MATH 8450 – LECTURE 4 – JAN 30, 2023

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Formal power series continued:

Remark 1. *Writing an explicit formula for the composition of formal power series is essentially equivalent to the Arbogast-Faà di Bruno formula, here in the multivariate case. This formula is the higher derivative version of the chain rule. See §1.1 of the notes on the theory of distributions posted on Canvas.*

The inverse function theorem for formal power series:

In Math 4330, one of the important theorems covered is the Inverse Function Theorem (InvFT) for multivariate differentiable functions. This is usually studied in combination with the Implicit Function Theorem (ImpFT). The InvFT is a particular case of the ImpFT, but one can also prove the ImpFT from the InvFT, using a trick of adding variables. The following can be seen as a FPS analogue or variant of the InvFT.

Theorem 1. *Let $n, p \geq 1$. Let $f_1, \ldots, f_n \in \mathbb{C}[[X_1, \ldots, X_n]]$. Let $h_1, \ldots, h_n \in \mathbb{C}[[Y_1, \ldots, Y_p]]$. Suppose that*

$$\forall k \in [n], \quad [X^0]f_k = [Y^0]h_k \ ,$$

and suppose that the matrix

$$L = \left(\left. \frac{\partial f_k}{\partial X_i} \right|_{X:=0} \right)_{1 \leq k,i \leq n}$$

is invertible. Then, there exists a unique n-tuple of power series $g_1, \ldots, g_n$ with zero constant term in $\mathbb{C}[[Y_1, \ldots, Y_p]]$, for which the following system holds:

$$\left\{ \begin{array}{rcl} f_1(g_1(Y), \ldots, g_n(Y)) & = & h_1(Y) \\ & \vdots & \\ f_n(g_1(Y), \ldots, g_n(Y)) & = & h_n(Y) \ . \end{array} \right. \tag{1}$$

Note that, in writing FPS's like $f_k(g_1(Y), \ldots, g_n(Y))$, we are using the composition operation from Lecture 3, n times. The partial derivative $\partial f_k / \partial X_i$ was also defined in the previous lecture, as well as the evaluation at the origin $\cdots|_{X:=0}$, i.e., the extraction of the constant coefficient. As a result $L = (L_{k,i})$ is just a matrix of complex numbers of $n \times n$ format. It is the analogue of the Jacobian matrix of a map at the origin.

Proof: To fix notations for the coefficients of the various FPS's involved, let us write:

$$f_k = \sum_{\alpha \in \mathbb{N}^n} f_{k,\alpha} X^\alpha \ , \quad \text{for } 1 \leq k \leq n \ ,$$

$$g_i = \sum_{\beta \in \mathbb{N}^p} g_{i,\beta} Y^\beta \ , \quad \text{for } 1 \leq i \leq n \ ,$$

and

$$h_k = \sum_{\gamma \in \mathbb{N}^p} h_{k,\gamma} Y^\gamma \ , \quad \text{for } 1 \le k \le n \ .$$

From the previous lecture on composition of FPS's, we see that the system (1) is equivalent to having, for all $k \in [n]$ and for all $\gamma \in \mathbb{N}^p$,

$$h_{k,\gamma} = \sum_{\alpha \in \mathbb{N}^n} f_{k,\alpha} \sum_{\mathbb{B} \in (\mathbb{N}^p)^{\mathbb{D}_\alpha}} \mathbb{1}\left\{ \sum_{(i,j)\in\mathbb{D}_\alpha} \beta(i,j) = \gamma \right\} \times \prod_{(i,j)\in\mathbb{D}_\alpha} g_{i,\beta(i,j)} \ , \tag{2}$$

where

$$\mathbb{D}_\alpha = \{(i,j) \in [n] \times \mathbb{Z}_{>0} \mid 1 \le j \le \alpha_i\} \ ,$$

as in Lecture 3.

