta})_{\alpha \in \mathbb{N}_0^a, \beta \in \mathbb{N}_0^b} \in S'(\mathbb{N}_0^{a+b})$. z is mapped to $\phi \in S'(\mathbb{R}^{a+b})$: $\forall\ h \in S(\mathbb{R}^{a+b}),\ \phi(h) = \sum_{\alpha,\beta} z_{\alpha,\beta} \quad < h_{\alpha,\beta}, h >_{L_2}$. Now $\phi \mapsto B$, a bilinear form:

$$\begin{aligned} B(f,g) &= \phi(f \otimes g) \\ &= \sum_{\alpha,\beta} z_{\alpha,\beta} \quad < h_\alpha \otimes h_\beta, f \otimes g >_{L_2} \\ &\overset{Fubini}{=} \sum_{\alpha,\beta} z_{\alpha,\beta} < h_\alpha, f >_{L_2} < h_\beta, g >_{L_2} \end{aligned}$$

Next $B \mapsto l \in S(\mathbb{N}_0^a) \times S(\mathbb{N}_0^b)$: $l(x,y) = B(\Gamma_a^{-1}(x), \Gamma_b^{-1}(y))$ where $x = (x_\alpha)_{\alpha\in\mathbb{N}_0^a}, y = (y_\beta)_{\beta\in\mathbb{N}_0^b}$. Note that we have,

$< h_\alpha, \Gamma_a^{-1}x >_{L_2} = [\Gamma_a(\Gamma_a^{-1}(x))]_\alpha = x_\alpha$,

$< h_\beta, \Gamma_b^{-1}y >_{L_2} = [\Gamma_b(\Gamma_b^{-1}(y))]_\beta = y_\beta$.

Thus, $z \overset{k_{a,b}}{\mapsto} l$; $l(x,y) = \sum_{\alpha,\beta} z_{\alpha,\beta}x_\alpha y_\beta$. Hence z is the matrix of the bilinear form l.

We want to show now that $k_{a,b}$ is a vector space isomorphism.

Injectivity follows from $l(e_\alpha, e_\beta) = z_{\alpha,\beta}$.

$\phi(f\otimes g)$ is continuous implies $K_{a,b}$ is well-defined, so $k_{a,b}$ is well-defined.

Surjectivity: Suppose we have a continuous bilinear form l. Let $z_{\alpha,\beta} := l(e_\alpha, e_\beta)$. Now l continuous implies there exist ρ_a, ρ_b continuous semi-norms on $S(\mathbb{N}_0^a, \mathbb{N}_0^b)$ such that $\forall\ x, y,\ |l(x,y) \le \rho_a(x)\rho_b(y)$. Now we have the following chain of inequalities, $||\cdot||_{\infty,0} \le ||\cdot||_{\infty,1} \le \dots$.

Thus $\exists\, C > 0, \exists\, n \in \mathbb{N}_0, \forall\, x \in S(\mathbb{N}_0^a), \forall\, y \in S(\mathbb{N}_0^b, l(x,y)| \le C||x||_{\infty,n}||y||_{\infty,n}$.

Take $x = e_\alpha, y = e_\beta$. So, $|z_{\alpha,\beta}| \le C||e_\alpha||_{\infty,n}||e_\beta||_{\infty,n}$.

$||e_\alpha||_{\infty,n} = \sup_{\alpha'\in\mathbb{N}_0^a} < \alpha' >^n = < \alpha >^n$. Thus we have

$\forall\ \alpha, \beta,\ |z_{\alpha,\beta}| \le C < \alpha >^n < \beta >^n \le C < \alpha, \beta >^{2n}$ (since $< \alpha, \beta > = \sqrt{1 + |\alpha|^2 + |\beta|^2} \ge \sqrt{1 + |\alpha|^2}$) $\implies z \in S'(\mathbb{N}_0^{a+b})$. Note by construction, $k_{a,b}(z) = l$. □

5.3 Tensor Product of Distributions

Theorem 5.3.1. *Let $\varphi(x) \in \mathcal{S}'_x(\mathbb{R}^a)$, $\psi(y) \in \mathcal{S}'_y(\mathbb{R}^b)$. Then there exists a unique $T(x,y) \in \mathcal{S}'_{x,y}(\mathbb{R}^{a+b})$ such that for any $f(x) \in \mathcal{S}_x(\mathbb{R}^a)$, $g(y) \in \mathcal{S}_y(\mathbb{R}^b)$*

$$\langle T(x,y)\,,\, f(x)g(y)\rangle_{x,y} = \langle \varphi(x)\,,\, f(x)\rangle_x \langle \psi(y)\,,\, g(y)\rangle_y. \tag{5.1}$$

Proof. The map $(f,g) \mapsto \varphi(f)\psi(g)$ continuous and bilinear. By the Nuclear Theorem (Theorem 5.2.1), there exists a unique map T such that $T(f\otimes g) = \varphi(f)\psi(g)$. This gives the desired result. □

The following theorem is "Fubini's theorem for distributions".

Theorem 5.3.2. *For any $\varphi(x) \in \mathcal{S}'_x(\mathbb{R}^a)$, $\psi(y) \in \mathcal{S}'_y(\mathbb{R}^b)$, $h(x,y) \in \mathcal{S}_{x,y}(\mathbb{R}^{a+b})$,*

$$\begin{aligned}\langle (\varphi\otimes\psi)(x,y)\,,\, h(x,y)\rangle_{x,y} &= \langle \varphi(x)\,,\, \langle \psi(y)\,,\, h(x,y)\rangle_y\rangle_x \\ &= \langle \psi(y)\,,\, \langle \varphi(x)\,,\, h(x,y)\rangle_x\rangle_y.\end{aligned} \tag{5.2}$$