The Combinatorics of Classical Invariant Theory Revisited by Modern Physics

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I - Invariants, Covariants of Binary Forms
II - Feynman Graphs, Symbolic Method
III - Quantum Theory of Angular Momentum
IV - Gordan 1868

References:

- A.A., J. Chipalkatti, Adv. Math. 208 (2007), 491-520
- A.A., J. Chipalkatti, Transform. Groups 11 (2006), 341-370
- A.A., J. Algebra, 303 (2006), 771-788
- A.A., J. Chipalkatti, J. Pure App. Alg. 210 (2007), 43-61
- A.A., appendix of math.AG/0601705 by C. D'Andrea, J. Chipalkatti, to appear in Collect. Math.

all on arXiv.

My goal:

Doing Algebra 1 with Algebra 2 (Rota)

Physical mathematics (Itzykson)

Binary form :

$$F(x) = F(x_1, x_2) = \sum_{i=0}^{d} \binom{d}{i} \alpha_i \, x_1^{d-i} \, x_2^{i}$$

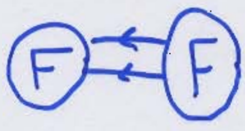
Ex : binary quadratic $(d=2)$

$$F(x) = \alpha_0 x_1^2 + 2\alpha_1 x_1 x_2 + \alpha_2 x_2^2$$

Invariant :

$$\Delta = \Delta(\alpha_0, \alpha_1, \alpha_2) = \Delta(F)$$

$$= \boxed{\alpha_1^2 - \alpha_0 \alpha_2}$$

$\Delta = -\frac{1}{2}$ 

$x = (x_1, x_2) \iff$ point on $\mathbb{P}^1(\mathbb{C})$

$g \in GL_2(\mathbb{C}) \qquad x \to gx$

$F \to gF \quad : \qquad (gF)(x) = F(g^{-1}x)$

Covariant:

polynomial $\quad C(F, x) = C(a_0, \dots, a_d ; x_1, x_2)$

$$\boxed{C(gF, gx) = (\det g)^{-w}\, C(F, x)}$$

- deg % a : degree
- deg % x : order
- w : weight

Invariant: Covariant of order $= 0$

Rings: Invariants $\subset$ Covariants $\subset \mathbb{C}[a_0, \dots, a_d, x_1, x_2]$

- Tensors : matrices with not necessarily two indices
- Contraction of indices → Einstein convention → Feynman graphs

Here indices = 1 or 2

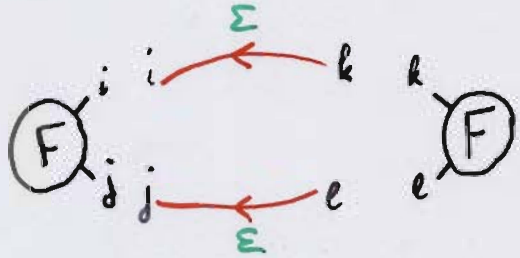
ex: $\mathcal{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\mathcal{E}_{ij})_{1 \le i,j \le 2}$ anti sym

$F = \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_1 & \alpha_2 \end{pmatrix} = (F_{ij})_{1 \le i,j \le 2}$ sym

$$\Delta = \alpha_1^2 - \alpha_0 \alpha_2$$

$$= -\frac{1}{2} \sum_{i,j,k,l=1}^{2} F_{ij}\, \mathcal{E}_{ik}\, \mathcal{E}_{jl}\, F_{kl}$$

$$= -\frac{1}{2} \sum_{i,j,k,l=1}^{2}$$



$$= -\frac{1}{2}$$

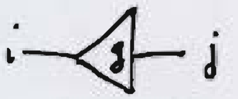
Graphs:

i —— j $= \delta_{ij} \Leftrightarrow \begin{pmatrix}1&0\\0&1\end{pmatrix}$

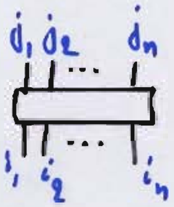
$i \longleftarrow j = \varepsilon_{ij} \Leftrightarrow \begin{pmatrix}0&1\\-1&0\end{pmatrix}$

i—ⓧ $= x_i$

$= \alpha$ (# i's = 2) $\Leftrightarrow$ form F



$= g_{ij} \Leftrightarrow g \in GL_2(\mathbb{C})$



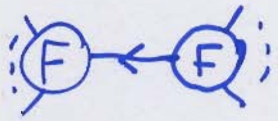
$= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \delta_{i_1 j_{\sigma(1)}} \cdots \delta_{i_n j_{\sigma(n)}}$ symmetrizer