Let us first consider the case where $\gamma = 0$, i.e., is the zero multiindex. We claim that, in this situation, the right-hand side (RHS) of (2) reduces to $f_{k,0}$. If $\alpha \neq 0$, then $\mathbb{D}_\alpha \neq \varnothing$. Pick an element $(i,j) \in (i,j) \in \mathbb{D}_\alpha$. For the term corresponding to $\mathbb{B}$ to be nonzero, we would need the effectively present factor $g_{i,\beta(i,j)}$ to be nonzero, but this implies $\beta(i,j) \neq 0$ because the g's have no constant terms. The condition/constraint inside the indicator function $\mathbb{1}\{\cdots\}$, then implies the component-wise inequality $\beta(i,j) \le \gamma$. This would force $\gamma \neq 0$ which is excluded by assuption. Therefore, the only contribution to the RHS comes from $\alpha = 0$. In that case, $\mathbb{D}_\alpha = \varnothing$. The sum over $\mathbb{B}$ ranges over the summation set $(\mathbb{N}^p)^{\mathbb{D}_\alpha} = \{\varnothing\}$ whose only element is the empty map $\mathbb{B} = \varnothing$. The condition inside the indicator reduces to the equality of multiindices $0 = 0$, because, on the left, the sum is empty and, on the right, γ is zero. The indicator is thus equal to 1. The product over g's is empty and therefore also equal to 1. In sum, the RHS reduces to $f_{k,0}$. When $\gamma = 0$, the equation (2) simplifies to $h_{k,0} = f_{k,0}$ which holds by the hypotheses of the theorem.

Now we move on to the case of $\gamma \neq 0$ and examine more closely the sum over α, by separating the cases $\alpha = 0$, $|\alpha| = 1$ and $|\alpha| \ge 2$. If $\alpha = 0$, the indicator function vanishes. Indeed, the sum being empty, $\sum_{(i,j)\in\mathbb{D}_\alpha} \beta(i,j) = 0$, and it cannot be equal to $\gamma \neq 0$. Now take the case $|\alpha| = 1$. This means that $\alpha = e_i$ for some $i \in [n]$, where e_i is the canonical basis vector with a 1 in the i-th position and zeros everywhere else. Then, $\mathbb{D}_\alpha = \{(i,1)\}$ and

$$\sum_{\mathbb{B} \in (\mathbb{N}^p)^{\mathbb{D}_\alpha}} \mathbb{1}\left\{ \sum_{(i,j)\in\mathbb{D}_\alpha} \beta(i,j) = \gamma \right\} \times \prod_{(i,j)\in\mathbb{D}_\alpha} g_{i,\beta(i,j)} = \sum_{\beta(i,1)\in\mathbb{N}^p} \mathbb{1}\left\{\beta(i,1) = \gamma\right\} g_{i,\beta(i,1)} \ ,$$

i.e., the sum over the, in general complicated, "spreadsheet" $\mathbb{B}$ degenerates into a sum over a single multiindex $\beta(i,1)$ in $\mathbb{N}^p$. The last sum further simplifies to $g_{i,\gamma}$. As a result, the part of the sum on the RHS with $|\alpha| = 1$ is exactly

$$\sum_{i=1}^n L_{k,i} \ g_{i,\gamma}$$

where $L_{k,i} = f_{k,e_i}$.

We can now rewrite (2) as

$$h_{k,\gamma} = \sum_{i=1}^n L_{k,i} \ g_{i,\gamma} + \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\ge 2}} f_{k,\alpha} \sum_{\mathbb{B} \in (\mathbb{N}^p)^{\mathbb{D}_\alpha}} \mathbb{1}\left\{ \sum_{(i,j)\in\mathbb{D}_\alpha} \beta(i,j) = \gamma \right\} \times \prod_{(i,j)\in\mathbb{D}_\alpha} g_{i,\beta(i,j)} \ ,$$

or equivalently as

$$\sum_{i=1}^{n} L_{k,i}\ g_{i,\gamma} = h_{k,\gamma} + R_{k,\gamma}\ , \tag{3}$$

