MATH 8450 – LECTURE 6 – FEB 6, 2023

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TREE EXPANSIONS AND EXPLICIT POWER SERIES INVERSION

Consider the problem of solving for the variables $X = (X_1, \ldots, X_n)$ in terms of the variables $Y = (Y_1, \ldots, Y_n)$ in the system of equations

$$X_i - \frac{1}{2} Q_i(X) = Y_i \ , \qquad (1)$$

with $1 \le i \le n$. Here, $Q_i(X)$ is a quadratic homogeneous polynomial in $\mathbb{C}[X_1, \ldots, X_n] \subset \mathbb{C}[[X_1, \ldots, X_n]]$, of the form

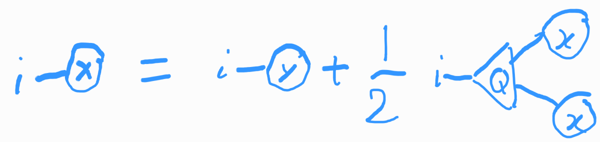
$$Q_i(X_1, \ldots, X_n) = \sum_{j,k \in [n]} Q_{ijk} X_j X_k$$

where the tensor $Q = (Q_{ijk})_{i,j,k \in [n]}$ is symmetric with respect to the last two indices j, k, i.e., satisfies

$$\forall i, j, k \in [n], \ Q_{ijk} = Q_{ikj} \ .$$

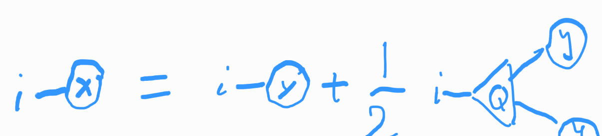
We will denote this tensor graphically by

After putting the quadratic part on the other side, turning (1) into a fixed point equation for X, and after translating it to graphical form, we get

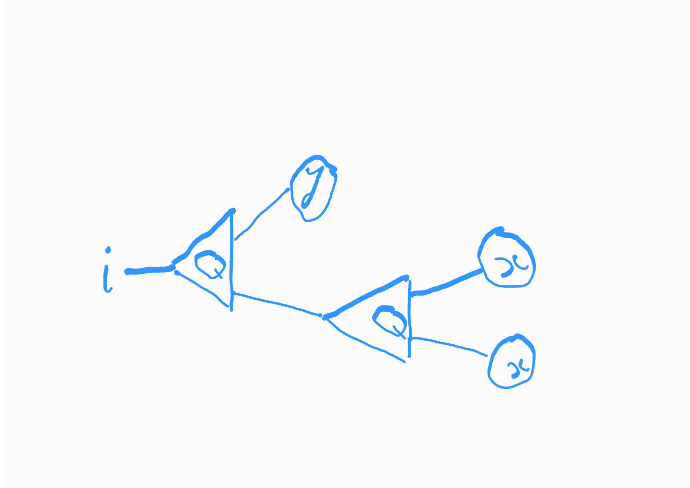


(2)

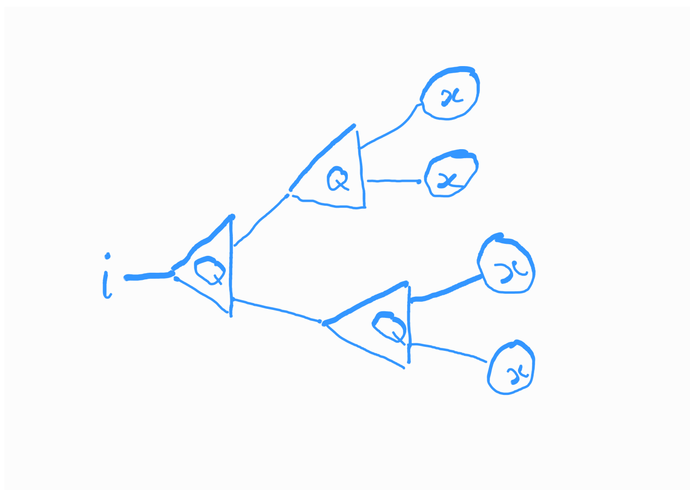
We can then iterate, namely replace the x blobs on the right-hand side by the equation we just wrote. This gives



