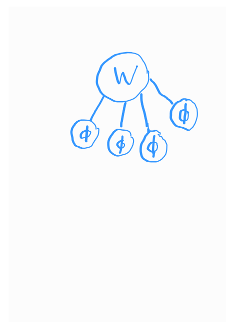
MATH 8450 – LECTURE 11 – FEB 22, 2023

ABDELMALEK ABDESSELAM

Perturbation theory with Feynman diagrams cont'd:

Picking up the thread from the previous lecture, our next task is to write a Feynman diagram expansion for

$$\mathcal{Z} := \int_{\mathbb{R}^\Lambda} \mathrm{d}\mu_C(\phi)\ \exp\left(\frac{1}{4!} \right.$$

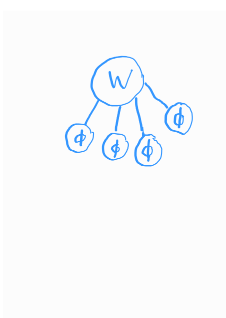


$$\left. \vphantom{\frac{1}{4!}} \right) ,$$

where Λ is a finite set, $A = (A(x,y))_{x,y\in\Lambda}$ is a real symmetric positive definite matrix with rows and columns indexed by Λ, $C = A^{-1}$, and

$$\mathrm{d}\mu_C(\phi) := \frac{e^{-\frac{1}{2}\phi^{\mathrm{T}} A\phi}}{(2\pi)^{\frac{|\Lambda|}{2}}\sqrt{\det(C)}}\ \mathrm{d}^\Lambda\phi\ .$$

We also have



$$= \sum_{x_1,x_2,x_3,x_4\in\Lambda} W(x_1,x_2,x_3,x_4)\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\ ,$$

where the symmetric tensor W is such that the tensor elements $W(x_1,x_2,x_3,x_4)$ belong to some ring of formal power series $\mathbb{C}[[Y_1,\ldots,Y_p]] =: \mathbb{C}[[Y]]$.

Theorem 1. *Suppose that $W = (W(x_1,x_2,x_3,x_4))_{x_1,x_2,x_3,x_4\in\Lambda}$ is such that all tensor elements $W(x_1,x_2,x_3,x_4)$ are formal power series with zero constant term, then we have the representation*

$$\mathcal{Z} = \sum_{[E,F]} \frac{\mathcal{A}(E,F)}{|\mathrm{Aut}(E,F)|}\ .$$

which converges in the ring $\mathbb{C}[[Y]]$.

Most of what follows is about explaining the different concepts and notations featuring in the above theorem.

We first define a functor $\mathscr{F}$ from the category FinBij of finite sets with bijections into itself. Given a finite set E, we define a Feynman diagram structure on E to be a pair $F = (V, M)$ made of two set partitions of E satisfying the following two conditions.

(1) The partition V is only made of blocks of size 4.
(2) The partition M is only made of blocks of size 2, i.e., it is a perfect matching on E.

We will denote the set of Feynman diagrams on a given finite set E by $\mathscr{F}(E)$. We will call V a vertex structure, and we will call M a Wick contraction scheme. For example, consider the finite set

$$E = \{1, 3, 5, 6, 7, 10, \mathrm{A}, \mathrm{B}, \mathrm{C}, \clubsuit, \diamondsuit, \heartsuit\}\ ,$$

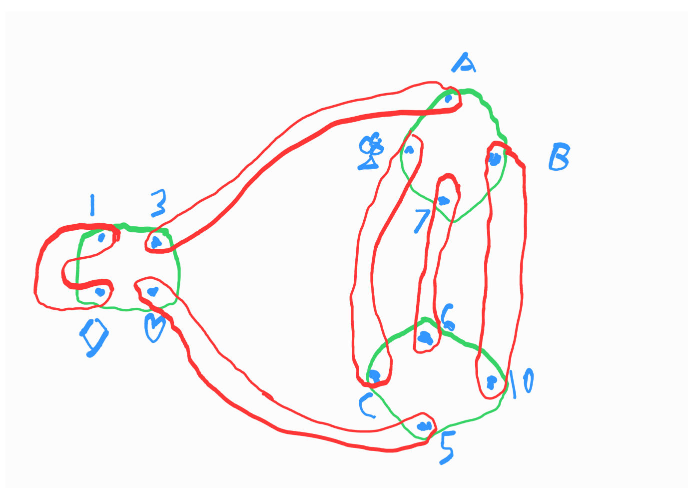
and define the set partitions

$$V = \{\ \{1, 3, \diamondsuit, \heartsuit\}, \{\mathrm{A}, \mathrm{B}, \clubsuit, 7\}, \{6, 5, 10, \mathrm{C}\}\ \}\ ,$$