with the remainder term defined as

$$R_{k,\gamma} := -\sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\geq 2}} f_{k,\alpha} \sum_{\mathbb{B}\in(\mathbb{N}^p)^{\mathbb{D}_\alpha}} \mathbb{1}\left\{ \sum_{(i,j)\in\mathbb{D}_\alpha} \beta(i,j) = \gamma \right\} \times \prod_{(i,j)\in\mathbb{D}_\alpha} g_{i,\beta(i,j)}\ . \tag{4}$$

Now looking at (4), we notice that each $\mathbb{D}_\alpha$ which appears has at least two elements/cells. As a result, for $\mathbb{B}$ to have a nonzero contribution, we must have

$$\forall (i,j)\in\mathbb{D}_\alpha,\ |\beta(i,j)| < |\gamma|\ .$$

Indeed, let (i,j) be some element of $\mathbb{D}_\alpha$, then a nonzero contribution needs the product of g's to be nonzero, and therefore the individual factor $g_{i,\beta(i,j)}$ must be nonzero. By the no constant term hypothesis, we deduce $|\beta(i,j)|\geq 1$. Since $\mathbb{D}_\alpha$ has at least two elements, we can pick another element $(i',j')\neq(i,j)$ in $\mathbb{D}_\alpha$. By repeating the previous argument, we must also have $|\beta(i',j')|\geq 1$. Now invoking the nonvanishing of the indicator, we get

$$|\beta(i,j)| + |\beta(i',j')| \leq \sum_{(i'',j'')\in\mathbb{D}_\alpha} |\beta(i'',j'')| = |\gamma|$$

and therefore

$$|\beta(i,j)| \leq |\gamma| - |\beta(i',j')| \leq |\gamma| - 1\ .$$

We showed that the remainder $R_{k,\gamma}$ only involves $g_{i,\beta}$'s with β of strictly smaller length than γ. We can show existence and uniqueness by an inductive procedure on the length of multiindices in $\mathbb{N}^p$.

Existence: For $\ell\geq 1$, let us denote by Γ_ℓ, the subcollection of unknowns

$$\Gamma_\ell = (g_{k,\beta})_{\substack{1\leq k\leq n \\ \beta\in\mathbb{N}^p, |\beta|=\ell}}\ .$$

For each fixed $\gamma\in\mathbb{N}^p$ with $|\gamma|=1$, we must have $R_{k,\gamma}=0$, and we can solve the $n\times n$ linear system of equations

$$\sum_{i=1}^{n} L_{k,i} g_{i,\gamma} = h_{k,\gamma}$$

with $1\leq k\leq n$. We define $g_{1,\gamma},\ldots,g_{n,\gamma}$ as the solutions of this system, which exist because L is invertible. We collect the results over all γ's with length 1 and this gives us the subcollection Γ_1. Now suppose that, for $\ell\geq 1$, we have already constructed $\Gamma_1,\Gamma_2,\ldots,\Gamma_\ell$. Let now $\gamma\in\mathbb{N}^p$ with $|\gamma|=\ell+1$. For $1\leq k\leq n$, the remainder $R_{k,\gamma}$ is determined by the g coefficients accounted for in $\Gamma_1,\Gamma_2,\ldots,\Gamma_\ell$. We then again use the system given by (3), for $1\leq k\leq n$, in order to solve for $g_{1,\gamma},\ldots,g_{n,\gamma}$. Doing this for each $\gamma\in\mathbb{N}^p$ produces $\Gamma_{\ell+1}$. At the end of this procedure we obtain an infinite collection of coefficients $(g_{k,\gamma})_{k\in[n],\gamma\in\mathbb{N}^p}$. One could use DC (the Axiom of Dependent Choice to justify this last step, but as Zac noted this is not necessary). It is easy to see that, by construction, this collection satisfies the system of equations (3) and the corresponding power series $g_1(Y),\ldots,g_n(Y)$ satisfy (1).