i—① $= \delta_{i1}$

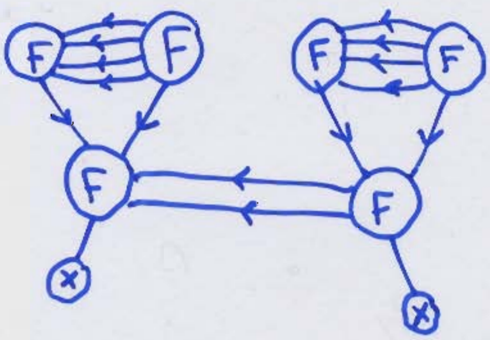
i—② $= \delta_{i2}$

i—(∂x) $= \dfrac{\partial}{\partial x_i}$

FFT

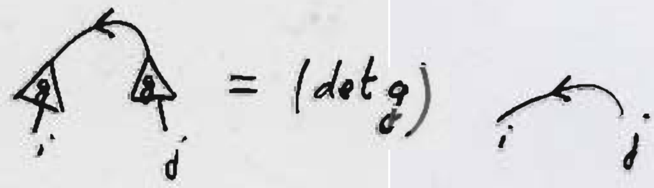
Covariants $\Longleftrightarrow$ $\mathbb{C}$-linear combinations of graphs made of 
with two kinds of edges 
or 

Ex:

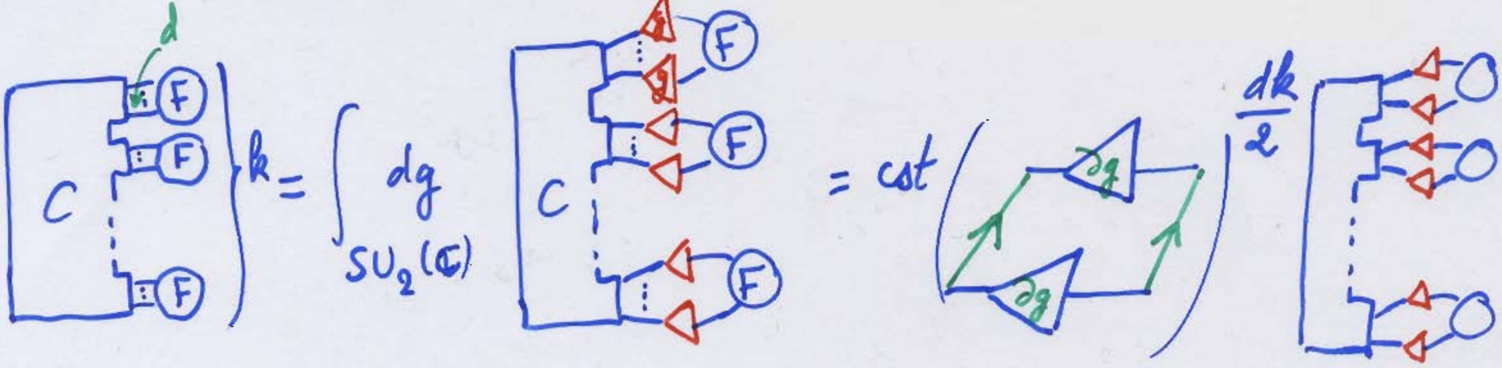


covariant of binary quintic $\begin{cases} \text{degree } 6 \\ \text{order } 2 \\ \text{weight } 14 \end{cases}$

Proof:

