MATH 8450 – LECTURE 13 – MAR 1, 2023

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Proof of the main theorem about combinatorial species:

We now prove Theorem 1 from Lecture 12. There it was stated for two particular combinatorial species $\mathscr{F}$ (Feynman diagrams) and $\mathscr{V}$ (vertex structures), but we will now do this in full generality. We assume we have two arbitrary combinatorial species $\mathscr{F},\mathscr{V}$, i.e., endofunctors of the category FinBij. We assume we have a definition of amplitude, i.e., for each pair (E,F) where E is a finite set and $F\in\mathscr{F}(E)$, we associate an element $\mathcal{A}(E,F)$ in some fixed formal power series ring $\mathbb{C}[[Y]]$. Like before, for pairs (E,F) and $(E,'F,')$ of finite sets equipped with an instance of a type $\mathscr{F}$ combinatorial structure, we have an equivalence relation $(E,F)\sim(E',F')$ defined by $\exists\sigma\in\mathrm{Hom}(E,E')$, $F'=\mathscr{F}[\sigma](F)$. We assume that the amplitude $\mathcal{A}(E,F)$ only depends on the equivalence class $[E,F]$ of a pair (E,F). We assume we have a natural transformation (weak sense explained in last lecture) ρ from $\mathscr{F}$ to $\mathscr{V}$, i.e., for each finite set E we are given a map (not necessarily bijective) ρ_E from $\mathscr{F}(E)$ to $\mathscr{V}(E)$. These maps, by hypothesis, satisfy

$$\mathscr{V}[\sigma]\circ\rho_E=\rho_{E'}\circ\mathscr{F}[\sigma]\ ,$$

for all finite sets E,E' and bijection $\sigma:E\to E'$. Since the notions of equivalence classes and automorphism groups depend on which combinatorial species we are considering, we will make liberal use of $\mathscr{F}$ or $\mathscr{V}$ subscripts, in order to avoid any ambiguity.

We now recall the statement of the theorem.

Theorem 1. *We have the equality*

$$\sum_{[E,F]_{\mathscr{F}}}\frac{\mathcal{A}(E,F)}{|\mathrm{Aut}_{\mathscr{F}}(E,F)|}=\sum_{[E,V]_{\mathscr{V}}}\frac{1}{|\mathrm{Aut}_{\mathscr{V}}(E,V)|}\left(\sum_{F\in\rho_E^{-1}(\{V\})}\mathcal{A}(E,F)\right)\ . \qquad (1)$$

More precisely, if the LHS converges in the ambient ring of formal power series, then so does the RHS and equality holds.

Proof: Suppose we have a pair equivalence $(E,F)\sim(E',F')$ in the context of the species $\mathscr{F}$, then we can pick a bijection $\sigma:E\to E'$ for which $F'=\mathscr{F}[\sigma](F)$. By the hypotheses on ρ, this implies

$$\rho_{E'}(F')=(\rho_{E'}\circ\mathscr{F}[\sigma])(F)=(\mathscr{V}[\sigma]\circ\rho_E)(F)=\mathscr{V}[\sigma](\rho_E(F))\ .$$

This shows that the $\mathscr{V}$-equivalence class $[E,\rho_E(F)]_{\mathscr{V}}$ only depends on the $\mathscr{F}$-equivalence class $[E,F]^{\mathscr{F}}$ of a pair (E,F). In other words, we have a "map"

$$[E,F]_{\mathscr{F}}\rightsquigarrow[E,\rho_E(F)]_{\mathscr{V}}$$

and we can group terms in the sum on the LHS of (1) according to the value of $[E,\rho_E(F)]_{\mathscr{V}}$. Namely, we have

$$\mathrm{LHS} = \sum_{[E,V]_{\mathscr{V}}} \sum_{[E',F']_{\mathscr{F}} \rightsquigarrow [E,V]_{\mathscr{V}}} \frac{\mathcal{A}(E',F')}{|\mathrm{Aut}_{\mathscr{F}}(E',F')|} \ .$$

The above step is the only one in this proof which involves issues of convergence in $\mathbb{C}[[Y]]$, and it is legitimate because we assumed the full sum on the LHS is convergent. The theorem thus reduces to showing that for each fixed class $[E,V]_{\mathscr{V}}$, we have the equality

$$\sum_{[E',F']_{\mathscr{F}} \rightsquigarrow [E,V]_{\mathscr{V}}} \frac{\mathcal{A}(E',F')}{|\mathrm{Aut}_{\mathscr{F}}(E',F')|} = \frac{1}{|\mathrm{Aut}_{\mathscr{V}}(E,V)|} \left(\sum_{F \in \rho_E^{-1}(\{V\})} \mathcal{A}(E,F) \right) \ . \tag{2}$$

So we now fix the finite set E and some $V \in \mathscr{V}(E)$ and consider an arbitrary equivalence class $[E',F']_{\mathscr{F}}$ such that $[E',F']_{\mathscr{F}} \rightsquigarrow [E,V]_{\mathscr{V}}$. This means $(E,V) \sim_{\mathscr{V}} (E',\rho_{E'}(F'))$, and therefore there exists a bijection $\sigma : E \to E'$ such that $\rho_{E'}(F') = \mathscr{V}[\sigma](V)$. Define $F := \mathscr{F}[\sigma^{-1}](F')$. By construction $(E',F') \sim_{\mathscr{F}} (E,F)$ and therefore $[E',F']_{\mathscr{F}} = [E,F]_{\mathscr{F}}$. Moreover,

$$\rho_E(F) = \rho_E(\mathscr{F}[\sigma^{-1}](F')) = \mathscr{V}[\sigma^{-1}](\rho_{E'}(F')) = \mathscr{V}[\sigma^{-1}](\mathscr{V}[\sigma](V)) = \mathscr{V}[\sigma^{-1}\circ\sigma](V) = V \ ,$$

so that $F \in \rho_E^{-1}(\{V\})$. We thus have a surjective parametrization $F \mapsto [E,F]_{\mathscr{F}}$, $F \in \rho_E^{-1}(\{V\}) \subset \mathscr{F}(E)$, for classes $[E',F']_{\mathscr{F}}$ such that $[E',F']_{\mathscr{F}} \rightsquigarrow [E,V]_{\mathscr{V}}$. However, this is not necessarily injective and may lead to overcounting if we simply replace the sum over $[E',F']_{\mathscr{F}}$ by a sum over F on the LHS of (2). To fix this, we introduce the following relation on the set (fiber) $\rho_E^{-1}(\{V\})$. For $F_1, F_2 \in \rho_E^{-1}(\{V\})$, we let $F_1 \approx F_2$, iff $(E,F_1) \sim_{\mathscr{F}} (E,F_2)$, i.e., $\exists \sigma \in \mathfrak{S}_E := \mathrm{Hom}_{\mathsf{FinBij}}(E,E)$, $F_2 = \mathscr{F}[\sigma](F_1)$. It is easy to see that this an equivalence relation on $\rho_E^{-1}(\{V\})$ and we will just write $[F]$ for the equivalence class of some $F \in \rho_E^{-1}(\{V\})$. We now have a bijective parametrization $[F] \mapsto [E,F]_{\mathscr{F}}$ for classes $[E',F']_{\mathscr{F}}$ such that $[E',F']_{\mathscr{F}} \rightsquigarrow [E,V]_{\mathscr{V}}$. We can now change the summation "index", in order to write the LHS of (2)

$$\sum_{[E',F']_{\mathscr{F}} \rightsquigarrow [E,V]_{\mathscr{V}}} \frac{\mathcal{A}(E',F')}{|\mathrm{Aut}_{\mathscr{F}}(E',F')|} = \sum_{[F_1]} \frac{\mathcal{A}(E,F_1)}{|\mathrm{Aut}_{\mathscr{F}}(E,F_1)|} \ ,$$

where we used F_1 instead of F because we will soon need another $\mathscr{F}$-structure F_2. Meanwhile, the RHS of (2) is given by

$$\begin{aligned} \sum_{F_2 \in \rho_E^{-1}(\{V\})} \frac{\mathcal{A}(E,F_2)}{|\mathrm{Aut}_{\mathscr{V}}(E,V)|} &= \sum_{[F_1]} \sum_{F_2 \in [F_1]} \frac{\mathcal{A}(E,F_2)}{|\mathrm{Aut}_{\mathscr{V}}(E,V)|} \\ &= \sum_{[F_1]} \mathcal{A}(E,F_2) \times \sum_{[F_1]} \sum_{F_2 \in [F_1]} \frac{1}{|\mathrm{Aut}_{\mathscr{V}}(E,V)|} \ . \end{aligned}$$

In the 1st line, we just organized the sum over $\rho_E^{-1}(\{V\}$ according to equivalence classes for $\approx$. In the 2nd line, we noted that $F_2 \in [F_1]$ means $(E,F_2) \sim_{\mathscr{F}} (E,F_1)$ which implies $\mathcal{A}(E,F_2) = \mathcal{A}(E,F_1)$, and the amplitude can be factored out of the inner sum. As a result, the equality (2) will hold, if we can show that for all $F_1 \in \rho_E^{-1}(\{V\})$,

$$\frac{1}{|\mathrm{Aut}_{\mathscr{F}}(E,F_1)|} = \sum_{F_2 \in [F_1]} \frac{1}{|\mathrm{Aut}_{\mathscr{V}}(E,V)|} \ ,$$

i.e.,

$$|[F_1]| = \frac{|\mathrm{Aut}_{\mathscr{V}}(E,V)|}{|\mathrm{Aut}_{\mathscr{F}}(E,F_1)|}\ .$$

However, this is a staightforward application of the Orbit-Stabilizer Theorem in basic group theory (see following review). ☐

Brief review of group actions:

We recall (or introduce) some notions about group actions on sets, leading to the Orbit-Stabilizer Theorem invoked in the previous proof.

Definition 1. *Let G be a group with operation denoted multiplicatively and neutral element e, and let X be a set. A left-group action of G on X is a map $L : G \times X \to X$ which satisfies:*

(1) $\forall x \in X,\ L(e,x) = x$,
(2) $\forall g,h \in G,\ \forall x \in X,\ L(gh,x) = L(g, L(h,x))$.

In practice, we write $L(g,x) = gx$, so that the above properties become $ex = x$ and $(gh)x = g(hx)$. The latter looks similar to the associative property.

Definition 2. *Let G be a group with operation denoted multiplicatively and neutral element e, and let X be a set. A right-group action of G on X is a map $R : X \times G \to X$ which satisfies:*

(1) $\forall x \in X,\ R(x,e) = x$,
(2) $\forall g,h \in G,\ \forall x \in X,\ R(x,gh) = R(R(x,g),h)$.

In practice, we write $R(g,x) = xg$, so that the above properties become $xe = x$ and $x(gh) = (xg)h$.

Example 1. *Let G be the symmetric group $\mathfrak{S}_n$ and let $X = [n]$. Then we have a left-action $(\sigma, j) \mapsto \sigma(j)$.*

Example 2. *Let $G = \mathrm{SO}(N)$, $N \geq 2$, namely the group of rotations in N-dimensional (real) space. Here G is the set of orthogonal $N \times N$ matrices, with matrix multiplication as group operation. Let $X = \mathbb{R}^N$ seen as the space of column vectors with N components. Then the matrix product $(R,x) \mapsto Rx$ is a left-action of G on X.*

Example 3.

Note that if $(g,x) \mapsto gx$ is left action of a group G on a set X and if H is a subgroup of G, then by restricting it to $H \times X$, we get a left group action of H on X, and similarly for right actions. One has similar versions for right actions for all the statements/definitions in this review, so we will not keep repeating this "and similarly". Given a left group action of G on X, we define a relation on X as follows. If $x,y \in X$, then we say $x \sim y$, iff $\exists g \in G,\ y = gx$. It is easy to see that this an equivalence relation on X. The equivalence classes are called the *orbits* of the group action. In the $\mathrm{SO}(N)$ example, these would be spheres centered at the origin of radius $r \geq 0$. The case $r = 0$ is a degenerate situation where the orbit reduces to a point, the origin. For a left action, the orbit of a point $x \in X$ is denoted by

$$Gx := \{gx \mid g \in G\}\ .$$

Example 4. *If G is a group, then it has a right action on itself $X \times G \to X$, $(x, g) \mapsto xg$, with $X = G$ and where x, g are arbitrary in G. By taking the restriction to a subgroup H, we get a right action of H on the set G, i.e., $(g, h) \mapsto gh$. The orbits for this action are subsets of G which are of the form $gH := \{gh \mid h \in H\}$, which are called left cosets of H in G. The set of left cosets is denoted by ${}^{G}\!/_{H}$.*

When G (and therefore the subgroup H) is finite, the number of left cosets $[G : H] := \left|{}^{G}\!/_{H}\right|$ called the index of H in G is given by

$$[G : H] = \frac{|G|}{|H|}$$

which is known as Lagrange's Theorem.

If we have a left action of a group G on a set X and if $x \in X$, then the set

$$G_x := \{g \in G \mid gx = x\}$$

forms a subgroup of G, called the *stabilizer* of x. The map

$$\begin{aligned} {}^{G}\!/_{G_x} &\longrightarrow Gx \\ gG_x &\longmapsto gx \end{aligned}$$

is bijective. When G is finite, and using Lagrange's Theorem, we obtain the Orbit-Stabilizer Theorem which gives the cardinality of the orbit of an element $x \in X$:

$$|Gx| = \frac{|G|}{|G_x|} \ .$$

In the previous proof, we have the group $\mathfrak{S}_E = \mathrm{Hom}_{\mathsf{FinBij}}(E, E)$ of bijections σ from E to E, with the operation of composition. It has a left action on $\mathscr{F}(E)$ given by

$$(\sigma, F) \longmapsto \sigma F := \mathscr{F}[\sigma](F) \ .$$

It also has a left action on $\mathscr{V}(E)$ given by

$$(\sigma, V) \longmapsto \sigma V := \mathscr{V}[\sigma](V) \ .$$

Given the fixed $V \in \mathscr{V}(E)$ considered in the previous proof, we see that the stabilizer of V for the action of $\mathfrak{S}_E$ is $\mathrm{Aut}_{\mathscr{V}}(E, V)$. When restricting the action of $\mathfrak{S}_E$ on $\mathscr{F}(E)$, to the subgroup $\mathrm{Aut}_{\mathscr{V}}(E, V)$, we see that the subset $\rho_E^{-1}(V) \subset \mathscr{F}(E)$ is invariant (as a whole, not pointwise) by this last action. We can thus restrict the action further (on the set rather than group side) to a left action of $\mathrm{Aut}_{\mathscr{V}}(E, V)$ on $\rho_E^{-1}(V)$. The orbit, for this very last action, of F_1 is precisely $[F_1]$, whereas the stabilizer of F_1 is $\mathrm{Aut}_{\mathscr{F}}(E, F_1)$, seen as a subgroup of $\mathrm{Aut}_{\mathscr{V}}(E, V)$, instead of the even bigger group $\mathfrak{S}_E$. Therefore the last equation in the previous proof,

$$|[F_1]| = \frac{|\mathrm{Aut}_{\mathscr{V}}(E, V)|}{|\mathrm{Aut}_{\mathscr{F}}(E, F_1)|} \ ,$$

follows from the Orbit-Stabilizer Theorem.