Uniqueness: Suppose we have two collections of power series $g_1(Y),\ldots,g_n(Y)$ as well as $g'_1(Y),\ldots,g'_n(Y)$ which both satisfy the system (1), then we will have two collections of coefficients $(g_{k,\gamma})_{k\in[n],\gamma\in\mathbb{N}^p}$ and $(g'_{k,\gamma})_{k\in[n],\gamma\in\mathbb{N}^p}$ which satisfy the system of equations (3). We can organize the latter into subcollections $\Gamma_1, \Gamma_2, \ldots$ and $\Gamma'_1, \Gamma'_2, \ldots$ respectively, as described earlier. If $|\gamma| = 1$ then, for all $k \in [n]$, we have

$$\sum_{i=1}^{n} L_{k,i}\ g_{i,\gamma} = h_{k,\gamma} = \sum_{i=1}^{n} L_{k,i}\ g'_{i,\gamma}$$

and therefore $\forall k \in [n], g_{k,\gamma} = g'_{k,\gamma}$, because L is invertible. Hence, $\Gamma_1 = \Gamma'_1$. By induction on $\ell \geq 1$, we show that $\Gamma_1 = \Gamma'_1$, $\Gamma_2 = \Gamma'_2, \ldots$, and $\Gamma_\ell = \Gamma'_\ell$. Indeed, when considering a multiindex $\gamma \in \mathbb{N}^p$ with $|\gamma| = \ell + 1$, we have for all $k \in [n]$,

$$\sum_{i=1}^{n} L_{k,i}\ g_{i,\gamma} = h_{k,\gamma} + R_{k,\gamma} = h_{k,\gamma} + R'_{k,\gamma} = \sum_{i=1}^{n} L_{k,i}\ g'_{i,\gamma}\ ,$$

because of the equality of the remainder terms $R_{k,\gamma} = R'_{k,\gamma}$ computed, by the same formula (4), using $\Gamma_1, \ldots, \Gamma_\ell$ and $\Gamma'_1, \ldots, \Gamma'_\ell$ respectively, and the latter must be equal by the induction hypothesis. This proof by induction shows that the collections of g and g' coefficients are the same, and similarly $g_i(Y) = g'_i(Y)$, for all $i \in [n]$, so uniqueness holds. $\square$

Remark 2. *Existence and uniqueness is good to know, but we will in fact need explicit formulas (rather than an inductive construction) for the wanted g power series. Continuing with the same "sum of monomials" framework would make this objective harder to achieve. We will need to switch to a different "tensorial" point of view. The wanted explicit formulas will involve an infinite expansion using tree graphs. An example of application in QFT, is to precisely relate the generating function $\Gamma(\phi)$ of 1PI (1-particle-irreducible) Feynman diagrams to that $W(J)$ of connected Feynman diagrams. One has to solve for $J = (J_i)_i$ in the system of equations*

$$\phi_k = \frac{\partial W}{\partial J_k}(J)\ ,$$

for all lattice sites k. Then, if $J(\phi)$ denotes the solution, one has

$$\Gamma(\phi) = \phi \cdot J(\phi) - W(J(\phi))\ .$$

This is an example of Legendre-Fenchel transform which, in QFT books, is done in the sense of FPS's, i.e., without worrying about convergence. For basics on the Legendre-Fenchel transform, see the two files Homework9MATH7310F12.pdf (Extra Problem 3, with no differentiability hypothesis) and RealAnalysisGeneralExamAug2021.pdf (Question 5, with assumed dfferentiability and application to Cramer's Theorem and large deviations).