$$+2 \times \frac{1}{2} \times \frac{1}{2}$$



$$+\frac{1}{8}$$



In the third term each of the two Q blobs brings with it a factor $\frac{1}{2}$. The factor of 2 in the numerator accounts for two choices of which previous x blob was replaced by a y and which got replaced by a Q carriying two new x's. By symmetry of the Q tensor both choices lead to pictures with the same evaluation. Note that the first two terms are "dead", i.e., do not contain x's and will remain as they are if we continue the iteration of (2). The last two terms are "alive" (will continue to grow) because they still contain x's which can be substituted with the RHS of (2). Clearly we can continue ad infinitum and thus generate an expansion in terms of diagrams which look like trees carrying y blobs as leaves. This series will be infinite, and we thus need some care in defining what we mean by trees, how to evaluate them, and what precise combinatorial factors like $\frac{1}{8}$ are they weighted by.

Some basics about graphs:

Let V be a finite set. We will denote the set of unordered pairs in V as

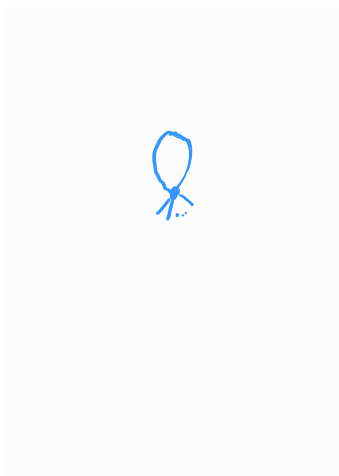
$$V^{(2)} := \{e \in \mathscr{P}(V) \mid |e| = 2\} \ .$$

Elements of $V^{(2)}$ are of the form $e = \{i, j\} = \{j, i\}$ with $i, j \in V$ and $i \neq j$. Of course,

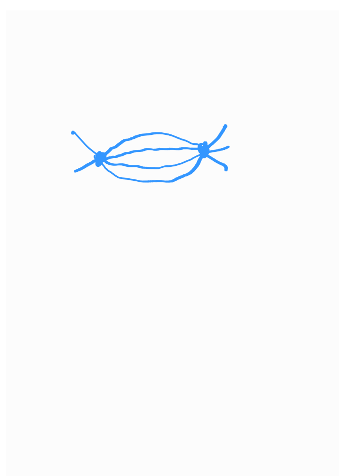
$$\left|V^{(2)}\right| = \binom{|V|}{2} \ .$$

Let G be any subset of $V^{(2)}$. We will call G a graph on V. The elements of G are called the edges of the graph, while the elements of V are called the vertices.

Remark 1. *What we defined is technically called a simple graph without loops in the combinatorics literature. A loop (in combinatorics) or tadpole (in physics) is an edge which starts and ends at the same vertex as in*



Simple means we do not allow multiple edges as in



So our definition excludes the last two features. Note that clashes in terminology will be hard to avoid since the following pertains to graph theory in combinatorics but also the calculus of Feynman diagrams in the physics QFT literature. For instance what physicists call loops (as in "perturbation theory up to the order of five loops") is what combinatorialists would call circuits or cycles, to be defined below.

Given a graph G on a set V we define the following relation $\leftrightarrow$ between elements of V. For $a, b \in V$, we write $a \leftrightarrow b$ and say that a is connected to b by the graph G iff $\exists k \in \mathbb{N}$, $\exists v_0, \ldots, v_k \in V$, such that

(1) $v_0 = a$,
(2) $v_k = b$,
(3) for all i with $0 \leq i < k$, $\{v_i, v_{i+1}\} \in G$.

It is easy to see that this is an equivalence relation. The quotient set is the set of connected components of the graph G. A sequence $v_0, \ldots, v_k$ as above will be called a path/walk of length k from a to b.

A bit of review:

If X is a set, a relation R on X is any subset of the Cartesian product $X \times X$. Instead of writing $(x, y) \in R$, we write $x\ R\ y$. A relation R is called an equivalence relation iff

(1) It is reflexive, i.e., $\forall x \in X, x\ R\ x$.
(2) It is symmetric, i.e., $\forall x, y \in X, x\ R\ y \Rightarrow y\ R\ x$.
(3) It is transitive, i.e., $\forall x, y, z \in X, (x\ R\ y \text{ and } y\ R\ z) \Rightarrow x\ R\ z$.

For such a relation and for any $x \in X$, the subset

$$\overline{x} := \{y \in X \mid x\ R\ y\}$$

is called the equivalence class of x. The set of equivalence classes is the quotient set ${}^{X}/_{R}$. By construction, it is a set partition of X.

