MATH 8450 – LECTURE 7 – FEB 8, 2023

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Tree expansions and explicit FPS inversion cont'd:

We can now state the explicit solution of the FPS inversion problem from last lecture, i.e., the system of equations, for $1 \le i \le n$,

$$X_i - \frac{1}{2}Q_i(X) = Y_i \ . \qquad (1)$$

This is a particular case of Theorem 1 of Lecture 4, where $f_i(X) = X_i - \frac{1}{2}Q_i(X)$ and $h_i(Y) = Y_i$. The invertible linear part is given by the matrix $L = I$, the identity matrix. We know there is a unique solution $X_i = g_i(Y)$ where the formal power series $g_i(Y)$ are in $\mathbb{C}[[Y_1, \ldots, Y_n]]$. The g_i can be written explicitly as follows.

$$g_i(Y) = \sum_{m,\ell \ge 0} \frac{1}{m!\ell!} \sum_T \mathscr{A}_i(T) \ . \qquad (2)$$

We will explain the above formula in a semi-formal (human-readable?) way, and in the next lecture we will see how to give a more formal (computer-readable?) presentation.

For given $m, l \in \mathbb{N}$, we consider the set $V_{m,\ell} := [m + \ell + 1]$ of integers from 1 to $m + \ell + 1$, and we partition it into the following three consecutive subsets

(1) $V^{\mathrm{R}}_{m,\ell} := \{1\}$ corresponding to the label for the univalent (of degree 1) root vertex,
(2) $V^{\mathrm{I}}_{m,\ell} := \{2, 3, \ldots, m + 1\}$ corresponding to the labels for the internal trivalent (of degree 3) vertices,
(3) $V^{\mathrm{L}}_{m,\ell} := \{m + 2, m + 3, \ldots, m + \ell + 1\}$ corresponding to the labels for the univalent leaf vertices.

The sum over T is a sum over all trees on $V_{m,\ell}$ such that vertices in $V^{\mathrm{I}}_{m,\ell}$ have degree 3 and all vertices in $V^{\mathrm{R}}_{m,\ell} \cup V^{\mathrm{L}}_{m,\ell}$ have degree 1. By Cayley's Theorem, the number of such trees is exactly

$$\frac{[(m + \ell + 1) - 2]!}{(1 - 1)! \times (3 - 1)!^m \times (1 - 1)!^\ell} = \frac{(m + \ell - 1)!}{2^m} \ .$$

Note that the sum over trees would be empty unless the relation between the degree sum and the cardinality $|V_{m,\ell}|$ holds. This requires

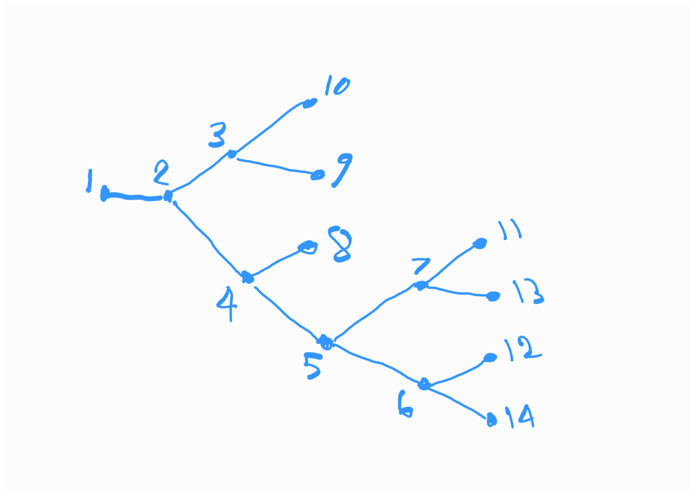
$$1 + 3m + \ell = 2(m + \ell + 1) - 2 \ ,$$

namely,

$$\ell = m + 1 \ .$$

Hence, the two summation indices m, ℓ are not independent, but rather bound by the above relation.

