MATH 8450 – LECTURE 12 – FEB 27, 2023

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Perturbation theory cont'd:

We continue our study of the partition function

$$\mathcal{Z} = \int_{\mathbb{R}^\Lambda} \mathrm{d}\mu_C(\phi)\ \exp\left(\frac{1}{4!} W \cdot \phi^4\right)\ , \qquad (1)$$

where we used the shorthand notation

$$W \cdot \phi^4 = \sum_{x_1, x_2, x_3, x_4 \in \Lambda} W(x_1, x_2, x_3, x_4)\ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\ .$$

The tensor elements $W(x_1, x_2, x_3, x_4)$ are assumed to be formal power series in some $\mathbb{C}[[Y]] = \mathbb{C}[[Y_1, \ldots, Y_p]]$, with no constant term. We first provide more details on the definition of the RHS of (1). The integrand is defined using the composition of formal power series as in Lecture 3. The exponential is given by its usual power series expansion

$$\exp(u) = \sum_{N=0}^{\infty} \frac{1}{N!} u^N\ \in\ \mathbb{C}[[u]]\ ,$$

and we then substitute $u := \frac{1}{4!} W \cdot \phi^4$. So, as an element of $\mathbb{C}[[Y]]$, we have the convergent sum representation

$$\exp\left(\frac{1}{4!} W \cdot \phi^4\right) = \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{1}{4!} W \cdot \phi^4\right)\ . \qquad (2)$$

Suppose we are given a map $\mathbb{R}^\Lambda \to \mathbb{C}[[Y]]$,

$$\phi \longmapsto \sum_{\alpha \in \mathbb{N}^p} c_\alpha(\phi) Y^\alpha\ .$$

We can then define the integral $\int \mathrm{d}\mu(\phi)$ of this map, as the element of $\mathbb{C}[[Y]]$ given by

$$\sum_{\alpha \in \mathbb{N}^p} \left(\int_{\mathbb{R}^\Lambda} \mathrm{d}\mu(\phi)\ c_\alpha(\phi) \right)\ Y^\alpha\ ,$$

namely, in a term-by-term fashion. Of course, this needs each function $\phi \mapsto c_\alpha(\phi)$ to be integrable. In the case of (2), and because of the no-constant term hypothesis for the W tensor elements, we have that all the $c_\alpha(\phi)$ are multivariate polynomials in the components $\phi(x)$, $x \in \Lambda$, of the vector variable ϕ. As seen in Lecture 8, these functions are integrable with respect to the Gaussian measure

$$\mathrm{d}\mu_C(\phi) := \frac{e^{-\frac{1}{2}\phi^{\mathrm{T}} C^{-1} \phi}}{(2\pi)^{\frac{|\Lambda|}{2}} \sqrt{\det(C)}}\ \mathrm{d}^\Lambda \phi\ .$$

Moreover, it is easy to see, using the definition of convergence of infinite sums in rings of formal power series from Lecture 3, that

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\mathbb{R}^\Lambda} \mathrm{d}\mu(\phi) \left(\frac{1}{4!} W \cdot \phi^4\right)^N .$$

Having defined the LHS, we now turn to the RHS of the main equation to be established

$$\mathcal{Z} = \sum_{[E,F]} \frac{\mathcal{A}(E,F)}{|\mathrm{Aut}(E,F)|} , \tag{3}$$

and provide the general definition of the amplitude $\mathcal{A}(E,F)$. Given any finite set I with 4 elements and any map $x : I \to \Lambda$, we let

$$W(x) := W(x(i_1), x(i_2), x(i_3), x(i_4)) ,$$

where i_1, i_2, i_3, i_4 form an enumeration of the elements of I. Namely, i_1, i_2, i_3, i_4 are distinct and $I = \{i_1, i_2, i_3, i_4\}$. The above is well defined because W is a symmetric tensor and different enumerations differ by simple reordering. Likewise, for I a set with 2 elements and for any map $x : I \to \Lambda$, we let

$$C(x) := C(x(i_1), x(i_2)) ,$$

where i_1, i_2 is any of two possible enumerations of the elements of I. Again, this is meaningful because the tensor, or rather matrix, $C = A^{-1}$ is symmetric, since A is symmetric. Now if E is a finite set, and $F = (V, M) \in \mathscr{F}(E)$ is a Feynman diagram structure on E, we define its amplitude as

$$\mathcal{A}(E,F) := \sum_{x \in \Lambda^E} \left(\prod_{A \in V} W(x|_A)\right) \times \left(\prod_{B \in M} C(x|_B)\right) ,$$

where the standard notation $x|_A$ refers to the restriction of the map x to the subset A. It is easy to see that this generalizes the definition given for the explicit example of Feynman diagram with $|E| = 12$, in Lecture 11. We now show that the amplitude only depends on the equivalence class $[E,F]$.

