MATH 8450 – LECTURE 1 – JAN 18, 2023

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General introduction to the course:

In this course, we will learn some techniques which are useful to the rigorous mathematical study of models of quantum field theory (QFT). We will focus on one model, the so called Euclidean ϕ^4_d model in the Feynman path integral formulation. The plan of the course is in three parts, with if time permits, a possible fourth part on the construction of the model in the two-dimensional case.

- Part I: Combinatorial analysis.
- Part II: The infinite volume limit and the method of cluster expansions (originally introduced in [1]).
- Part III: Perturbative renormalization.

In most physics QFT textbooks one encounters expressions like the following:

$$C_n(z_1,\ldots,z_n) = \langle \phi(z_1)\cdots\phi(z_n)\rangle := \frac{1}{Z}\int \mathrm{D}\phi\ \phi(z_1)\cdots\phi(z_n)\ e^{-S(\phi)}\ , \qquad (1)$$

usually after many pages of introductory material (around p. 293 for instance in [2], and p. 185 in [3]). The above are called the n-point correlation functions of the Euclidean version of the ϕ^4_d model. The latter is related via analytic continuation to the Minkowski version which is a QFT describing a bosonic scalar field with quartic self-interactions in d-dimensional spacetime. Quantum excitations of the field correspond to particles of spin zero and say mass $m \ge 0$.

Warning: Everything in the next few paragraphs is heuristic and is not to be taken too seriously. These are desiderata rather than mathematical definitions.

In (1), the integral $\int \mathrm{D}\phi$ is over a vector space $\mathscr{F}$ given by the set of all functions $\phi : \mathbb{R}^d \to \mathbb{R}$ (with as usual, addition and scalar multiplication being define pointwise). An element $\phi \in \mathscr{F}$ is called a classical field configuration or simply a field. The symbol $\mathrm{D}\phi$ stands for the volume element for the Lebesgue measure on $\mathscr{F}$, namely,

$$\mathrm{D}\phi = \prod_{x\in\mathbb{R}^d} \mathrm{d}\phi_x = \prod_{x\in\mathbb{R}^d} \mathrm{d}\phi(x)\ .$$

Note that $\phi = (\phi_x)_{x\in\mathbb{R}^d}$ is a package made of uncountably many real numbers ϕ_x, indexed/labeled/named by the location $x \in \mathbb{R}^d$. So as not to strain our eyes, we will use the (treacherous) notation $\phi(x)$, but it is better to think of it as ϕ_x. For instance, the volume element factor $\mathrm{d}\phi(x)$ has nothing to do with the notion of differential of ϕ, traditionally written the same, and given by

$$\mathrm{d}\phi(x) = \sum_{i=1}^{d} \frac{\partial\phi}{\partial x_i}(x)\ \mathrm{d}x_i\ ,$$

as seen in courses on differential calculus and differential geometry.

Also in (1), the Euclidean action $S(\phi)$ is given by

$$S(\phi) = \int_{\mathbb{R}^d} \mathrm{d}^d x \ \left[\frac{1}{2} \sum_{i=1}^{d} \left(\frac{\partial \phi}{\partial x_i}(x) \right)^2 + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right]$$

which features the mass m, as well as the coupling constant $\lambda \geq 0$ which indicates the amount of intereaction between the underlying particles of the QFT. Note that S is an example of functional, i.e., a function or map which eats a function as argument and here returns a number. Another example of functional is the integrand of the functional integral in (1), namely, the map

$$\left\{ \begin{array}{lcl} \mathscr{F} & \longrightarrow & \mathbb{R} \\ \phi & \longmapsto & \phi(z_1)\cdots\phi(z_n)\ e^{-S(\phi)} \end{array} \right.$$

which involves a given fixed collection of n points $z_1, \ldots, z_n$ in $\mathbb{R}^d$ and the evaluations of the field ϕ at those points.

Finally Z is a normalization factor,

$$Z = \int \mathrm{D}\phi\ e^{-S(\phi)}\ .$$

As a result $\frac{1}{Z}e^{-S(\phi)}$ can be thought of as the density (with respect to the Lebesgue measure $\mathrm{D}\phi$) of a probability measure on $\mathscr{F}$, an infinite-dimensional analogue of say

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x\ ,$$

the probability density over $\mathbb{R}$ for the standard normal/Gaussian random variable (bell curve). This justifies the notation $\langle \cdots \rangle$ which is the statistical mechanics way of writing statistical averages/expectations, which probability theorists would denote by $\mathbb{E}(\cdots)$. One of the reasons why the Euclidean version of QFT is important is that it establishes a connection with statistical mechanics/physics. In the latter (e.g. for the Ising model), one usually studies random functions ϕ with domain a discrete lattice/grid like $\mathbb{Z}^d$ instead of the continuum $\mathbb{R}^d$. Euclidean QFT, a.k.a statistical field theory (see [4]), is the same, but with the additional difficulty of working in the continuum versus on a lattice.

Now let us put our mathematician's hat and examine more closely the formulas written above. Almost nothing makes rigorous mathematical sense. Textbooks on integration tell us how to define and compute integrals $\int_{\mathbb{R}^N} \cdots \mathrm{d}^N x$ over finite-dimensional spaces with the Lebesgue measure $\mathrm{d}^N x = \mathrm{d}x_1 \cdots \mathrm{d}x_N$, even if $N = 10^{100}$, but here we want to integrate over $\mathscr{F}$ which is infinite-dimensional. Those who took graduate probability have perhaps seen the measure associated to Brownian motion which is a measure on an infinite-dimensional space of continuous functions, but it is very different from the Lebesgue measure $\mathrm{D}\phi$ which does not exist.

Even if one could make sense of the measure, there is no reason to trust that the ϕ's being integrated over are differentiable, and therefore

$$\sum_{i=1}^{d} \left(\frac{\partial \phi}{\partial x_i}(x) \right)^2$$

which we will also denote by $(\nabla\phi(x))^2$ or $(\partial\phi)^2(x)$, will be undefined. This is an issue due to the short distance behavior and the regularity of the field ϕ. We call that the UV or ultraviolet problem because high frequency/Fourier modes correspond to short distances. Even if one ignores this issue, there is no reason to believe that ϕ would decay at infinity fast enough for the integral over $\mathbb{R}^d$ defining $S(\phi)$ to converge. We will call this the IR or infrared problem or infinite volume problem which has to do with long distances, i.e., growth or decay at infinity for the field ϕ.

Notation: If A, B are two sets, we will denote the set of maps/functions $f : A \to B$ by B^A. We will have no use for the notation $\mathbb{N}_0$ and adopt the convention that $\mathbb{N} := \{0, 1, 2, \ldots\}$. The set of positive integers will be denoted as $\mathbb{Z}_{>0} := \{1, 2, 3, \ldots\}$. The set of relative integers will be denoted as usual by $\mathbb{Z}$, and we will, if needed, use the self-explanatory notations $\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\leq 0}, \mathbb{Z}_{<0}$. Note the duplicate $\mathbb{Z}_{\geq 0} = \mathbb{N}$. For a finite set A, we will denote the cardinality or number of elements of A by $|A|$. For $n \in \mathbb{N}$, we let $[n] := \{1, 2, \ldots, n\} = \{i \in \mathbb{N} \mid 1 \leq i \leq n\}$. Of course $|[n]| = n$, and note in particular that $[0] = \varnothing$. If we write an inclusion $A \subset B$, we allow equality, so we will not use the notation $\subseteq$.

As a first step to address the UV problem, we will replace the very infinite-dimensional space $\mathscr{F} = \mathbb{R}^{\mathbb{R}^d}$ by the still infinite-dimensionl (in fact even still uncountably so) but less scary $\mathbb{R}^{(L^r\mathbb{Z})^d}$. Here L is a fixed integer $L > 1$ which for technical reason we will choose to be odd. One could take $L = 3$, for instance, throughout the remainder of the discussion. On the other hand $r \in \mathbb{Z}$ will vary and will ultimately be taken to $-\infty$. What we are doing here is similar to what engineers and numerical analysits, solving PDEs on the computer, typically do, namely, replace the continuous domain $\mathbb{R}^d$ of ϕ by a grid/lattice which is a discrete set

$$(L^r\mathbb{Z})^d = L^r(\mathbb{Z}^d) = \{(L^r n_1, \ldots, L^r n_d) \mid k_1 \in \mathbb{Z}, \ldots, k_d \in \mathbb{Z}\} \subset \mathbb{R}^d$$

The UV cutoff r is a parameter which controls the mesh L^r or spacing between neighboring lattice sites/nodes. In the formula for the action $S(\phi)$, we now replace the partial derivative $\frac{\partial \phi}{\partial x_i}(x)$ by the lattice-adapted finite difference

$$\frac{\phi(x + L^r e_i) - \phi(x)}{L^r}$$

where we used the notation $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$,... for the canonical basis of $\mathbb{R}^d$. The improper integral $\int_{\mathbb{R}^d} \cdots \mathrm{d}^d x$ in the definition of $S(\phi)$ will be replaced by its Riemann sum approximation

$$\sum_{x \in (L^r\mathbb{Z})^d} L^{dr} \cdots$$

where L^{dr} is the volume of a (hyper)cubic cell cut by the lattice.

Note that the last sum is still infinite because the original integral is an improper one, i.e., we have yet to deal with the IR or infinite volume problem. We now introduce another parameter $s \in \mathbb{Z}$, with $r \leq s$, which we call the IR cutoff and which controls the size of the system. Namely instead of $\mathbb{R}^d$ we will work in a finite cubic box of linear size L^s, centered at the origin. We thus make another change to the domain of ϕ and replace the discrete but infinite grid $(L^r\mathbb{Z})^d$ by the finite portion

$$\Lambda_{r,s} = (L^r\mathbb{Z})^d \cap \left[-\frac{L^s}{2}, \frac{L^s}{2}\right]^d .$$

Hence the infinite sum $\sum_{x \in (L^r\mathbb{Z})^d} L^{dr} \cdots$ now becomes a finite sum $\sum_{x \in \Lambda_{r,s}} L^{dr} \cdots$, and the daunting infinite-dimensional space $\mathscr{F}$ for the original functional integrals has been replaced by a finite-dimensional space $\mathbb{R}^N$ with

$$N = |\Lambda_{r,s}| = L^{d(s-r)}$$

as a simple counting of points shows. Note that we could have open or semiopen intervals like $\left[-\frac{L}{2}, \frac{L}{2}\right)$ without making any difference because we chose L to be an odd integer.

Finally, we have replaced our original quantities of interest $C_n(z_1, \ldots, z_n)$ by (first attempt)

$$C_{n,r,s}(z_1, \ldots, z_n) = \frac{C^{\mathrm{U}}_{n,r,s}(z_1, \ldots, z_n)}{C^{\mathrm{U}}_{0,r,s}}$$

where

$$C^{\mathrm{U}}_{n,r,s}(z_1, \ldots, z_n) = \int_{\mathbb{R}^{\Lambda_{r,s}}} \prod_{x \in \Lambda_{r,s}} \mathrm{d}\phi(x) \quad \phi(z_1) \cdots \phi(z_n)$$

$$\times \exp\left(- \sum_{x \in \Lambda_{r,s}} L^{dr} \left[\frac{1}{2} \sum_{i=1}^{d} \left(\frac{\phi(x + L^r e_i) - \phi(x)}{L^r} \right)^2 + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{24}\phi(x)^4 \right] \right) \ .$$

The "U" superscript stands of "unnormalized" correlations, i.e., without the division by the factor $Z = C^{\mathrm{U}}_{0,r,s}$. Note that the latter has $n = 0$ arguments (no z_i's) and is just a real number.

I said first attempt because there are still some outstanding issues. If x is a point near the end/boundary of the box, the expression $\phi(x + L^r e_i)$ is as yet undefined. We choose to impose periodic boundary conditions. This means that if we go out of the box one way we return from the opposite side. Equivalently the d-dimensional box of size L^s is thought of as a d-dimensional torus by identifying opposite ($(d-1)$-dimensional) faces. More precisely, there is a bijection

$$\Lambda_{r,s} \longrightarrow {(L^r\mathbb{Z})^d}\big/{(L^s\mathbb{Z})^d}$$

from a set to a finite Abelian group (for addition), which sends a tuple to it class/coset modulo the subgroup $(L^s\mathbb{Z})^d$. By definition, we understand the addition involved in $x + L^r e_i$ as the operation in the above finite quotient group.