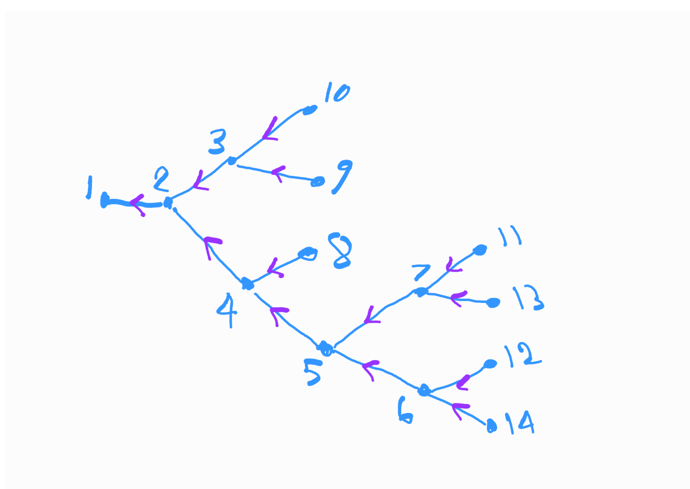
We now explain the evaluation of the "amplitude" $\mathscr{A}_i(T)$ of a tree T, on a sufficiently general explicit example. Consider the following tree



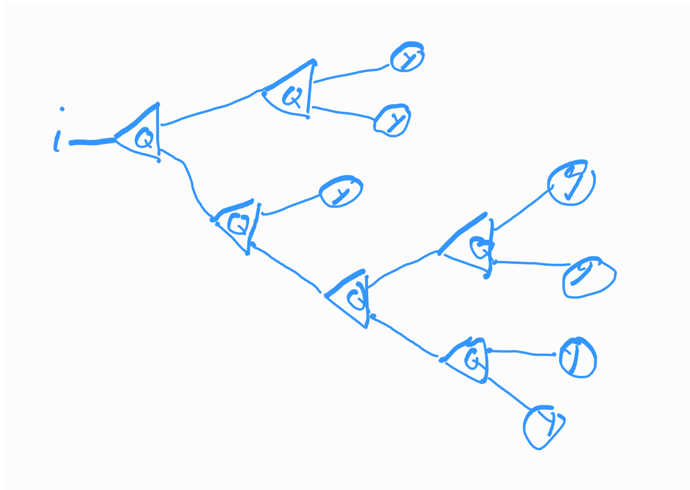
namely,

$$T = \{ \ \{1,2\}, \{2,3\}, \{3,10\}, \{9,3\}, \{2,4\}, \{4,8\},$$
$$\{4,5\}, \{5,7\}, \{11,7\}, \{13,7\}, \{6,12\}, \{6,14\}, \{5,6\} \ \}$$

where we put some items on nonintuitive order so as to emphasize that the set of pairs is unordered and the pairs are unordered. Here the number of internal vertices is $m = 6$ and the number of leaves is $\ell = 7$ for a total number of vertices $|V_{m,\ell}| = 14$. The first step is to orient all the edges towards the root vertex 1, which gives



The second step is to turn the leaves into y blobs and the internal vertices into Q tensor elements, while making sure the pointy part of the triangle representing the Q tensors feeds into the outgoing arrow indicated by the above picture. This results in the tensor diagram

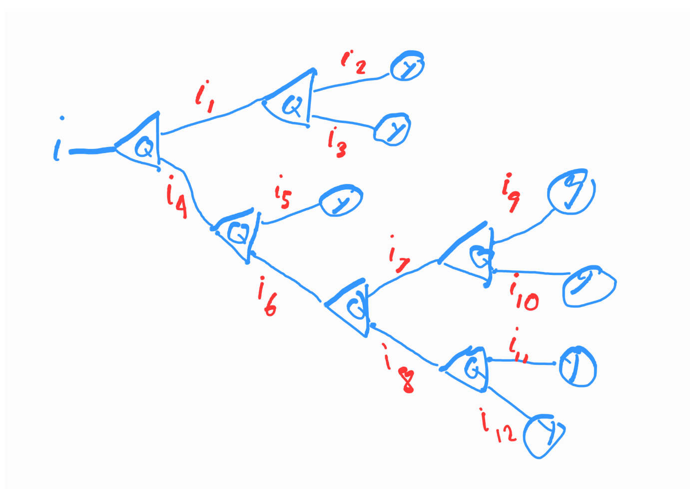


The leg incident to the root does not involve summing over an index, since it is assigned the fixed value i, when evaluating the i-th series $g_i(Y)$. Finally, the evaluation of the tree amplitude $\mathscr{A}_i(T)$ is, by definition, the evaluation of the last tensor diagram, as seen in Lectures 4 and 5. More precisely, in the above example, we have