Proposition 1. *If $(E,F) \sim (E',F')$, then $\mathcal{A}(E,F) = \mathcal{A}(E',F')$.*

Proof: By hypothesis, there exists a bijection $\sigma : E \to E'$, such that $\mathscr{F}[\sigma](F) = F'$. Namely, if $F' = (V', M')$, then, using direct images, $V' = \{\sigma(A) \ |A \in V\}$ and $M' = \{\sigma(B) \ |B \in M\}$.

We now have

$$\begin{aligned}
\mathcal{A}(E',F') &= \sum_{x'\in\Lambda^{E'}} \left(\prod_{A'\in V'} W(x'|_{A'})\right) \times \left(\prod_{B'\in M'} C(x'|_{B'})\right) \\
&= \sum_{x'\in\Lambda^{E'}} \left(\prod_{A\in V} W(x'|_{\sigma(A)})\right) \times \left(\prod_{B\in M} C(x'|_{\sigma(B)})\right) \\
&= \sum_{x'\in\Lambda^{E'}} \left(\prod_{A\in V} W((x'\circ\sigma)|_{A})\right) \times \left(\prod_{B\in M} C((x'\circ\sigma)|_{B})\right) \\
&= \sum_{x\in\Lambda^{E}} \left(\prod_{A\in V} W(x|_{A})\right) \times \left(\prod_{B\in M} C(x|_{B})\right) \\
&= \mathcal{A}(E,F)\ .
\end{aligned}$$

In the 1st line, we used the definition of amplitude. In the 2nd line, we used the bijective parametrization $A \mapsto \sigma(A)$ of V' by the elements of V, and likewise for M', in order to change the indices in the products. The 3rd line is just basic theory of sets and maps. In the 4th line, we introduced a new summation index $x := x' \circ \sigma \in \Lambda^E$ which is in bijective correspondence with the old summation index $x' \in \Lambda^{E'}$, in order to rewrite the sum. Finally, we again used the definition of amplitude. $\square$

Proposition 2. *The series*

$$\sum_{[E,F]} \frac{\mathcal{A}(E,F)}{|\mathrm{Aut}(E,F)|}$$

is well defined as an element of $\mathbb{C}[[Y]]$.

Proof: Note that, for fixed $k \in \mathbb{N}$, there is only finitely many classes $[E,F]$ with $|E| = k$. Indeed, if E has k elements, then there exists a bijection $\sigma : [k] \to E$. We can then define $F_0 := \mathscr{F}[\sigma^{-1}](F)$, and by construction we will have $(E,F) \sim ([k],F_0)$, and hence, $[E,F] = [[k],F_0]$. The summation over $[E,F]$ is therefore over a countable set, namely, the union over $k \in \mathbb{N}$, of finite sets (of cardinality at most $2^{2^{k+1}}$ as seen in the previous lecture). Note that a class $[E,F]$ does not contribute to the series unless E has at least one Feynman diagram structure on it, i.e., F, and this requires $|E|$ to be divisible by 4. Also note that because the W's have no constant term, the total degree $|\alpha|$ of a monomial Y^α present in $\mathcal{A}(E,F)$ must be at least equal to the number of W factors, i.e., $|V| = |E|/4$. So for any fixed multiindex α, if $|E| > 4|\alpha|$, then $[Y^\alpha]\mathcal{A}(E,F) = 0$. This implies the convergence of the series in the proposition, in the ring of formal power series $\mathcal{A}(E,F)$. $\square$

At this point, both sides of (1) are well defined and we just need to prove the equality. This will require some additional notions from category theory.

Category theory cont'd:

Just like maps, functors can be composed.