We now go back to graphs and give the definition of connected graph.

Definition 1. *A graph G on a set V is called connected iff there is exactly one connected component. Equivalently, the graph is connected (or connects the underlying set of vertices V) iff $V \neq \varnothing$ and $\forall a, b \in V, a \leftrightarrow b$.*

Remark 2. *Contrary to the usual definition in point set topology, we exclude the empty set from our notion of connectedness. This will become more natural later when discussing (combinatorics) exponential generating functions of connect versus not necessarily connected object or (physics) the linked-cluster theorem.*

Let G be a connected graph on V, we then define a distance (called the graph distance) on the set V as follows. If $a, b \in V$, we define their distance $d_G(a, b)$ as the minimal $k \geq 0$ for which there is a path of length k from a to b. It is easy to see that this turns V into a metric space. Note that is $a = b$, a minimal length path (unique) must be such that $k = 0$ and $v_0 = a = b$. If $a \neq b$, a (not necessarily unique) minimal length path must be such that $k \geq 1$ and $v_0, \ldots, v_k$ are all distinct. Indeed, it is easy to see that if $v_i = v_j$ with $i < j$ then we can remove the part of the path between v_i and v_j and get a strictly shorter path. More precisely, we would replace $v_0, \ldots, v_i, \ldots, v_j, \ldots, v_k$ by $v_0, \ldots, v_i, v_{j+1}, \ldots, v_k$.

Remark 3. *The above procedure is an example of loop erasure. An important topic in probability theory is the study of the loop-erased random walk (LERW) where one starts from an arbitrary path/walk and then one recursively removes all loops in the order in which they appear. One of the main developers of this theory is Greg Lawler (UVa alumnus and McShane Prize 1976). Notice that mathematicians in probability theory side with physicists against mathematicians in combinatorics as to the meaning of "loop". The reason the LERW* *is important is that it is a toy model for the even more important self-avoiding walk (SAW) related to the physics of long polymer chains and soft condensed matter. The relevant theory for the SAW is in fact the* ϕ^4 *model of QFT mentioned in Lecture 1, with a couple of twists: 1) the field* ϕ *is valued in* $\mathbb{R}^N$ *instead of* $\mathbb{R}$*, 2) one has to take the number of field components to be* $N = 0$*. The main developer of this theory is Pierre-Gilles de Gennes (Nobel Prize 1991). An alternative approach uses supersymmetry (mixing Bosons and Fermions) and has been developed by Giorgio Parisi (Nobel Prize 2021) and Nicolas Sourlas. Finally, the LERW is also described by the* N*-compoment* ϕ^4 *QFT model, but this time with* $N = -2$*.*

We will also need the notion of vertex degree or valence. Given a graph G (not necessarily connected) on V, and for any $v \in V$ we define the degree of v as

$$\delta_G(v) := |\{e \in G \mid v \in e\}| \ ,$$

namely, the number of immediate neighbors of v, or the number of edges incident to v.

Lemma 1. *Let* G *be a connected graph on a set* V *with* $|V| \geq 2$ *and let* $v \in V$*, then* $\delta_G(v) \geq 1$*.*

Proof: Since $|V| \geq 2$, one can pick some other vertex $w \neq v$ in V. Since G is connected, we have $v \leftrightarrow w$ and there exists a path $u_0, u_1, \ldots, u_k$ in the graph which goes from $u_0 = v$ to $u_k = w$. Since $v \neq w$, we have $k \geq 1$. Therefore, $v \in e := \{u_0, u_1\} \in G$, which implies $|\delta_G(v) \geq 1$. □

Definition 2. *Let* G *be a graph on* V*. A circuit/cycle is a sequence of distinct vertices* $v_1, \ldots, v_k$ *with* $k \geq 3$*, such that* $\forall i \in [k-1]$*,* $\{v_i, v_{i+1}\} \in G$*, and such that* $\{v_k, v_1\} \in G$*.*

Definition 3. *Let* T *be a graph on a finite set* V*. We call* T *a tree on* V *iff* T *is connected and has no circuits.*

Clearly, we will only use this notion for nonempty sets V. The above notion is also called a Cayley tree or vertex-labeled tree, to be distinguished from the notion of tree shape. This is important for us because we will be counting trees and trying to prove convergence of series, and such details make the difference between having to fight a factor of $n!$ or not.

Proposition 1. *Let* T *be a tree on a set* $V \neq \varnothing$*, then* $|T| = |V| - 1$*.*

Proof: See Lecture 8.

Proposition 2. *Let* T *be a tree on a set* $V \neq \varnothing$*, then*

$$\sum_{v \in V} \delta_T(v) = 2|V| - 2 \ .$$

Proof: By Proposition 2,

$$\begin{aligned} 2|V|-2 &= 2|T| \\ &= \sum_{e\in T} 2 \\ &= \sum_{e\in T}\sum_{v\in V} \mathbb{1}\{v\in e\} \\ &= \sum_{v\in V}\sum_{e\in T} \mathbb{1}\{v\in e\} \\ &= \sum_{v\in V} \delta_T(v)\ . \end{aligned}$$

□

Theorem 1. *(Cayley) Let V be a set with $|V| =: n \geq 2$. Let $(d_v)_{v\in V} \in (\mathbb{Z}_{>0})^V$ be such that $\sum_{v\in V} d_v = 2n-2$. Then,*

$$|\{T \text{ tree on } V \mid \forall v \in V, \delta_T(v) = d_v\}| = \frac{(n-2)!}{\prod_{v\in V}(d_v-1)!}\ .$$

Proof: See Lecture 8.

The above theorem gives the total number of trees that one can put on the vertex set V, if the degrees/valences of the vertices have been decided in advance. If one does not fix the vertex degrees, then the tree count is provided by the following corollary.

Corollary 1. *(Cayley) Let V be a set with $|V| =: n \geq 1$. Then, the total number of trees that one can put on V is n^{n-2}.*

Proof of the corollary: If $n = 1$, then $V^{(2)} = \varnothing$ and therefore the only candidate for a tree on V is $T = \varnothing$. It is easy to see that it satisfies the definition of tree and therefore the number of trees is 1 which coincides with 1^{1-2}. We now suppose $n \geq 2$ and organize/condition the sum over trees according to the collection of vertex degrees. Since a tree is a connected graph, the $n \geq 2$ hypothesis and Lemma 1 imply that all vertex degrees of a tree on V are in $\mathbb{Z}_{>0}$. We have, again with perhaps too much detail,

$$
\begin{aligned}
|\{T \text{ tree on } V\}| &= \sum_{T \text{ tree on } V} 1 \\
&= \sum_{T \text{ tree on } V} \ \sum_{(d_v)_{v\in V}\in(\mathbb{Z}_{>0})^V} \mathbb{1}\{\forall v\in V, \delta_T(v)=d_v\} \\
&= \sum_{(d_v)_{v\in V}\in(\mathbb{Z}_{>0})^V} \ \sum_{T \text{ tree on } V} \mathbb{1}\{\forall v\in V, \delta_T(v)=d_v\} \\
&= \sum_{(d_v)_{v\in V}\in(\mathbb{Z}_{>0})^V} \mathbb{1}\left\{\sum_{v\in V} d_v = 2n-2\right\} \sum_{T \text{ tree on } V} \mathbb{1}\{\forall v\in V, \delta_T(v)=d_v\} \\
&= \sum_{(d_v)_{v\in V}\in(\mathbb{Z}_{>0})^V} \mathbb{1}\left\{\sum_{v\in V} d_v = 2n-2\right\} \times \frac{(n-2)!}{\prod_{v\in V}(d_v-1)!} \\
&= \sum_{(k_v)_{v\in V}\in\mathbb{N}^V} \mathbb{1}\left\{\sum_{v\in V} k_v = n-2\right\} \times \frac{(n-2)!}{\prod_{v\in V}(k_v)!} \\
&= \left(\sum_{v\in V} 1\right)^{n-2} \\
&= n^{n-2} \ .
\end{aligned}
$$

□

Note that the above is a calculation in formal logic as well as a numerical calculation. In the first three line we use the standard trick to do conditioning (from probability), i.e., writing 1 in a more complicated way and then switching the order of summation. In the fourth line, we actually used Proposition 2: either the sum over T is zero and there is no harm inserting a new indicator factor, or it is not which means there exists a tree with that degree specification and the proposition implies the indicator is equal to 1. The fifth line used Theorem 1. Finally, the last lines used the summation change of variable $k_v = d_v - 1$, for all $v \in V$, and the multinomial theorem.