$$\mathscr{A}_i(T) := \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\sum_{i_4=1}^{n}\sum_{i_5=1}^{n}\sum_{i_6=1}^{n}\sum_{i_7=1}^{n}\sum_{i_8=1}^{n}\sum_{i_9=1}^{n}\sum_{i_{10}=1}^{n}\sum_{i_{11}=1}^{n}\sum_{i_{12}=1}^{n}$$

$$Q_{i,i_1,i_4}Q_{i_1,i_2,i_3}Q_{i_4,i_5,i_6}Q_{i_6,i_7,i_8}Q_{i_7,i_9,i_{10}}Q_{i_8,i_{11},i_{12}}\ y_{i_2}y_{i_3}y_{i_5}y_{i_9}y_{i_{10}}y_{i_{11}}y_{i_{12}}\ . \quad (3)$$

To help parse the above formula, we indicated the location of the numbered summation indices in the picture below



This concludes our semi-formal statement of the explicit solution to the inversion problem (1).

A very simple example/sanity check:
All of the above should work in the one-dimensional case where $n = 1$. All sums over indices degenerate to just one term since the only possible index value is 1. Let use simplify our notations to $x := X_1$, $y := Y_1$ and $q := Q_{1,1,1}$. Then the "system" (1) to be solved is just

$$x - \frac{q}{2}x^2 = y$$

Namely, we need to solve the high school quadratic equation

$$\frac{q}{2}x^2 - x + y = 0 \ . \tag{4}$$

We will compare the previous tree expansion solution to the easy explicit solutions with the square roots of the discriminant.

Consider any tree T which appears in the formula (2). Because there are no real/nontrivial sums over indices, it is immediate that

$$\mathscr{A}_i(T) = Q_{1,1,1}^m Y_1^\ell = q^m y^\ell \ .$$

Using the relation $\ell = m + 1$ and the Cayley count of trees, we get

$$\begin{aligned} g(y) := g_1(Y) &= \sum_{m=0}^{\infty} \frac{1}{m!(m+1)!} \sum_T q^m y^{m+1} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!(m+1)!} \times \frac{(2m)!}{2^m} \times q^m y^{m+1} \ . \end{aligned}$$

Remark 1. *Note the appearance of the quantity*

$$C_m := \frac{(2m)!}{m!(m+1)!} = \frac{1}{m+1}\binom{2m}{m}$$

called the m-th Catalan number. This is one of the most famous integer sequences in combinatorics. There are lots of combinatorial objects counted by the C_m, most notably, Dyck paths, and complete planar binary trees which are related to, yet different from the Cayley trees we have been considering. A great reference on this area of enumerative combinatorics is the book [1].

We now consider the roots of (4):

$$x = \frac{1 \pm \sqrt{1 - 2qy}}{q} \ ,$$

and look at their expansion in powers of y, say for y small. Recall Newton's series generalization of the binomial theorem for real exponent α:

$$\begin{aligned} (1+u)^\alpha &= 1 + \alpha u + \frac{\alpha(\alpha-1)}{2}u^2 + \cdots \\ &= \sum_{\ell=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-\ell+1)}{\ell!} u^\ell \end{aligned}$$

which converges for $|u| < 1$. For $u = -2qy$ and $\alpha = \frac{1}{2}$, this gives

$$\sqrt{1-2qy} = \sum_{\ell=0}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-\ell+1\right)}{\ell!} (-2qy)^{\ell} \ .$$

After treating separately the first $\ell = 0$ term, and using $(-2)^{\ell}$ to clear denominators and change the sign of the individual factors in the numerator, we get

$$\sqrt{1-2qy} = 1 - \sum_{\ell=1}^{\infty} \frac{1 \times 1 \times 3 \times \cdots \times (2\ell-3)}{\ell!} (qy)^{\ell}$$

or, after changing the summation index to $m = \ell - 1$,

$$\sqrt{1-2qy} = 1 - \sum_{m=1}^{\infty} \frac{1 \times 3 \times \cdots \times (2m-1)}{(m+1)!} q^{m+1} y^{m+1} \ .$$