Definition 1. *Let* $\mathsf{A},\mathsf{B},\mathsf{C}$ *be somes categories. Let* $\mathscr{F}$ *be a functor from* A *to* B*, and let* $\mathscr{G}$ *be a functor from* B *to* C*. We can define the composition* $\mathscr{G}\circ\mathscr{F}$ *as the functor from* A *to* C *obtained as follows. For any object* $A \in \mathrm{Ob}(\mathsf{A})$*, we let* $(\mathscr{G}\circ\mathscr{F})(A) := \mathscr{G}(\mathscr{F}(A)) \in \mathrm{Ob}(\mathsf{C})$*. For any* *objects* $A_1, A_2 \in \mathrm{Ob}(\mathsf{A})$*, and any morphism* $f \in \mathrm{Hom}_{\mathsf{A}}(A_1, A_2)$*, we let* $(\mathscr{G} \circ \mathscr{F})[f] := \mathscr{G}[\mathscr{F}[f]]$*, which belongs to* $\mathrm{Hom}_{\mathsf{C}}((\mathscr{G} \circ \mathscr{F})(A_1), (\mathscr{G} \circ \mathscr{F})(A_2))$.

Similarly to sets, there is a notion of inclusion for categories.

Definition 2. *Let* A, B *be some categories. We will say that* A *is a subcategory of* B*, if every object of* A *is also an object of* B*, and if, for all objects* $A_1, A_2 \in \mathrm{Ob}(\mathsf{A})$*, we have the inclusion*

$$\mathrm{Hom}_{\mathsf{A}}(A_1, A_2) \subset \mathrm{Hom}_{\mathsf{B}}(A_1, A_2) \ .$$

If A *is a subcategory of* B*, we have an obvious inclusion functor* $\mathscr{I}_{\mathsf{A},\mathsf{B}}$ *from* A *to* B*. It sends objects and morphisms to themselves, seen in reference to the larger category* B*.*

Definition 3. *If* A *is a subcategory of* B*, and we have that, for all objects* $A_1, A_2 \in \mathrm{Ob}(\mathsf{A})$*,*

$$\mathrm{Hom}_{\mathsf{A}}(A_1, A_2) = \mathrm{Hom}_{\mathsf{B}}(A_1, A_2) \ ,$$

then we say that A *is a full subcategory of* B*. Namely, we may lose some objects but we retain all morphisms between the objects we kept, when going to the subcategory.*

Definition 4. *If* A *is a subcategory of* B*, and every object of* B *is an object of* A*, then we say that* A *is a wide subcategory of* B*. Namely, we keep all objects, but we allow the loss of some morphisms when going to the subcategory.*

For instance, the category $\mathsf{Metrizable}$ of metrizable topological spaces with continuous maps is a full subcategory of Top (topological spaces with continuous maps). From the list of examples from Lecture 9 on metric spaces, we see, e.g., that MetLip is a wide subcategory of $\mathsf{MetUnif}$, which itself is a wide subcategory of $\mathsf{MetCont}$.

The last notion we will need from basic category theory is that of natural transformation (or morphism of functors). Suppose A, B are some categories and suppose that $\mathscr{F}$ and $\mathscr{G}$ are both functors from A to B. Then a natural transformation from $\mathscr{F}$ to $\mathscr{G}$, as in the schematic

$$\begin{array}{ccc} & \mathscr{F} & \\ & \longrightarrow & \\ \mathsf{A} & \downarrow \rho & \mathsf{B} \\ & \longrightarrow & \\ & \mathscr{G} & \end{array}$$

is defined by the following data and coherence/compatibility axiom.

The data needed to specify ρ is a choice, for every object $A \in \mathrm{Ob}(\mathsf{A})$, of a morphism

$$\rho_A \in \mathrm{Hom}_{\mathsf{B}}(\mathscr{F}(A), \mathscr{G}(A)) \ .$$

The axiom that needs to be satisfied, for ρ to qualify as natural transformation, is that for all $A_1, A_2 \in \mathrm{Ob}(A)$, and for all $f \in \mathrm{Hom}_{\mathsf{A}}(A_1, A_2)$, we have

$$\mathscr{G}[f] \circ \rho_{A_1} = \rho_{A_2} \circ \mathscr{F}[f] \ ,$$

namely, the diagram

$$\begin{array}{ccccc} & & \rho_{A_1} & & \\ & \mathscr{F}(A_1) & \longrightarrow & \mathscr{G}(A_1) & \\ \mathscr{F}[f] & \downarrow & & \downarrow & \mathscr{G}[f] \\ & \mathscr{F}(A_2) & \longrightarrow & \mathscr{G}(A_2) & \\ & & \rho_{A_2} & & \end{array}$$

commutes.