Exercise 1. *(as per Joe's question in class) Show that if we used backward finite differences* $L^{-r}(\phi(x) - \phi(x - L^r e_i))$ *instead of the forward finite differences* $L^{-r}(\phi(x + L^r e_i) - \phi(x))$*, as a replacement for partial derivatives, the finite discrete approximation to* $S(\phi)$ *stays the same. In other words, prove the identity*

$$\sum_{x \in \Lambda_{r,s}} L^{dr} \left[\frac{1}{2} \sum_{i=1}^{d} \left(\frac{\phi(x) - \phi(x - L^r e_i)}{L^r} \right)^2 + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{24}\phi(x)^4 \right] =$$

$$\sum_{x \in \Lambda_{r,s}} L^{dr} \left[\frac{1}{2} \sum_{i=1}^{d} \left(\frac{\phi(x + L^r e_i) - \phi(x)}{L^r} \right)^2 + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{24}\phi(x)^4 \right] \ .$$

The other issue is the product $\phi(z_1) \cdots \phi(z_n)$ where the z_j's are points that are arbitrarily placed in $\mathbb{R}^d$ and may well not belong to the lattice $(L^r\mathbb{Z})^d$ let alone the portion $\Lambda_{r,s}$. To remedy this problem, for each point z_j we replace the evaluation $\phi(z_j)$ exactly at z_j by a weighted "average"

$$\int_{\mathbb{R}^d} \mathrm{d}^d y_j \ f_j(y_j)\phi(y_j) \ ,$$

where f_j is C^∞ and rapidly decaying, and mostly peaked around z_j. Then, in order to accommodate the discrete setting we replace the last integral by the lattice Riemann sum version

$$\sum_{y_j \in \Lambda_{r,s}} L^{dr} f_j(y_j)\phi(y_j)$$

so the correct definition to be used for the unnormalized correlations is

$$C^{\mathrm{U}}_{n,r,s}(f_1,\ldots,f_n) := \int_{\mathbb{R}^{\Lambda_{r,s}}} \prod_{x\in\Lambda_{r,s}} \mathrm{d}\phi(x) \quad \prod_{j=1}^{n} \left(\sum_{y_j\in\Lambda_{r,s}} L^{dr} f_j(y_j)\phi(y_j) \right) \times \exp\left(-\sum_{\Lambda_{r,s}} L^{dr} \left[\frac{1}{2}\sum_{i=1}^{d} \left(\frac{\phi(x+L^r e_i)-\phi(x)}{L^r} \right)^2 + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{24}\phi(x)^4 \right] \right) \ . \quad (2)$$

We can now state the main problem related to the construction of the ϕ^4 model, which is to construct the limits

$$C_n(f_1,\ldots,f_n) = \lim_{r\to-\infty} \lim_{s\to\infty} \frac{C^{\mathrm{U}}_{n,r,s}(f_1,\ldots,f_n)}{C^{\mathrm{U}}_{0,r,s}}$$

for all $n\in\mathbb{N}$ and all test functions $f_1,\ldots,f_n$.

In fact we will have to be a bit more flexible and allow some parameters like m and λ to depend on r, in order to get interesting limits. One can think of the collection of multilinear maps C_n obtained at the end of the day as a QFT. This problem is mostly interesting for $d = 2,3,4$.

Exercise 2. *Show that the integral in (2) is perfectly well defined and convergent when $m \geq 0, \lambda \geq 0$ and at least one of them is nonzero. Show that $C^{\mathrm{U}}_{0,r,s} > 0$ so the division makes sense. Show that for n odd, the $C^{\mathrm{U}}_{n,r,s}$ vanish identically.*

Exercise 3. *What becomes of (2) when $d = 0$? Recall that $\mathbb{R}^0 = \mathbb{R}^{[0]} = \mathbb{R}^{\varnothing}$ is not empty and has one element.*

Next lecture(s): We will look at formal power series (multivariate), how to compose and invert them. Then we will learn about tensors ("matrices" with not necessarily only two indices), and the graphical calculus for computations with tensors.

References

[1] J. Glimm, A. Jaffe, and T. Spencer, The Wightman axioms and particle structure in the $\mathscr{P}(\phi)_2$ quantum field model. Ann. of Math. (2) **100** (1974), 585–632.

[2] M. E. Peskin, and D. V. Schroeder, *An introduction to quantum field theory.* Edited and with a foreword by David Pines. Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1995.

[3] M. Srednicki, *Quantum Field Theory*, Cambridge University Press, Cambridge, 2006.

[4] D. Tong, *Lectures on Statistical Field Theory*, `http://www.damtp.cam.ac.uk/user/tong/sft.html` , retrieved Jan 20, 2023.