TENSORS AND DIAGRAMS

Tensors feature in many areas of mathematics and physics. In this course, and for lack of better word, we will use the word tensor to just mean a multidimensional array of (complex) numbers. For example, an array

$$T = (T_{i,j,k})_{1 \leq i,j,k \leq n}$$

is a tensor with three indices. This is also the same as a map $T : [n]^3 \to \mathbb{C}$. We are here adopting the same point of view as that of matrix algebra as opposed to abstract linear algebra which talks about vector spaces, linear transformations, etc. In this course, tensors are to be understood as generalizations of matrices, where the number of indices is not just 1 (column or row vectors) or 2 (matrices), but can be any nonnegative integer (zero indices means the tensor is just a single number).

Remark 3. *Our definition is different from that in most abstract parts of mathematics where a tensor would be an element of a tensor product of vector spaces (or modules over a ring) $V_1 \otimes \cdots \otimes V_d$. Our definition is also different from the most common one used in physics, i.e., the notion of tensor fields which for mathematicians, in particular differential geometers, corresponds to a section of a vector bundle over a manifold M built by taking the tensor product of copies of the tangent and the cotangent bundles of M.*

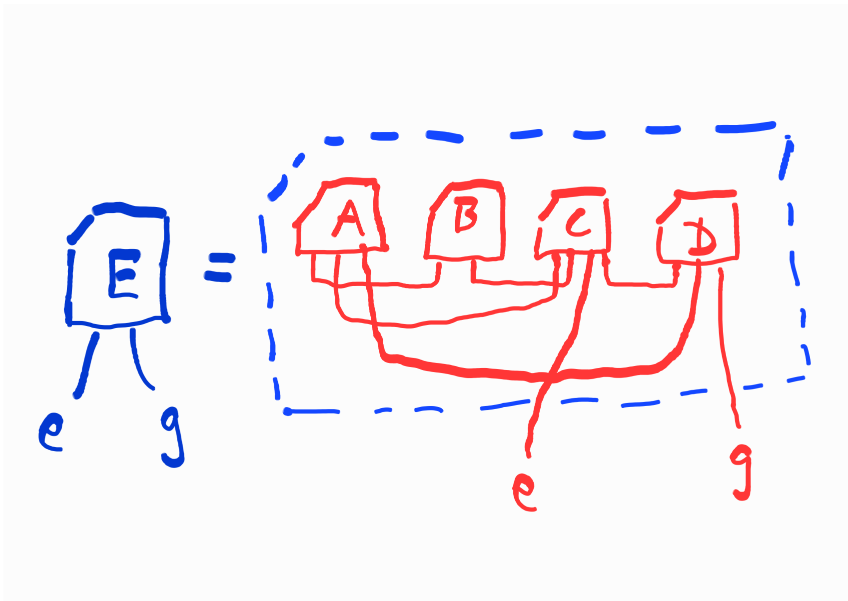
A key feature of tensors is that they can be combined, using the operation of contraction of indices, in order to build new tensors. For example, let A, B, C, D be tensors with 3, 2, 4, and 3 indices respectively, all ranging over the set of index values $[n]$. Then we can define a new tensor $E = (E_{e,g})_{1 \leq e,g \leq n}$ by letting, for all $e, g \in [n]$,

$$E_{e,g} := \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \sum_{d=1}^{n} \sum_{f=1}^{n} A_{a,b,c} B_{a,d} C_{b,d,e,f} D_{f,c,g} \ . \tag{5}$$

Needless to say, such expressions can quickly become very cumbersome. As shorthand for the above expression, one can simply omit the sums and write

$$E_{e,g} := A_{a,b,c} B_{a,d} C_{b,d,e,f} D_{f,c,g}$$

with the understanding that repeated (or contracted) indices are to be summed over. This is Einstein's convention. However, this is still cumbersome, and calls for an even more compact notation which uses pictures and diagrams. Namely, not only do we not write the sums, but we will not even write the indices. In this graphical notation, the equation (5) is written:

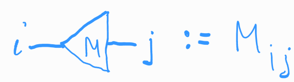


To use this graphical calculus one has to specify a number of elementary building blocks which here look like boxes with some legs coming out of them and carrying indices. For example, the LHS of the previous graphical equation is by definition equal to the entry $E_{e,g}$ of the tensor E. It has two legs, one assigned the index e and the other assigned the index g in $[n]$. On the RHS there are similar boxes denoting entries of the tensors A, B, C, D. However, some of the legs have been glued. When we see such a gluing of two legs, we are supposed to introduce an index, assign it to both legs which have been glued, and finally sum over it. Note that we have complete artistic freedom in choosing the shape of the boxes used in our graphical computations. In order to avoid any ambiguity, we just need to make sure our notation makes it clear which leg corresponds to the first index of a tensor, which leg corresponds to the second index, etc. The choice of shape in the above picture can be called the SIM card notation because of the notch which sits above the leg corresponding to the first index (the notch on a SIM card is why there is only one way to put it in the SIM card tray). We will use the next little while to get some practice with this diagrammatic calculus, and also see some perhaps surprising applications.

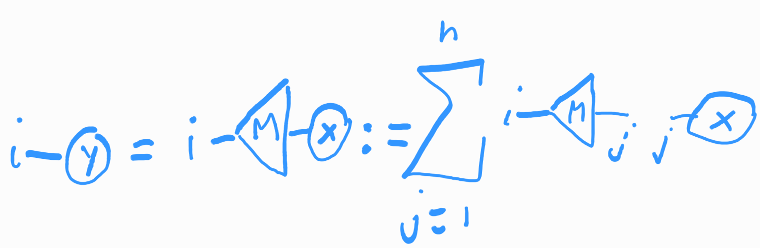
Vectors and matrices: Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ be a vector. We introduce the graphical notation

for the i-th entry of the vector. We would use the same notation regardless of whether the vector is a row or a column vector.

Let $M = (M_{ij})_{1 \le i,j \le n}$ be a matrix. We likewise introduce for it the notation



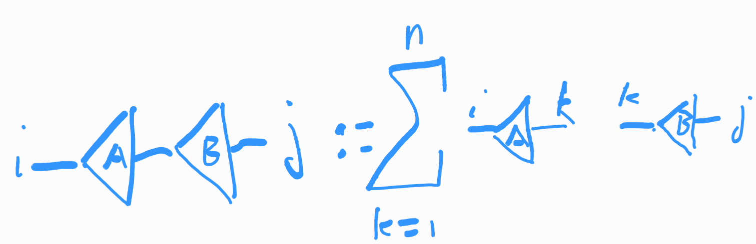
Let now X, Y be column vectors, then the equation $Y = MX$ would be written graphically as



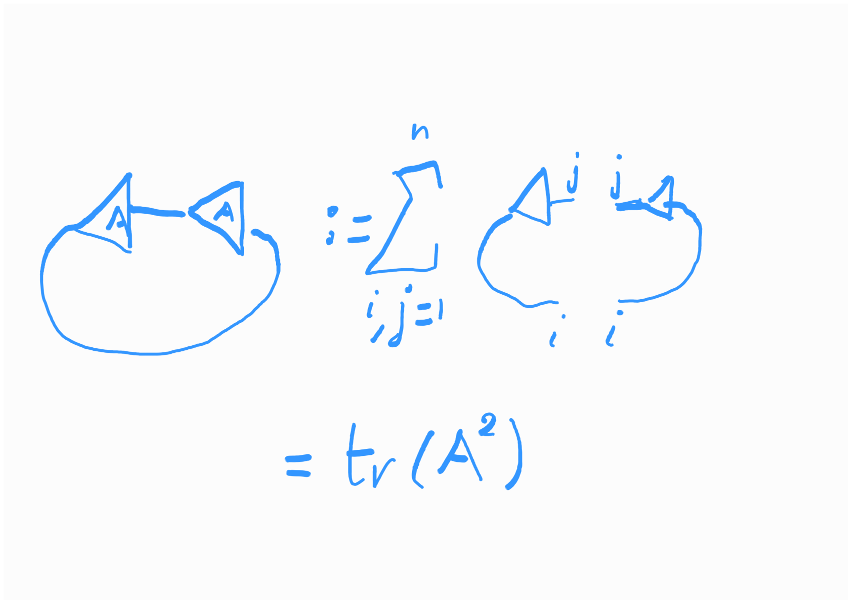
Note that when a diagram is made of disconnected pieces, the evaluation of the whole is by definition the product of the evaluations of the pieces. So the last graphical equation is just a different notation for saying, for all $i \in [n]$,