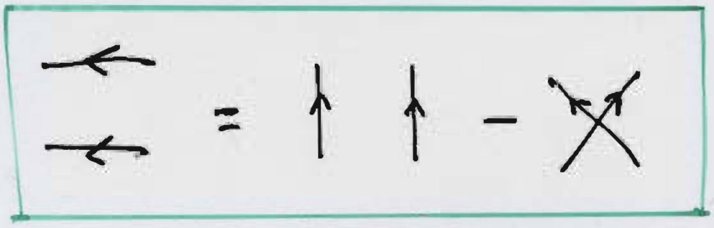
$\Longleftarrow$ Cayley 1845 

$\Longrightarrow$ Clebsch 1861



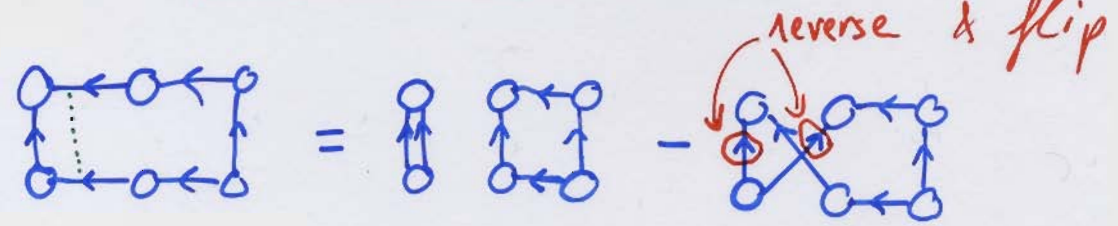
Rings of invariants and covariants are finitely generated (Gordan 1868, Hilbert 1890)

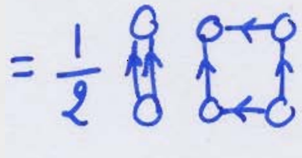
$\Longleftrightarrow$ There is a finite collection of graphs $\mathcal{G}_1, \dots, \mathcal{G}_p$, such that any graph $\mathcal{G}$ can be broken into connected pieces $\mathcal{G}_i$ using the relation



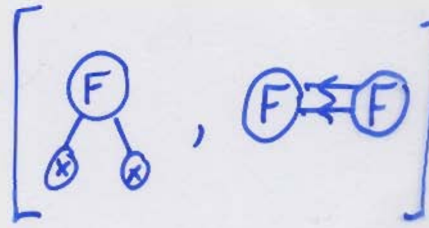
(IHX, GP, ...)

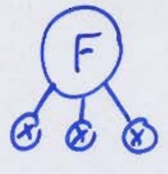
ex:



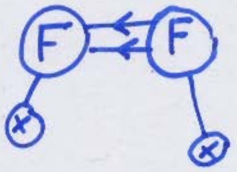


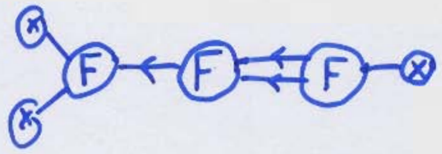
etc ... $\Rightarrow$ Covariants of quadratic $= \mathbb{C}$

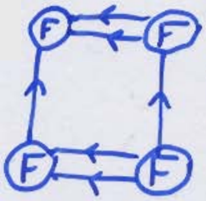


Covariants of cubic: 

Hessian

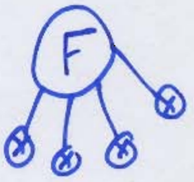
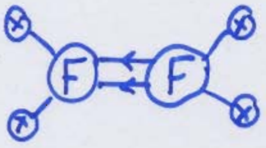
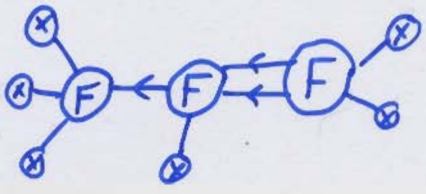


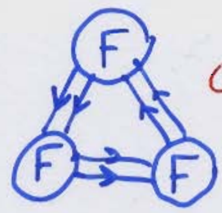




Discriminant

Covariants of quartic:

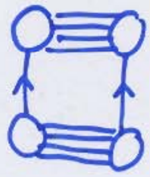
$S = \frac{1}{2}$  $T = \frac{1}{6}$  catalecticant

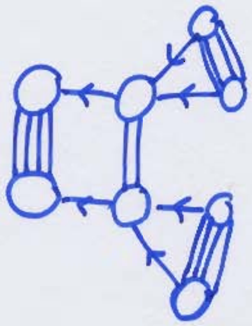
Rq: j-invariant of elliptic curves $= \dfrac{S^3}{S^3 - 27T^2}$ ← Discriminant

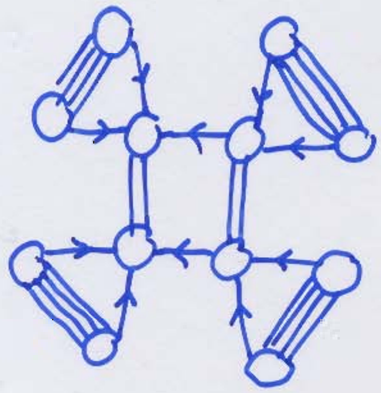
Covariants of quintic:

