MATH 8450 – LECTURE 10 – FEB 20, 2023

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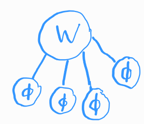
Perturbation theory with Feynman diagrams:

The problem we will look at is that of writing explicit series expansions for a quantities such as the *partition function*

$$\mathcal{Z} := \int_{\mathbb{R}^\Lambda} \mathrm{d}\mu_C(\phi)\ \exp\left(\frac{1}{4!} \right. \quad$$



$$\left. \right) \ ,$$

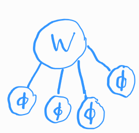
where Λ is a finite set, $A = (A(x,y))_{x,y\in\Lambda}$ is a real symmetric positive definite matrix with rows and columns indexed by Λ, $C = A^{-1}$, and

$$\mathrm{d}\mu_C(\phi) := \frac{e^{-\frac{1}{2}\phi^{\mathrm{T}}A\phi}}{(2\pi)^{\frac{|\Lambda|}{2}}\sqrt{\det(C)}}\ \mathrm{d}^\Lambda\phi\ .$$

The expression $\mathrm{d}\mu_C(\phi)$ can be seen as a Gaussian probability measure on $\mathbb{R}^\Lambda$ (if took Math 7310) or simply as an abbreviation for the expression on the RHS. As said before, we think of $\phi = (\phi(x))_{x\in\Lambda}$ as "column vector", so we in particular have

$$\phi^{\mathrm{T}}A\phi = \sum_{x,y\in\Lambda} \phi(x)\ A(x,y)\ \phi(y)\ .$$

We also used the graphical formalism from Lectures 4 and 5, in order to write



$$= \sum_{x_1,x_2,x_3,x_4\in\Lambda} W(x_1,x_2,x_3,x_4)\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\ ,$$

where the symmetric tensor W is given by

$$W(x_1,x_2,x_3,x_4) = -\lambda\ \mathbb{1}\{x_1 = x_2 = x_3 = x_4\}\ .$$

Remark 1. *In Lectures 4 and 5, we had tensor indices belong to the "preferred" finite set $[n]$. However, the graphical formalism works just as fine if indices belong to, and are summed over, any finite set, like Λ. The formalism also works if the tensor entries, instead of being complex numbers, take values in a commutative ring with unit and containing $\mathbb{Q}$ (so there is no issue dividing by some factorials), such as a rings of formal power series with complex coefficients. In the following, we will soon assume that the W entries belong to a ring of formal power series $\mathbb{C}[[Y]]$ in some collection $Y = (Y_1, \ldots, Y_p)$ of "'hidden" formal variables that are sitting in the background.*

In our situation of interest, we have $\Lambda = \Lambda_{r,s} = (L^r\mathbb{Z})^d \cap \left[-\frac{L^s}{2}, \frac{L^s}{2}\right]^d$, and the quadratic form defining the matrix A is

$$\begin{aligned}
\phi^{\mathrm{T}} A \phi &= 2\ S^{\mathrm{G}}_{r,s}(\phi) && (1)\\
&= \sum_{x\in\Lambda_{r,s}} L^{dr}\left[\sum_{i=1}^{d}\left(\frac{\phi(x+L^r e_i)-\phi(x)}{L^r}\right)^2 + m^2\phi(x)^2\right] && (2)\\
&= \sum_{x\in\Lambda_{r,s}} L^{dr}\left[m^2\phi(x)^2 + \sum_{i=1}^{d} L^{-2r}\left(\phi(x+L^r e_i)^2+\phi(x)^2-2\phi(x)\phi(x+L^r e_i)\right)\right] && (3)\\
&= \sum_{x\in\Lambda_{r,s}}\left(L^{dr}m^2+2dL^{(d-2)r}\right)\ \phi(x)^2\ -\ 2L^{(d-2)r}\sum_{x\in\Lambda_{r,s}}\sum_{i=1}^{d}\phi(x)\phi(x+L^r e_i)\ . && (4)
\end{aligned}$$

Note that we used the fact $x \mapsto x + L^r e_i$ is a bijection from $\Lambda_{r,s}$ to itself. So by a relabeling/discrete change of variables, a sum $\sum_{x\in\Lambda_{r,s}} \phi(x+L^r e_i)^2$ simply reduces to $\sum_{x\in\Lambda_{r,s}} \phi(x)^2$. By inserting the familiar preparatory step for conditioning, $1 = \sum_{y\in\Lambda_{r,s}} \mathbb{1}\{y = x+L^r e_i\}$, and symmetrizing over x, y (taking half the sum of an expression in x, y with the same expression with x, y exchanged), we can extract the matrix A as follows. We have

$$A(x,y) = (L^{dr}m^2 + L^{(d-2)r}\times 2d)\ \mathbb{1}\{x=y\} - L^{(d-2)r}\ \mathbb{1}\{x, y \text{ n.n.}\}\ ,$$

if $d \geq 0$ and $r < s$. The notation "n.n." stands for "nearest neighbors". By definition, $x, y \in \Lambda_{r,s}$ are nearest neighbors iff

$$x \neq y,\ \text{and}\ \exists i \in [d],\ x - y = \pm L^r e_i\ .$$

As said previously, addition is in the group $(L^r\mathbb{Z})^d/(L^s\mathbb{Z})^d$ which is identified with $\Lambda_{r,s}$, and e_i is the i-th canonical basis vector. Therefore, $L^r e_i \in (L^r\mathbb{Z})^d$ has a natural projection in the quotient group $(L^r\mathbb{Z})^d/(L^s\mathbb{Z})^d$, and this is what we refer to when writing expressions such as $x + L^r e_i$. Note that we are always assuming $r \leq s$ and, eventually, we will be interested in taking the limits $s \to \infty$ and $r \to -\infty$, so the limitation to $r < s$ is harmless. Nevertheless, one can also consider the case $d \geq 0$ and $r = s$, which gives

$$A(x,y) = L^{dr}m^2\mathbb{1}\{x=y\}\ ,$$

as could already be seen in (2), because $x + L^r e_i = x$ in this degenerate situation with $|\Lambda_{r,s}| = 1$. Note that our formula in the $r < s$ case uses our assumption that L is odd, because this excludes $x - y$ being at the same time of the form $+L^r e_i$ and $-L^r e_i$ which could happen if $s = r+1$ and say $L = 2$.

From now on, we will work in the massive case $m > 0$, as opposed to the more difficult massless QFT case where $m = 0$. From the positive mass condition $m > 0$, and by (2), we immediately see that the matrix A is indeed positive definite, which was needed for our Gaussian integrals to make sense. We will soon take $\lambda = \lambda_{r,s}$ to be a formal variable, so our perturbation expansion will take place in the ring of formal power series $\mathbb{C}[[Y]] = \mathbb{C}[[Y_1]]$ with $Y_1 = \lambda$. However, before doing that, we will look at the particularly simple case $d = 0$, while attempting to evaluate λ to some real or complex number.

The case $d = 0$, or QFT in zero-dimensional spacetime, as a case study in the issue of convergence of perturbation theory:

Set-theoretical constructions should sense, even in degenerate cases dealing with the empty set. When $d = 0$, we have

$$\mathbb{R}^d = \mathbb{R}^0 = \mathbb{R}^{\varnothing} = \{\varnothing\} = L^r\mathbb{Z}^d = \Lambda_{r,s}\ ,$$

since given any set X, there exists a unique map $\varnothing \to X$, namely, the empty map, i.e., the one whose graph is the empty subset of the Cartesian product $\varnothing \times X$. Here the vector $\phi = (\phi(x))_{x\in\Lambda}$ reduces to a single real number which we will continue to denote ϕ. The matrices A and C become 1×1 matrices or simply numbers given by

$$A = m^2\ ,\ C = m^{-2}$$

so that $\phi^{\mathrm{T}}A\phi = m^2\phi^2$. Let us introduce the quantity $\sigma = m^{-1}$ which has an interpretation as a standard deviation. We then have

$$\mathcal{Z} = \mathcal{Z}(\lambda) = \int_{\mathbb{R}} \frac{\mathrm{d}\phi}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2\sigma^2} - \frac{\lambda\phi^4}{4!}\right)\ . \tag{5}$$

It is easy to see that $\mathcal{Z}(\lambda)$ is a well-defined function $[0,\infty) \to \mathbb{R}$, because the integral above is convergent. Using the theorem of differentiation under the integral sign, repeatedly, it is also easy to see that this function is infinitely differentiable on $[0,\infty)$. Note that at the end-point 0, one must use right derivatives only in the previous differentiability statement. The Taylor series at the origin can be obtained by expanding the exponential of the "interaction" $\exp\left(-\frac{\lambda\phi^4}{4!}\right)$ and switching the sum and the integral, namely, by writing

$$\begin{aligned}
\mathcal{Z}(\lambda) \quad &= \quad \int_{\mathbb{R}} \frac{\mathrm{d}\phi}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2\sigma^2}\right) \times \sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{-\lambda}{4!}\right)^N \phi^{4N} \\
\text{“}=\text{”} \quad & \sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{-\lambda}{4!}\right)^N \int_{\mathbb{R}} \frac{\mathrm{d}\phi}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2\sigma^2}\right)\ \phi^{4N} \\
\text{“}=\text{”} \quad & \sum_{N=0}^{\infty} c_N\lambda^N\ ,
\end{aligned}$$

where the last expression is the Taylor series at the origin. Note the use of quotes, because, from the point of view of analysis (rigorous considerations of convergence etc.), the switching of the sum and the integral is *not* a legitimate operation. From basic calculus and the relationship between differentiable functions with their Taylor series, the best we can say is that

$$\mathcal{Z}(\lambda) \sim \sum_{N=0}^{\infty} c_N \lambda^N \ , \tag{6}$$

as an *asymptotic* series at the origin. More precisely, the above equation with the symbol $\sim$ means that for all $M \geq 0$,

$$\mathcal{Z}(\lambda) - \sum_{N=0}^{M} c_N \lambda^N = o(\lambda^M)$$

when $\lambda \to 0^+$. We used Landau's standard little "o" notation. Namely, the above just says

$$\lim_{\lambda \to 0^+} \frac{\mathcal{Z}(\lambda) - \sum_{N=0}^{M} c_N \lambda^N}{\lambda^M} = 0 \ .$$

Equation (6) is a statement relating, on the LHS, a true function $\lambda \mapsto \mathcal{Z}(\lambda)$ to a formal power series in $\mathbb{C}[[\lambda]]$, on the RHS. The coefficients of the Taylor series are given by

$$\begin{aligned} c_N &= \frac{(-1)^N}{N!\ 24^N} \int_{\mathbb{R}} \frac{\mathrm{d}\phi}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2\sigma^2}\right)\ \phi^{4N} \\ &= \frac{(-1)^N}{N!\ 24^N} \times \frac{(4N)!}{2^{2N}(2N)!} \times \sigma^{4N} \ , \end{aligned}$$

by the formula for the moments of a standard normal random variable, namely, the formula mentioned at the end of the proof of the Isserlis-Wick Theorem in Lecture 9. Recall Stirling's asymptotic formula which gives the asymptotic equivalent

$$N! \sim \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \ ,$$

when $N \to \infty$ (i.e., we are saying the limit of the ratio LHS/RHS is 1). From this formula, we immediately get

$$|c_N| \sim \frac{1}{\sqrt{\pi}} \left(\frac{8\sigma^4}{3e}\right)^N \ N^N \approx N! \ .$$

In the last equation, we simply indicated the order of growth, namely $\approx$ means asymptoticaly equivalent, up to irrelevant factors of the form Cst^N. As a result of this discussion, we see that perturbation theory has zero radius of convergence. The only truly numerical value (as opposed to a formal variable interpretation) we can give λ, for which the series $\sum_{N=0}^{\infty} c_N \lambda^N$ is convergent, is $\lambda = 0$.

That being said, it would be rather shortsighted to dismiss the perturbative series entirely as meaningless. In fact, one can recover the function from the series, if one uses a "smarter" summation procedure. Namely, to get $\mathcal{Z}(\lambda)$, one should not use the ordinary series summation recipe

$$\lim_{M \to \infty} \sum_{N=0}^{M} c_N \lambda^N \ ,$$

which is appropriate for recovering a holomorphic function from its Taylor series at a point λ_0 *in the interior* of its domain of analyticity. The expansion point here is $\lambda_0 = 0$ which is *on the boundary* of the analyticity domain, and this calls for a more sophisticated summation procedure called Borel summation. Let us go back to the original integral definition (5) for

$\mathcal{Z}(\lambda)$, restricted to the open interval $(0,\infty)$. It is easy to see that this function extends uniquely to a holomorphic function on the cut plane $\mathbb{C}\backslash(-\infty,0]$. For this, absorb the λ into the ϕ by a change of variable, and then expand the Gaussian part (the exponential of the quadratic in ϕ) instead of the exponential of ϕ^4. Commuting sum and integral is then legitimate (factorials now become our friends) and gives an entire analytic function of $\frac{1}{\sqrt{\lambda}}$. The ambiguity in defining the square root results in the need for a cut which can be placed rather arbitrarily, and which we choose to put on the negative real axis. See Homework 2 for more details, and the article by Sokal [4] for a definition of Borel summability, and how to use it here to recover the function $\lambda \mapsto \mathcal{Z}(\lambda)$ from the formal power series $\sum_{N=0}^{\infty} c_N \lambda^N \in \mathbb{C}[[\lambda]]$.

Remark 2. *If the original integral over ϕ was over a finite interval instead of $\mathbb{R}$, there would be no build-up of $N!$ in the $|c_N|$'s, and perturbation theory would be justified, namely, $\mathcal{Z}(\lambda)$ would be entire analytic and the Taylor series at the origin would converge everywhere. So the problem of divergence of perturbation theory is due to the so-called "large field problem" which is ubiquitous in constructive QFT for Bosonic models. Fermions do not have this problem, and even though they look scarier at first, because of the need for Grassmann or anticommuting variables etc., they are significantly easier to handle from the point of view of mathematical analysis.*

Remark 3. *In Part II of this course, we will learn how to control the thermodynamic (or infinite volume) limit $s \to \infty$ thanks to the method of cluster and Mayer expansions. This is a powerful method which gives very precise and explicit information on the limiting objects (infinite volume correlation functions). In particular, by pushing this method a little, one can prove that correlations are the Borel sums of their perturbation theory expansions in terms of Feynman diagrams that are familiar to physicists. One can also do that in the context of the UV limit too, since the latter is an infinite volume limit, but not in ordinary x-space. It is an infinite volume limit in "phase-space" (both direct and Fourier space) or even better, the (Euclidean) AdS bulk as studied in the context of the AdS/CFT (Anti-de Sitter/Conformal Field Theory) or more general gauge/gravity holographic correspondence in modern theoretical physics* [1]. *Borel summability for our ϕ^4 model, with no cutoffs (i.e., after taking the limits $s \to \infty$ and $r \to -\infty$), has been established in the $d = 2$ case in* [2] *and in the $d = 3$ case in* [3].

References

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[2] J.-P. Eckmann, J. Magnen, and R. Sénéor, Decay properties and Borel summability for the Schwinger functions in $P(\phi)_2$ theories. Comm. Math. Phys. **39** (1974/75), 251–271.

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[4] A. D. Sokal, An improvement of Watson's theorem on Borel summability. J. Math. Phys. **21** (1980), no. 2, 261–263.