$$Y_i = \sum_{j=1}^{n} M_{ij} X_j \ .$$

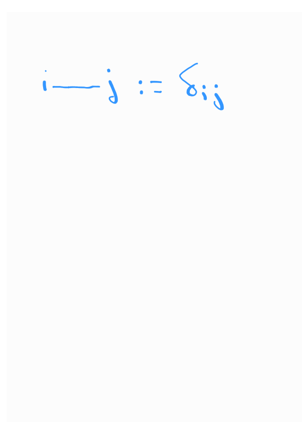
If we now have more than one matrix, say we have A and B, then the product AB has entries which can be written graphically as



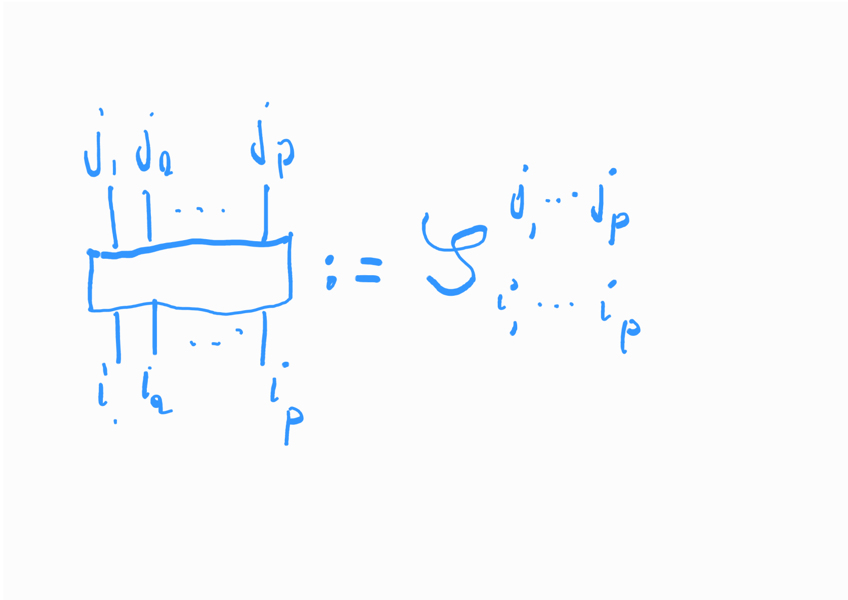
The graphical notation is powerful enough to express traces, for instance,



There are some particular tensors which are especially useful for the graphical calculus. The tensor corresponding to the identity matrix



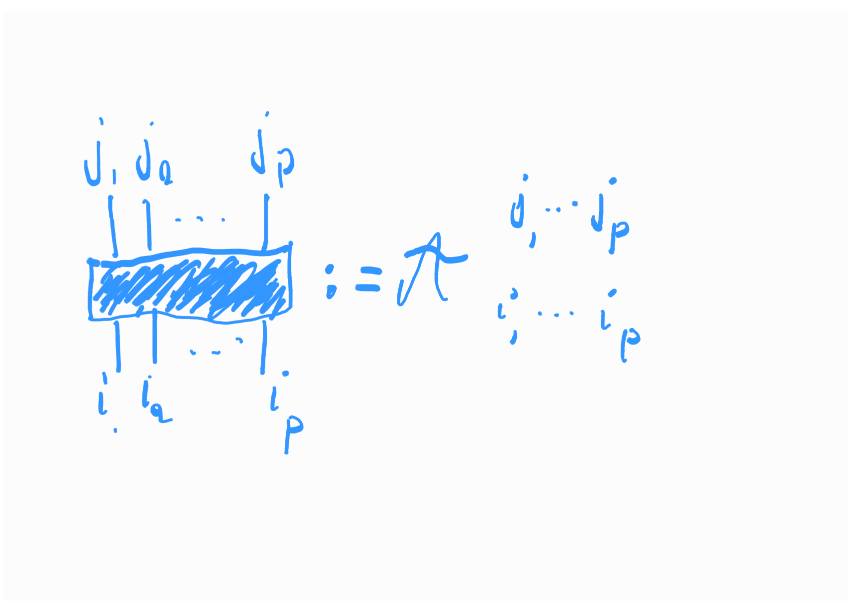
i.e., the Kronecker delta $\delta_{i,j} = \mathbb{1}\{i = j\}$. The symetrizer of size $p \geq 1$ is



where

$$\mathscr{S}^{j_1,\ldots,j_p}_{i_1,\ldots,i_p} := \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \delta_{i_1,j_{\sigma(1)}} \delta_{i_2,j_{\sigma(2)}} \cdots \delta_{i_p,j_{\sigma(p)}} \ .$$

Its evil twin, the antisymmetrizer of size p is



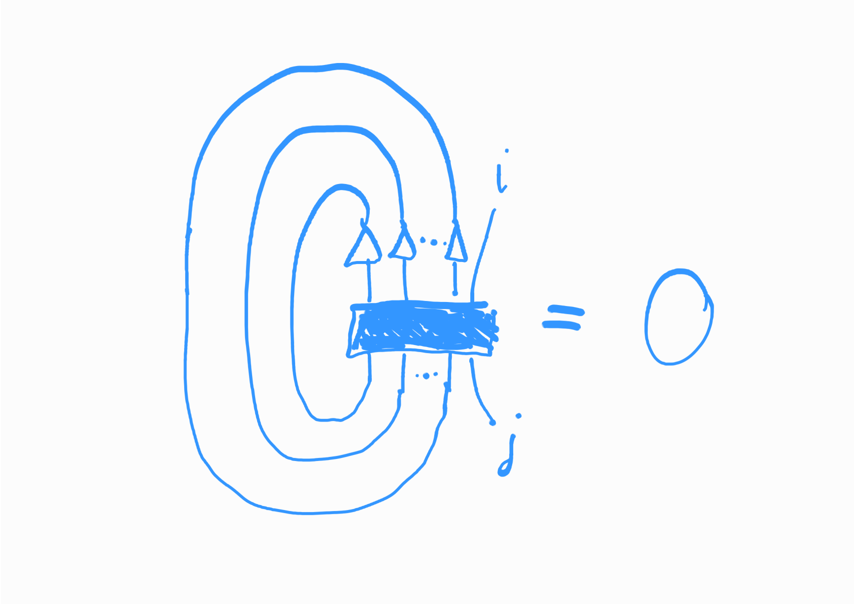
where

$$\mathscr{A}_{i_1,\ldots,i_p}^{j_1,\ldots,j_p} := \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \mathrm{sgn}(\sigma)\ \delta_{i_1,j_{\sigma(1)}}\ \delta_{i_2,j_{\sigma(2)}} \cdots \delta_{i_p,j_{\sigma(p)}}\ .$$

Note that we use a clear box for the symmetrizer and a dark box for the antisymmetrizer.

We can now state a very important and well known theorem.

Theorem 2. *For any $n \times n$ matrix A denoted by a triangle as above, for $i, j \in [n]$, we have*



where the antisymmetrizer is of size $n+1$.

Wait? What? How is this a famous theorem?

As correctly guessed by Pedro, this is the Cayley-Hamilton Theorem.