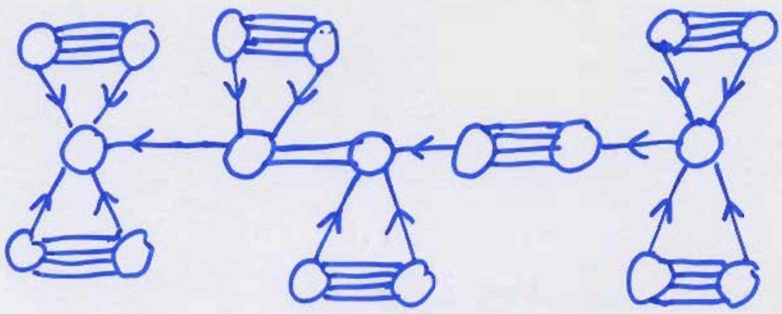
23 covariants $\supset$ 4 invariants

Invariants of quintic:

$J = -\frac{1}{2}$  16 monomials

$K = \frac{1}{8}$  68 monomials

$L = \frac{1}{96}$  228 terms

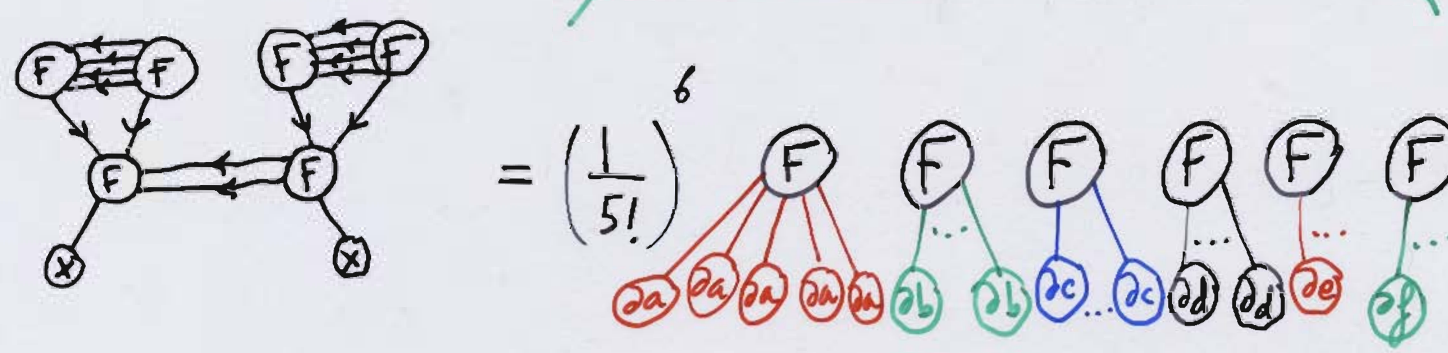
$H = -\frac{1}{384}$  848 terms

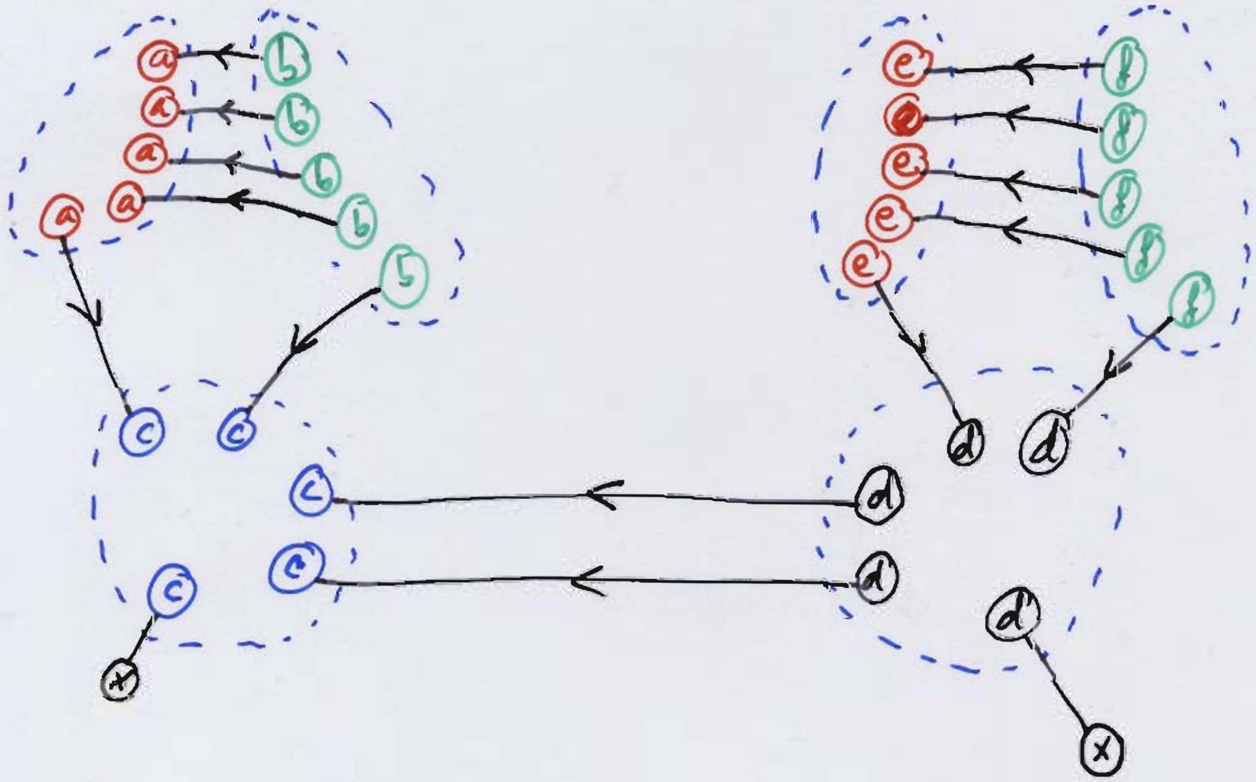
Hermite 1854

Symbolic method:

Ex: covariant of quintic

differential operator $\mathcal{D}$





symbolic expression $\mathcal{S}$