We now resume the discussion of (3). We will need to consider in addition to the species of Feynman diagrams $\mathscr{F}$, another combinatorial species or functor $\mathscr{V}$ for vertex structures. For a finite set E, recall that a vertex structure on E is any set partition V of E with only blocks of cardinality 4. The set of vertex structures on E will be denoted by $\mathscr{V}(E)$. If $\sigma : E \to E'$ is a bijection between finite sets, and if V is a vertex structure on E, we define the transport of V along σ as

$$\mathscr{V}[\sigma](V) := \{\sigma(A) \mid A \in V\} \ .$$

Clearly $\mathscr{V}[\sigma]$ is a bijection from $\mathscr{V}(E)$ onto $\mathscr{V}(E')$, and we just produced a functor $\mathscr{V}$ from FinBij to itself. We will now need a natural transformation ρ from $\mathscr{F}$ to $\mathscr{V}$. For all finite set E, we will have at our disposal a *map* ρ_E from $\mathscr{F}(E)$ to $\mathscr{V}(E)$. These maps will be such that for all finite sets E, E' and all bijections $\sigma : E \to E'$,

$$\mathscr{V}[\sigma] \circ \rho_E = \rho_{E'} \circ \mathscr{F}[\sigma] \ .$$

Remark 1. *Technically, ρ is not a natural transformation from $\mathscr{F}$ to $\mathscr{V}$ because these are endofunctors of FinBij and they would call for ρ_E's which are morphisms in FinBij, i.e., bijections, which is not good for our purposes. However, FinBij is a wide subcategory of Fin. Let us denote the corresponding inclusion functor by $\mathscr{I}$. The ρ mentioned above is, in the strict sense, a natural transformation from $\mathscr{I} \circ \mathscr{F}$ to $\mathscr{I} \circ \mathscr{V}$. This allows the ρ_E to be nonbijective maps.*

The concrete transformation ρ we will need is that of forgetting the Wick contraction scheme. Namely, if $F = (V, M)$ is Feynman diagram on E, we let $\rho_E(F) := V$, by definition. It is easy to see that this is a natural transformation. The following is the key result, as far as the use of category theory and combinatorial species for the handling of Feynman diagram expansions.

Theorem 1. *We have the equality*

$$\sum_{[E,F]} \frac{\mathcal{A}(E,F)}{|\mathrm{Aut}(E,F)|} = \sum_{[E,V]} \frac{1}{|\mathrm{Aut}(E,V)|} \left(\sum_{F \in \rho_E^{-1}(\{V\})} \mathcal{A}(E,F) \right) \ . \tag{4}$$

More precisely, if the LHS converges in the ambient ring of formal power series, then so does the RHS and equality holds.

We will prove this in the next lecture. In the remainder of this one, we will use this theorem to complete the proof of (3).

Remark 2. *The theorem is stated for the particular species $\mathscr{F}$ of Feynman diagrams, and $\mathscr{V}$ of vertex structures, as well as the forgetful transformation ρ, but it holds more generally for any combinatorial species and any natural transformation ρ (in the weak sense of the previous remark, i.e., allowing ρ_E's to just be maps instead of bijections). Finally, this more general theorem applies to any definition of of amplitude $\mathcal{A}(E,F)$, in some ring of formal power series $\mathbb{C}[[Y]]$, provided it only depends on the equivalence class $[E,F]$ with respect to the $\mathscr{F}$ species.*

Remark 3. *The definition of the equivalence $(E, F) \sim (E', F')$ as*

$$\exists \sigma \in \mathrm{Hom}_{\mathsf{FinBij}}(E, E'), \mathscr{F}[\sigma](F) = F'$$

is not specific to the species of Feynman diagrams, but can be used for any combinatorial species. In particular, it can be used for $\mathscr{V}$. This explains the meaning of the classes $[E, V]$ on the RHS of (4). It is easy to check also that a particular term

$$\frac{1}{|\mathrm{Aut}(E,V)|}\left(\sum_{F\in\rho_E^{-1}(\{V\})} \mathcal{A}(E,F)\right) ,$$

of the sum on the RHS, only depends on the class $[E, V]$ and not on the particular representative (E, V).

Going back to the RHS of (3), and after using Theorem 1 for our specific ρ, we get

$$\sum_{[E,F]} \frac{\mathcal{A}(E,F)}{|\mathrm{Aut}(E,F)|} = \sum_{[E,V]} \frac{1}{|\mathrm{Aut}(E,V)|}\left(\sum_{M} \mathcal{A}(E,(V,M))\right) , \tag{5}$$

where the sum over M is over all Wick contraction schemes on E. However, by the Isserlis-Wick Theorem, and the definition of the amplitude, we have

$$\begin{aligned}\sum_{M} \mathcal{A}(E,(V,M)) &= \int_{\mathbb{R}^\Lambda} \mathrm{d}\mu(\phi) \sum_{x\in\Lambda^E} \prod_{A\in V}\left[W(x|_A)\prod_{a\in A}\phi(x(a))\right] \\ &= \int_{\mathbb{R}^\Lambda} \mathrm{d}\mu(\phi) \prod_{A\in V}\left(\sum_{x_A\in\Lambda^A} W(x_A)\prod_{a\in A}\phi(x_A(a))\right) ,\end{aligned}$$

since summing over the map $x : E \to \Lambda$ can be done by summing independently on all its restrictions x_A to blocks A of the set partition V of E. Clearly,

$$\sum_{x_A\in\Lambda^A} W(x_A)\prod_{a\in A}\phi(x_A(a)) = W \cdot \phi^4 .$$

Thus

$$\sum_{M} \mathcal{A}(E,(V,M)) = \int_{\mathbb{R}^\Lambda} \mathrm{d}\mu(\phi) \left(W\cdot\phi^4\right)^{|V|} ,$$

and plugging back in (5), we obtain

$$\sum_{[E,F]} \frac{\mathcal{A}(E,F)}{|\mathrm{Aut}(E,F)|} = \sum_{[E,V]} \frac{1}{|\mathrm{Aut}(E,V)|}\int_{\mathbb{R}^\Lambda} \mathrm{d}\mu(\phi) \left(W\cdot\phi^4\right)^{|V|} .$$

We will now use Theorem 1 a second time but for another natural transformation, also using the idea of "forgetting". We define a "nothing" species $\mathscr{N}$ by letting $\mathscr{N}(E) := \{\varnothing\}$ for any finite set E. Also, for any bijection $\sigma : E \to E'$, we let $\mathscr{N}[\sigma]$ be the identity map on the set $\{\varnothing\}$. We also define a natural transformation (in the weak sense, as before) η from $\mathscr{V}$ to $\mathscr{N}$, given by the maps $\eta_E : \mathscr{V}(E) \to \mathscr{N}(E)$, $V \mapsto \varnothing$. We will also need a suitable amplitude for vertex structures. For E a finite set and $V \in \mathscr{V}(E)$, we define

$$\mathcal{B}(E,V) := \int_{\mathbb{R}^\Lambda} \mathrm{d}\mu(\phi) \left(W\cdot\phi^4\right)^{|V|} .$$

This clearly only depends on the class $[E,V]$. Now Theorem 1, used along the transformation η, gives us

$$\begin{aligned}
\sum_{[E,V]} \frac{\mathcal{B}(E,V)}{|\mathrm{Aut}(E,V)|} &= \sum_{[E,\varnothing]} \frac{1}{|\mathrm{Aut}(E,\varnothing)|} \left(\sum_{V\in\mathscr{V}(E)} \mathcal{B}(E,V) \right) \\
&= \sum_{N\in\mathbb{N}} \frac{1}{|\mathrm{Aut}([4N],\varnothing)|} \left(\sum_{V\in\mathscr{V}([4N])} \mathcal{B}([4N],V) \right) \\
&= \sum_{N\in\mathbb{N}} \frac{1}{(4N)!} \times \frac{(4N)!}{4!^N N!} \times \int_{\mathbb{R}^\Lambda} \mathrm{d}\mu(\phi) \left(W\cdot\phi^4\right)^N \\
&= \mathcal{Z}\ .
\end{aligned}$$

In the 1st line, sets E for which the sum $\sum_{V\in\mathscr{V}(E)}$ is nonempty must have cardinality divisible by 4. This allowed us, in the 2nd line, to parametrized the relevant classes $[E,\varnothing]$ with respect to the species $\mathscr{N}$, using the "preferred model" $E=[4N]$, $N\in\mathbb{N}$. The 3rd line used the fact $\mathrm{Aut}([4N],\varnothing)$ is the full symmetric group $\mathfrak{S}_{[4N]}$, and the number of vertex structures on $[4N]$ is exactly $\frac{(4N)!}{4!^N N!}$. This concludes the proof of (3) modulo that of Theorem 1.