and

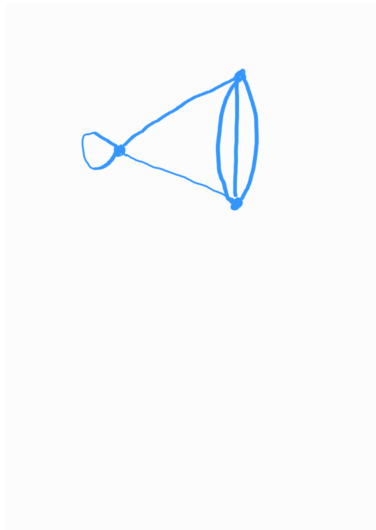
$$M = \{\ \{1, \diamondsuit\}, \{3, \mathrm{A}\}, \{\heartsuit, 5\}, \{\clubsuit, \mathrm{C}\}, \{7, 6\}, \{\mathrm{B}, 10\}\ \}\ .$$

Then $F = (V, M)$ is an example of a Feynman diagram on E, and it can be represented by the following picture



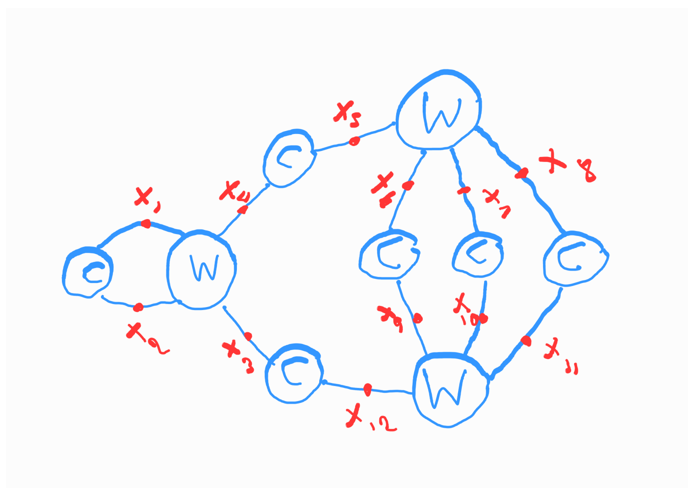
(1)

where the green subsets are the elements of the partition V, and the subsets in red are the elements of M. An abbreviated form of the above picture is



(2)

which is an example of a vacuum Feynman diagram for the ϕ^4 QFT, as can be seen in physics textbooks. The set E is a set of labels or names for the half-lines in the abbreviated picture (2). The purpose of such pictures is to encode a quantity called the amplitude of the diagram

$\mathcal{A}(E, F) :=$ 

which is to be computed according to the rules of the tensor diagram calculus from Lectures 4 and 5. We have indicated in red the junction points corresponding to contracted indices, here also indicated and named as x_1 to x_{12}. Namely,

$$\begin{aligned} \mathcal{A}(E, F) \; := & \sum_{x_1,\ldots,x_{12}\in\Lambda} W(x_1, x_2, x_3, x_4) W(x_5, x_6, x_7, x_8) W(x_9, x_{10}, x_{11} x_{12}) \\ & \times\, C(x_1, x_2) C(x_4, x_5) C(x_3, x_{12}) C(x_6, x_9) C(x_7, x_{10}) C(x_8, x_{11}) \ . \end{aligned}$$

In this sum, we continued our previous practice from Lectures 4 and 5 of naming indices using numbers, starting from 1 and ending with the number of indices needed, which is 12 in our example. However, since we already have a set of labels E, it is more natural, in this particular example, to label indices using the names provided by the elements of E, i.e., to use the picture (1) in order to write instead

$$\begin{aligned}\mathcal{A}(E,F) \;\; := & \sum_{(x_a)_{a\in E}\in\Lambda^E} W(x_1,x_3,x_\diamondsuit,x_\heartsuit)W(x_{\rm A},x_{\rm B},x_7,x_\clubsuit)W(x_5,x_6,x_{10},x_{\rm C}) \\ & \times C(x_1,x_\diamondsuit)C(x_3,x_{\rm A})C(x_\heartsuit,x_5)C(x_\clubsuit,x_{\rm C})C(x_7,x_6)C(x_{\rm B},x_{10})\ . \end{aligned} \qquad (3)$$

If $\sigma : E \to E'$ is a bijection between finite sets, and if $F = (V, M)$ is a Feynman diagram on E, we can define a Feynman diagram structure $F' = (V', M') =: \mathscr{F}[\sigma](F)$ on E' called the transport along the bijection σ of the Feynman diagram structure F on E. We use direct images of blocks of partitions to set

$$V' := \{\sigma(A)\ |\ A \in V\}\ ,$$

and

$$M' := \{\sigma(B)\ |\ B \in M\}\ .$$

It is easy to see that $\mathscr{F}$ is an endofunctor of the category FinBij, i.e., a (covariant) functor from the category FinBij into itself. Indeed, the set of Feynman diagram structures $\mathscr{F}(E)$ is a finite set, and one can give a very coarse bound

$$|\mathscr{F}(E)| \le |\mathscr{P}(\mathscr{P}(E))| \times |\mathscr{P}(\mathscr{P}(E))| = 2^{2^{|E|+1}}\ ,$$

because set partitions are elements of the iterated power set $\mathscr{P}(\mathscr{P}(E))$.

Definition 1. *An endofunctor of the category* FinBij *is called a combinatorial species.*

Example 1. *The functor $\mathscr{F}$ which, to a given finite set E, associates the set of all possible Feynman diagram structures on E, and sends morphisms (bijections) σ to morphisms (bijections too) $\mathscr{F}[\sigma]$, is an example of combinatorial species.*

Example 2. *Given a finite set V, we can define $\mathscr{T}(V)$ as the set of all trees on V. By taking direct images of edges seens as subsets, i.e., elements of $V^{(2)}$, we clearly have natural morphisms $\mathscr{T}[\sigma]$ sending trees on V to trees on V' whenever we have a bijection $\sigma : V \to V'$. This is another example of combinatorial species in the sense of Joyal.*

We now introduce some definitions which apply to general combinatorial species, but for the sake of concreteness we will do so on our running example provided by the functor $\mathscr{F}$ tailored for the notion of Feynman diagrams.

Let (E, F) be a pair consisting of a finite set E and a Feynman diagram F on E, i.e., $F \in \mathscr{F}(E)$. Let (E', F') be another such pair with $F' \in \mathscr{F}(E')$. We define the relation $(E, F) \sim (E', F')$ by the statement

$$\exists\sigma \in {\rm Hom}_{\mathsf{FinBij}}(E, E'),\ F' = \mathscr{F}[\sigma](F)\ .$$

In other words, two pairs are related if there is a relabeling/renaming bijection from E to E' which sends F to F', exactly. This clearly is an equivalence relation. By $[E, F]$ we denote the equivalence class of the pair (E, F). We will also introduce the notion of automorphism group of a such a pair. First, for any finite set E we will use $\mathfrak{S}_E$ for the group of all bijections from E to itself, with the composition operation. So $\mathfrak{S}_{[n]}$ is just the familiar symmetric group $\mathfrak{S}_n$. Note that we can also write

$$\mathfrak{S}_E = {\rm Hom}_{\mathsf{FinBij}}(E, E)\ ,$$

in a more category-theoretic fashion. Given a pair (E, F) made of a finite set and a Feynman diagram on it, we define the corresponding automorphism group

$$\mathrm{Aut}(E, F) := \{\sigma \in \mathfrak{S}_E \mid \mathscr{F}[\sigma](F) = F\} \ .$$

It is easy to see that this is indeed a group, in fact a subgroup of $\mathfrak{S}_E$. If $(E, F) \sim (E', F')$, then the corresponding automorphism groups are isomorphic. More precisely, suppose that $\sigma : E \to E'$ is a bijection such that $F' = \mathscr{F}[\sigma](F)$, then the map $\mathrm{Aut}(E, F) \to \mathrm{Aut}(E', F')$ given by

$$\tau \longmapsto \sigma \circ \tau \circ \sigma^{-1}$$

is a group isomorphism, i.e., it is a group homomorphism and it is bijective. To see that the map does indeed land in $\mathrm{Aut}(E', F')$, and to get some practice with these functors, one can write

$$\begin{aligned} \mathscr{F}[\sigma \circ \tau \circ \sigma^{-1}](F') &= \mathscr{F}[\sigma \circ \tau \circ \sigma^{-1}](\mathscr{F}[\sigma](F)) \\ &= \mathscr{F}[\sigma \circ \tau \circ \sigma^{-1} \circ \sigma](F) \\ &= \mathscr{F}[\sigma]\left(\mathscr{F}[\tau](F)\right) \\ &= \mathscr{F}[\sigma](F) \\ &= F' \ . \end{aligned}$$

In 1st line, we used the hypothesis on σ. In the 2nd and 3rd lines, we used the functorial properties of $\mathscr{F}$, i.e., its behavior with respect to composition of morphisms. In the 4th line, we used that τ is an automorphism of F. Finally, in the 5th line we again used the hypothesis on σ. We leave the checking of the other properties (homomorphism and bijection) as exercises. The above implies $|\mathrm{Aut}(E, F)| = |\mathrm{Aut}(E', F')|$. The number $|\mathrm{Aut}(E, F)|$ only depends on the equivalence class $[E, F]$ and is an important quantity called the *symmetry factor* of the Feynman diagram. Apart from taking care of index relabeling in an automated fashion, another advantage of the categorical species formalism is that it gives a very precise and careful definition of these symmetry factors which can be treacherous to compute and are important in practical QFT calculations.