$$\boxed{\mathcal{S} = (ab)^4 (ca)(cb)(cd)^2 (de)(df)(ef)^4 c_x d_x}$$

$a = (a_1, a_2)$
$b = (b_1, b_2)$ extra binary variables

$$(ab) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = (a_1 b_2 - a_2 b_1) \qquad \text{symbolic bracket}$$

$$a_x = a_1 x_1 + a_2 x_2$$

Interpretation / Umbral map / diff. op. $\mathcal{D}$:

expand $\mathcal{S}$, then

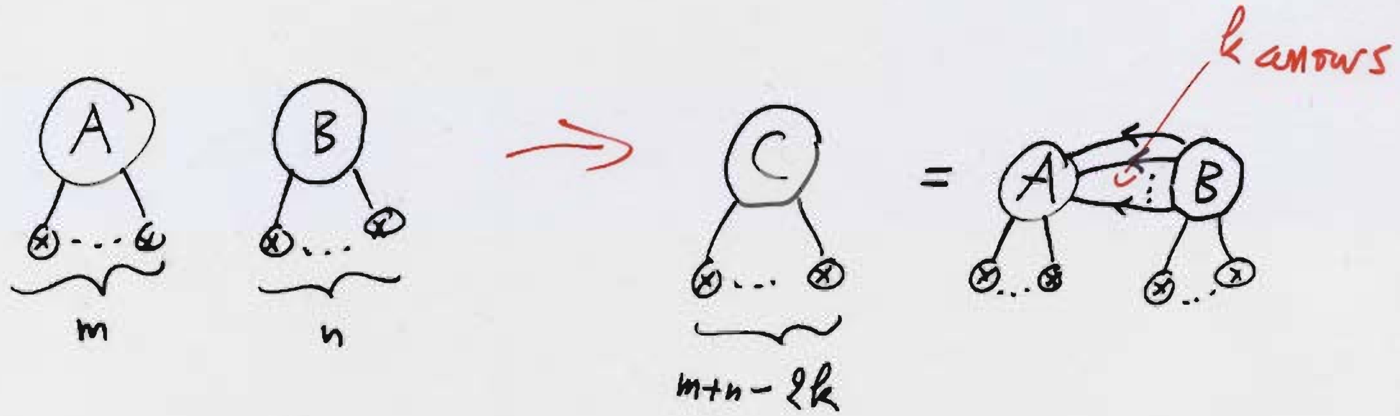
$$a_1^{5-i} a_2^{i} \longrightarrow \alpha_i$$