Note the appearance of the number of perfect matchings of $2m$ elements, a notion already encountered in Lecture 2:

$$1 \times 3 \times \cdots \times (2m-1) = \frac{(2m)!}{2^m m!} \tag{5}$$

sometimes denoted by $(2m-1)!!$, which we will avoid. Indeed, one can count perfect matchings on $[2m]$ as follows: one has $(2m-1)$ ways to choose a partner p_1 for the element 1, then $(2m-3)$ ways to choose a partner p_2 for the first remaining element, i.e., the smallest element of $[2m]\backslash\{1, p_1\}$, etc. This gives the expression on the LHS of (5). The expression on the RHS should be clear from Exercise 1 of Lecture 2. Alternatively, the equality (5) is trivial to prove by multiplying and dividing by the missing even factors

$$2 \times 4 \times \cdots \times (2m) = 2^m m! \ .$$

After cleaning up our last computation we see that

$$\frac{1-\sqrt{1-2qy}}{q} = \sum_{m=0}^{\infty} \frac{(2m)!}{m!(m+1)!} \times \frac{1}{2^m} \times q^m y^{m+1} = g(y) \ .$$

The tree expansion agrees exactly with the choice of root with minus the square root, i.e. the root closest to the origin when y is small, as opposed to the "far away" root which rather is close to $2/q$.

Remark 2. *The tree expansion method works for any degree of the Q_i. We can use it for solving $x^3 + px + q = 0$, and this would lead to the "exercise" of deriving Cardano's formula in that way. We can also use it for the Bring-Jerrard quintic equation $x^5 + px + q = 0$ (to which the general quintic can be reduced by a Tschirnhaus transformation). An interesting question raised by Zac would be to prove the Abel-Ruffini Theorem of impossibility of solution by radicals, using the "numerology" of the coefficients of the power series solution.*

Convergence of the tree expansion:

So far, we have only worked with FPS's which allowed us to sidestep any issues of convergence. We will now look at the such issues, on the example of the $g_i(Y)$ series, for general n this time. We will show that for Y evaluated at a small vector y in $\mathbb{C}^n$, these series converge absolutely. This topic is very important since it is our first (easy) example of "pin and sum" argument or the method of "summation along a tree" which is a common theme in many rigorous results about statistical physics and QFT models. The title of Part I of this course is "Combinatorial Analysis". Unfortunately, this expression is already taken and nowadays means anything related to combinatorics. Here, we really mean doing analysis (as in real or complex analysis) with the help of combinatorics, as the following example will show.

We start by defining some suitable norms in order to measure the size of our basic ingredients. For vectors $y = (y_1, \ldots, y_n)$ in $\mathbb{C}^n$ we will use the ℓ^∞ norm, namely,

$$\|y\| := \max_{1\le i\le n} |y_i| \ .$$

For tensors like (of the same format as) $Q = (Q_{i,j,k})_{1\le i,j,k\le n}$ we will use a mixed ℓ^∞-ℓ^1 norm

$$\|Q\| := \max_{1\le i\le n} \left(\sum_{j,k=1}^{n} |Q_{i,j,k}| \right) \ .$$

We now have the following proposition.

Proposition 1. *For any tree T appearing in (2) we have the bound*

$$|\mathscr{A}_i(T)| \le \|Q\|^m \|y\|^\ell \ .$$

Our proof will be semi-formal, and explained on a sufficiently general example, namely the previous tree amplitude (3) with $m = 6$, $\ell = 7$. We start by bounding each term by its absolute value/modulus:

$$|\mathscr{A}_i(T)| \le \sum_{i_1,\ldots,i_{12}=1}^{n} |Q_{i,i_1,i_4}|\ |Q_{i_1,i_2,i_3}|\ |Q_{i_4,i_5,i_6}|\ |Q_{i_6,i_7,i_8}|\ |Q_{i_7,i_9,i_{10}}|\ |Q_{i_8,i_{11},i_{12}}|$$
$$\times |y_{i_2}|\ |y_{i_3}|\ |y_{i_5}|\ |y_{i_9}|\ |y_{i_{10}}|\ |y_{i_{11}}|\ |y_{i_{12}}| \ .$$

Then, we bound each $|y_j|$ by $\|y\|$ and pull these norm factors out of the sum. Therefore,

$$|\mathscr{A}_i(T)| \le \|y\|^7 \times \sum_{i_1,\ldots,i_{12}=1}^{n} |Q_{i,i_1,i_4}|\ |Q_{i_1,i_2,i_3}|\ |Q_{i_4,i_5,i_6}|\ |Q_{i_6,i_7,i_8}|\ |Q_{i_7,i_9,i_{10}}|\ |Q_{i_8,i_{11},i_{12}}| \ .$$

We now recursively perform bounds and sums, starting from the leaves (or terminal Q branches) and progressing towards the root. We pick, for example, the leaf-like vertex/tensor $Q_{i_7,i_9,i_{10}}$ as the first one to be eliminated. To this effect, using Fubini's Theorem for finite sums, we rewrite the last bound as follows

$$|\mathscr{A}_i(T)| \le \|y\|^7 \times \sum_{i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8,i_{11},i_{12}=1}^{n} |Q_{i,i_1,i_4}|\ |Q_{i_1,i_2,i_3}|\ |Q_{i_4,i_5,i_6}|\ |Q_{i_6,i_7,i_8}|\ |Q_{i_8,i_{11},i_{12}}|$$
$$\times \left(\sum_{i_9,i_{10}=1}^{n} |Q_{i_7,i_9,i_{10}}| \right) \ .$$

The inner sum over i_9, i_{10}, for any fixed i_7 can be bounded by the constant factor $\|Q\|$ which we can then pull out. This gives

$$|\mathscr{A}_i(T)| \le \|y\|^7 \times \|Q\| \times \sum_{i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8,i_{11},i_{12}=1}^{n} |Q_{i,i_1,i_4}|\ |Q_{i_1,i_2,i_3}|\ |Q_{i_4,i_5,i_6}|\ |Q_{i_6,i_7,i_8}|\ |Q_{i_8,i_{11},i_{12}}| \ . \tag{6}$$

Now we repeat the process with say the $Q_{i_8,i_{11},i_{12}}$ vertex as the next one to be removed. Namely, we rewrite the RHS of (6) as

$$\|y\|^7 \times \|Q\| \times \sum_{i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8=1}^{n} |Q_{i,i_1,i_4}|\ |Q_{i_1,i_2,i_3}|\ |Q_{i_4,i_5,i_6}|\ |Q_{i_6,i_7,i_8}|\ \times \left(\sum_{i_{11},i_{12}=1}^{n} |Q_{i_8,i_{11},i_{12}}| \right)$$

$$\leq \|y\|^7 \times \|Q\|^2 \times \sum_{i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8=1}^{n} |Q_{i,i_1,i_4}|\ |Q_{i_1,i_2,i_3}|\ |Q_{i_4,i_5,i_6}|\ |Q_{i_6,i_7,i_8}|\ .$$

Clearly, we can progressively prune the tree and remove the Q's until none is left, at which point we get the final bound $|\mathscr{A}_i(T)| \leq \|y\|^7 \times \|Q\|^6$.

Theorem 1. *If $\|Q\|\ \|y\| < \frac{1}{2}$, then the tree series (2) converges absolutely.*

Proof: From the proposition, we have

$$\begin{aligned} \sum_{m,\ell\geq 0} \frac{1}{m!\ell!} \sum_{T} |\mathscr{A}_i(T)| \ &\leq\ \sum_{m,\ell\geq 0} \frac{1}{m!\ell!}\ \sum_{T} \|Q\|^m\ \|y\|^\ell \\ &=\ \sum_{m=0}^{\infty} \frac{(2m)!}{m!(m+1)!} \times \frac{1}{2^m} \times \|Q\|^m \|y\|^{m+1}\ . \end{aligned}$$

We finish by noting that

$$C_m \leq \binom{2m}{m} \leq 2^{2m}\ ,$$

since the number of subsets with m elements is bounded by the total number of subsets. As a result,

$$\sum_{m,\ell\geq 0} \frac{1}{m!\ell!} \sum_{T} |\mathscr{A}_i(T)| \leq \sum_{m=0}^{\infty} 2^m \|Q\|^m \|y\|^{m+1} < \infty\ .$$

□

References

[1] R. P. Stanley, *Catalan Numbers.* Cambridge University Press, New York, 2015.