For example, the automorphism group of the diagram (1) is

$$\mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}/_{2\mathbb{Z}} \times \mathfrak{S}_3 \ ,$$

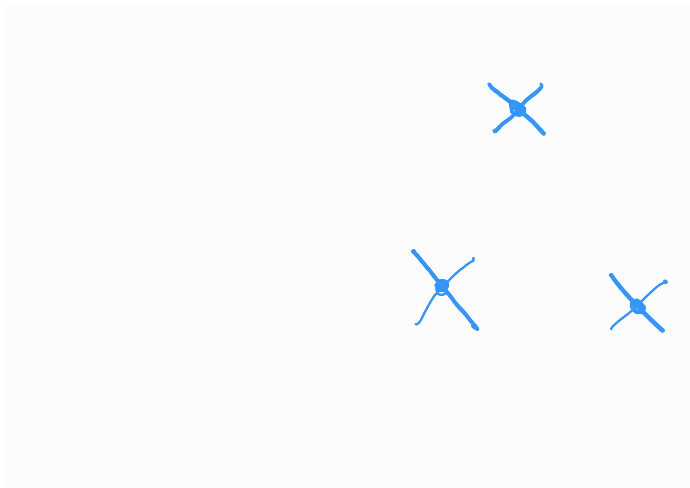
and the symmetry factor is $2 \times 2 \times 3! = 24$. Generators of the group are given, e.g., by the following permutations written in product (composition) of cycles (here transpositions) notation:

(1) the transposition $\sigma_1 = (1, \diamondsuit)$ which exchanges $1, \diamondsuit$ and leaves all the other elements of E fixed, i.e., it is the permutation which exchanges the two half line making the tadpole (edge with both ends at the same vertex) in picture (2),
(2) the permutation $\sigma_2 = (3, \heartsuit)\ (\mathrm{A}, 5)\ (\clubsuit, \mathrm{C})\ (7, 6)\ (\mathrm{B}, 10)$ which flips or switches the two ϕ^4 vertices on the right of (2),
(3) the permutation $\sigma_3 = (\clubsuit, 7)\ (\mathrm{C}, 6)$ which exchanges a pair of vertical edges on the right of (2),
(4) the permutation $\sigma_4 = (7, \mathrm{B})\ (6, 10)$ which exchanges another pair of vertical edges on the right of (2).

As a result, in the Feynman diagram expansion of $\mathcal{Z}$ given by Theorem 1, the expression (3) will appear with the weight $\frac{1}{24}$. This can be seen in a different more pedestrian way as follows. Since there are 3 vertices of ϕ^4 type (tetravalent vertices), this diagram appreas in third order perturbation theory in λ. From the expansion of the exponential defining $\mathcal{Z}$, we see that the diagram will come from the use of the Isserlis-Wick theorem when taking the Gaussian integral of

$$\frac{1}{3!}\left(\frac{1}{4!}\right)^3\left[\sum_{x_1,x_2,x_3,x_4\in\Lambda} W(x_1,x_2,x_3,x_4)\ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right]^3 ,$$

The expression between $[\cdots]$ can be represented graphically by



with "free-floating" vertices waiting to be "Wick-contracted" so as to obtain the diagram (2). So we need to count Wick contraction schemes which will produce that diagram ("up to topological equivalence" as can be read in QFT books). We have a factor of 3 for choosing which of the three vertices will carry the tadpole (or loop edge). Then we have a factor $\binom{4}{2}=6$, to place that tadpole. We pick one of the remaining half-edges of that vertex, and connect it to a half-edge on any of the two remaining vertices. This accounts for 8 possibilities. Then the last half-edge of the initial vertex has no choice but to connect to a half-edge of the so far uncontracted third vertex. This gives 4 possibilities. Finally, one has to connect the 6 remaining half-edges, in order to build the triple edge on the right of (2). This gives $3!=6$ possibilities. Altogether, this gives a combinatorial weight

$$\frac{1}{3!}\left(\frac{1}{4!}\right)^3\times 3\times 6\times 8\times 4\times 6=\frac{1}{24} ,$$

as expected.