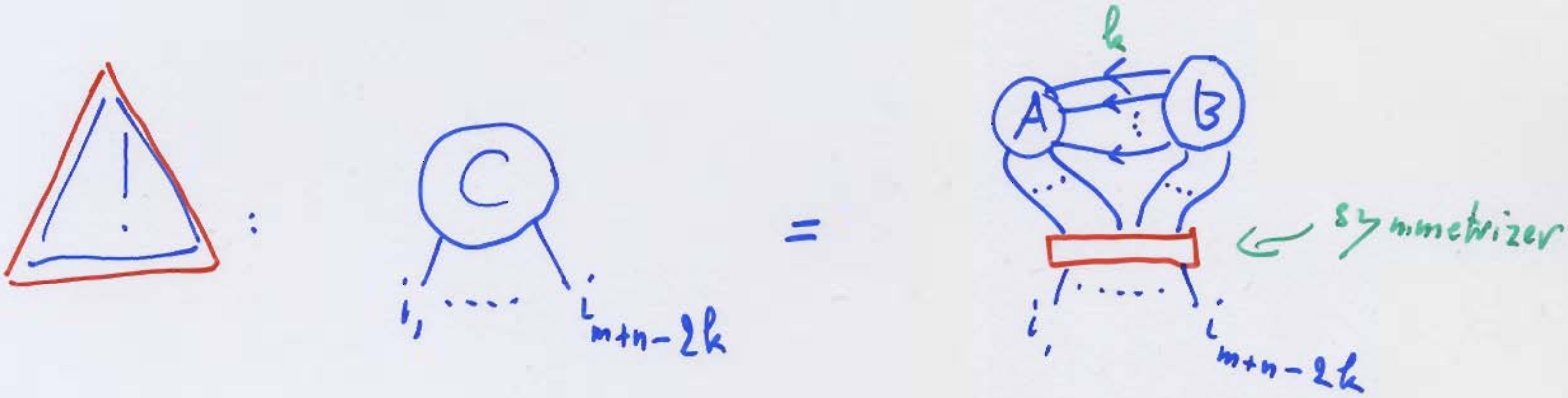
$$b_1^{5-j} b_2^{j} \longrightarrow \alpha_j$$

Transvectants:

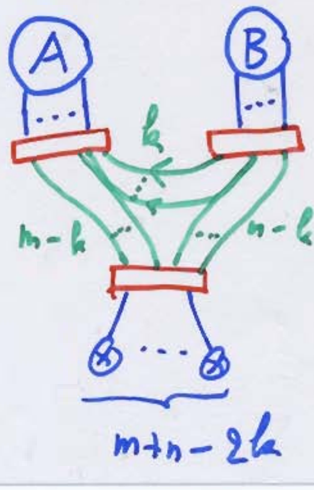
$0 \leq k \leq \min(m, n)$

$$\left.\begin{array}{l} A(x) \text{ form degree } m \\ B(x) \text{ form degree } n \end{array}\right\} \longrightarrow \text{form } (A,B)_k = C(x) \text{ degree } m+n-2k$$

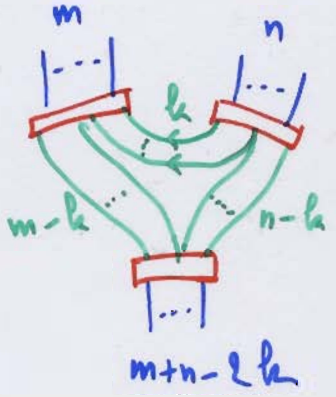




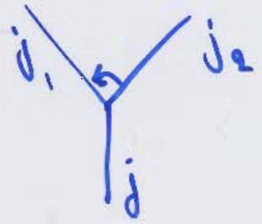
$(A, B)_k =$



Macroscopic notation $\Rightarrow$
- Jucys et al. graphical methods in the quantum theory of angular momentum
- Spin networks, Penrose ...



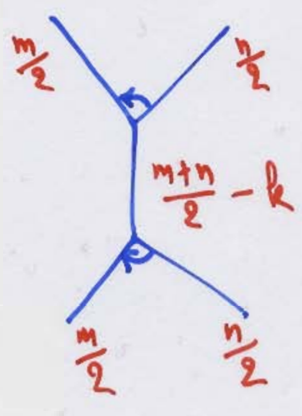
$\longrightarrow$



$$\left.\begin{aligned} j_1 &= \frac{m}{2} \\ j_2 &= \frac{n}{2} \\ j &= \frac{m+n}{2} - k \end{aligned}\right\} \in \frac{1}{2}\mathbb{N}$$

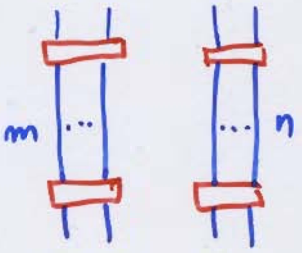
Gordan series: (turbo IHX)

$$= \sum_{k=0}^{\min(m,n)} \frac{\binom{m}{k}\binom{n}{k}}{\binom{m+n-k+1}{k}}$$

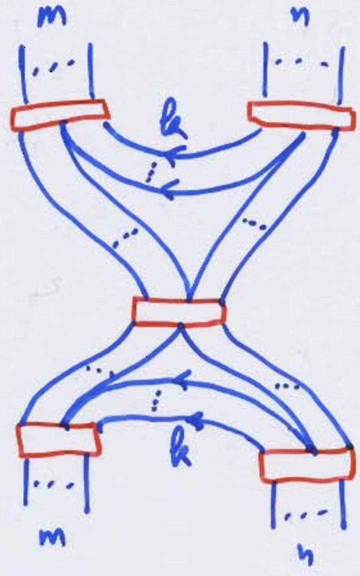


(Macroscopic)

Gordan series : (Microscopic view)

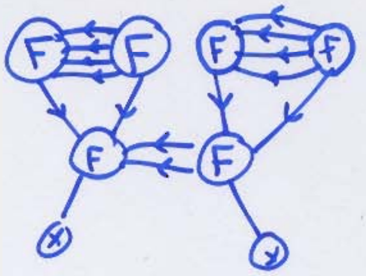


$$= \sum_{k=0}^{\min(m,n)} \frac{\binom{m}{k}\binom{n}{k}}{\binom{m+n-k+1}{k}}$$



GS $\Rightarrow$ every covariant = iterated transvectants

ex:



$$= \Big((F,(F,F)_4)_2 \ , \ (F,(F,F)_4)_2 \Big)_2$$

GS $\Rightarrow$ identities between iterated transvectants

recoupling $6j$, $9j$ ... symbols

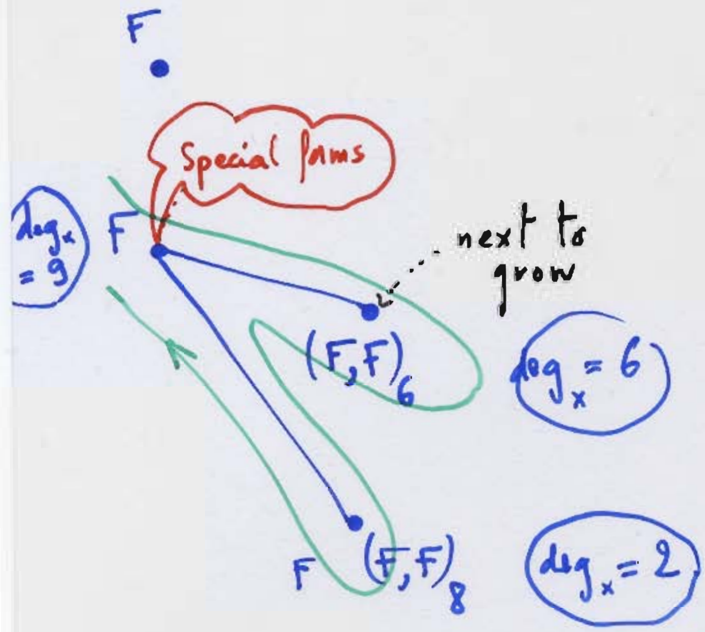
$\rightarrow$ hypergeometric series

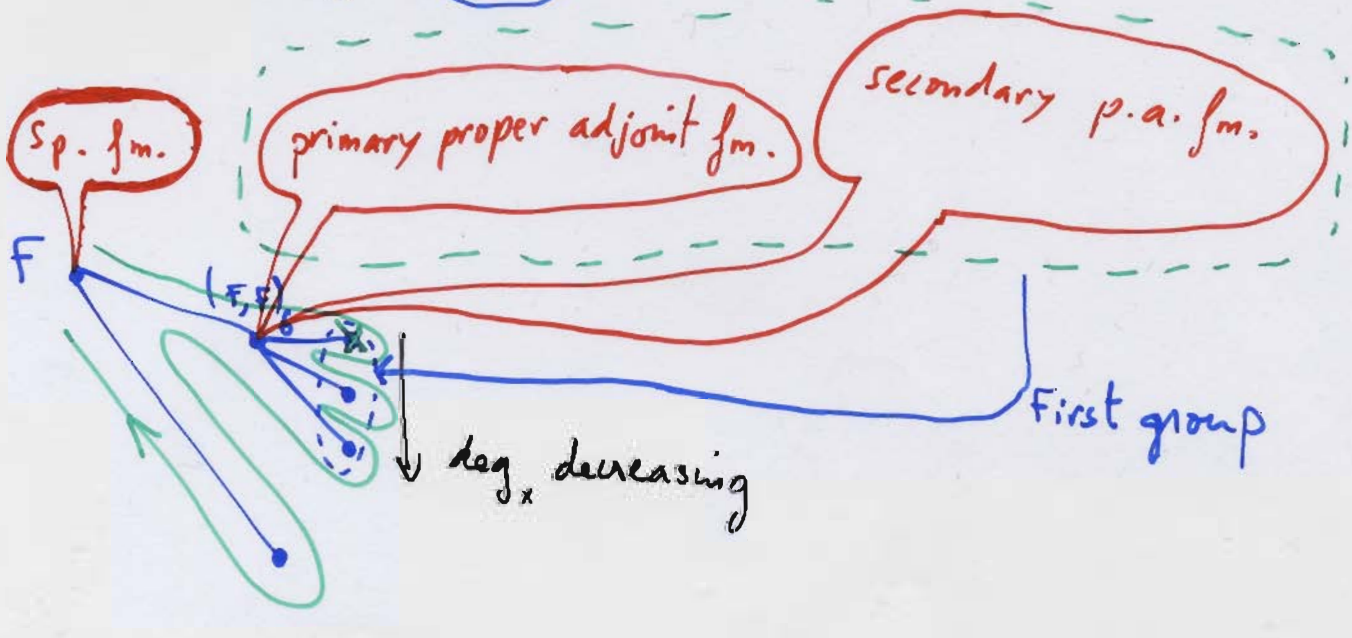
Gordan 1868 : (• see GDZ → original article • Translation K. Hoechsmann, Edit. A.A.)

Generators for covariants of forms of degree $< d$

$\Longrightarrow$ Generating system for F of degree d.

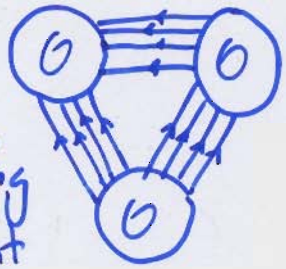
(ex: $d=9$) Growing decorated planar trees



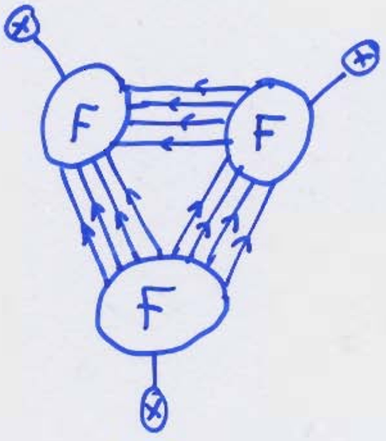


... it stops.

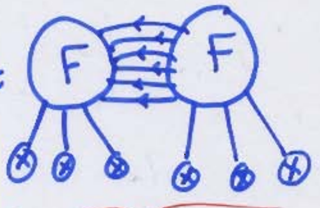
Special forms:

ex: 

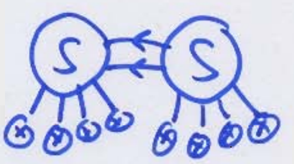
generating covariant of octavic



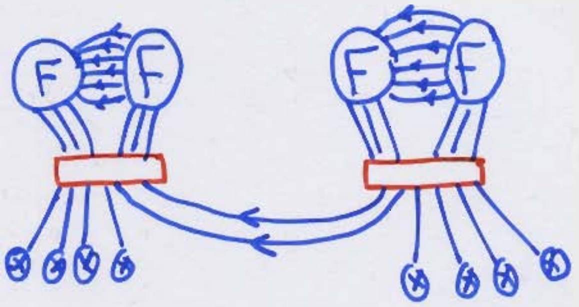
Primary proper adjoint forms:

for $(F,F)_6 =$ 

$\deg_x = 6$

generating covariant of sextic 

substitute for S



Secondary proper adjoint forms: