MATH 8450 – LECTURE 8 – FEB 13, 2023

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Some proofs about trees:

We now revisit the material from Lecture 6 and add proofs of some of the results therein, obtained solely from the given definition (Definition 3 from Lecture 6).

The main tool we will need is the following inductive structural lemma.

Lemma 1. *Let T be a tree on a set V with $|V| \geq 2$ and let $v_* \in V$. Then there exist $k \geq 1$, and subsets $V_1, \ldots, V_k$ of V and vertices $v_1, \ldots, v_k$ in V such that all the following properties hold.*

(1) $\forall i \in [k], v_i \in V_i$.
(2) $\forall i, j \in [k], i \neq j \to V_i \cap V_j = \varnothing$.
(3) $V_1 \cup \cdots \cup V_k = V \backslash \{v_*\}$.
(4) For all $i \in [k]$, $T_i := T \cap \mathscr{P}(V_i)$ is a tree on V_i.
(5) The $2k$ subsets of $V^{(2)}$ given by $\{\{v_*, v_1\}\}, \ldots, \{\{v_*, v_k\}\}$, and $T_1, \ldots, T_k$ are pairwise disjoint and their union is T.

Proof: Let $k := \delta_T(v_*)$. By Lemma 1 of Lecture 6, we have $k \geq 1$. Define the set neighbors

$$N := \{v \in V \mid \{v_*, v\} \in T\} ,$$

and the "star" of v_*

$$S := \{e \in T \mid v_* \in e\} .$$

By definition of the vertex degree, $|S| = k$. By definition, $T \subset V^{(2)}$ and elements of the latter are subsets of V with exactly two elements. So if $\{v_*, v\} \in T$, we must have $v \neq v_*$. Clearly, the map $N \to S$, $v \mapsto \{v_*, v\}$ is a bijection, and therefore $|N| = k$. Let us pick a numbering of the elements of N as $v_1, \ldots, v_k$. We therefore have that $v_*, v_1, \ldots, v_k$ are all distinct. Let $\widetilde{T} := T \backslash S$. For each $i \in [k]$, let V_i be the connected component of v_i for the graph $\widetilde{T}$ on V, so (1) is satisfied by construction.

We now note that the connected component V_* of v_* for the graph $\widetilde{T}$ is just $\{v_*\}$. Indeed, if some $w \neq v_*$ was in that connected component, we would have a path $v_* = u_0, u_1, \ldots, u_p = w$ in $\widetilde{T}$, and $w \neq v_*$ implies $p \geq 1$ and thus $e := \{u_0, u_1\} \in \widetilde{T} \subset T$. Since $v_* = u_0 \in e$, we have $e \in S$, but this contradicts $e \in \widetilde{T} = T \backslash S$. Hence $V_* = \{v_*\}$ is established. Since for all i, $v_i \neq v_*$, we have that v_i is not in V_* and therefore v_* is not in V_i, by the symmetry property of equivalence relations. This proves $V_1 \cup \cdots \cup V_k \subset V \backslash \{v_*\}$. Now let $w \in V \backslash \{v_*\}$. Since T is a tree, it connects v_* to w. Consider a path of minimal length $v_* = u_0, u_1, \ldots, u_p = w$ in T with necessarily $p \geq 1$ and all vertices $u_0, \ldots, u_p$ being distinct. Since $v_* \in \{u_0, u_1\} \in T$, and by definition and property of S and N, there exists $i \in [k]$, such that $u_1 = v_i$. Since the u's are distinct, none of the vertices $u_1, u_2, \ldots, u_p$ is equal to v_* and therefore the successive edges $\{u_\ell, u_{\ell+1}\}$, with $1 \leq \ell < p$, do not belong to S, i.e., are in $\widetilde{T}$. Hence $u_1, \ldots, u_p$ is a path in $\widetilde{T}$ from v_i to w, and therefore $w \in V_i$. This finishes the proof of item (3).

We now consider $i \neq j$ in $[k]$ and show that v_i and v_j are not connected by $\widetilde{T}$. Suppose for the sake of contradiction that there was a path $u_0 = v_i, u_1, \ldots, u_p = v_j$ in $\widetilde{T}$. Since $i \neq j$, we have $v_i \neq v_j$ and therefore $p \geq 1$. We can also assume the path has minimal length so that all vertices $u_0, \ldots, u_p$ are distinct. Since $p \geq 1$, for each $\ell \in [p]$, we have that at least one of $\{u_\ell, u_{\ell+1}\}$ or $\{u_{\ell-1}, u_\ell\}$ makes sense and is in $\widetilde{T} = T \backslash S$. By definition of S, this implies that u_ℓ cannot be equal to v_*. As a result, $v_*, u_0, u_1, \ldots, u_p$ are all distinct and form a set of $p+2 \geq 3$ vertices. Since $\{v_*, u_0\} = \{v_*, v_i\} \in T$ and $\{v_*, u_p\} = \{v_*, v_j\} \in T$, we see that $v_*, u_0, u_1, \ldots, u_p$ form a circuit in T which is a contradiction. Clearly, what we showed proves that the connected components V_i and V_j are distinct and therefore disjoint, and item (2) has been established.

Let $i \in [k]$, and recall that T_i is by definition the set of edges (unordered pairs of two vertices) which belong to T and which also are contained in V_i. Clearly, T_i can have no circuit (definition relative to the set of vertices V_i) because this would also be a circuit for T (definition relative to the vertex set V). Let $w \in V_i$ and consider a path $u_0 = v_i, u_1, \ldots, u_p = w$ in $\widetilde{T}$. Then for all $\ell \geq 0$, $u_0, \ldots, u_\ell$ is a path in $\widetilde{T}$ from v_i to u_ℓ. Hence, u_ℓ is in the connected component V_i. As a result, for all ℓ, with $1 \leq \ell < p$, we have $\{u_\ell, u_{\ell+1}\} \subset V_i$ and therefore $\{u_\ell, u_{\ell+1}\} \in T_i$. This shows that the graph T_i on $V_i \neq \varnothing$ is connected. We completed the proof of item (4).

We will now prove item (5). Note that if $i, j \in [k]$ and $i \neq j$, then $v_i \neq v_j$ and therefore $\{v_*, v_i\} \neq \{v_*, v_j\}$. This shows that the sets $\{\{v_*, v_i\}\}$, $1 \leq i \leq k$, are pairwise disjoint. Moreover, we clearly have $S = \cup_{i=1}^{k} \{\{v_*, v_i\}\}$.

Let $\{v, w\} \in T_i$, for some $i \in [k]$, so in particular we have $v \neq w$, and $v, w \in V_i$, as well as $\{v, w\} \in T$. Since $v_* \notin V_i$, then v, w are different from v_* and $\{v, w\} \notin S$. This shows the inclusion $\cup_{i=1}^{k} T_i \subset \widetilde{T}$, and also shows that for all $i, j \in [k]$, $\{\{v_*, v_i\}\} \cap T_j = \varnothing$, because $S \cap \widetilde{T} = \varnothing$ by definition of $\widetilde{T}$ as the complement of S in T. Now let $i, j \in [k]$ with $i \neq j$, and suppose $e \in T_i \cap T_j$. Then we would have $e \in \mathscr{P}(V_i) \cap \mathscr{P}(V_j)$, i.e., $e \subset V_i$ and $e \subset V_j$. This would imply $e \subset V_i \cap V_j = \varnothing$, by (2) which contradicts $|e| = 2$ coming from $e \in T \subset V^{(2)}$. We now completed the proof of pairwise disjointness for all $2k$ subsets involved.

What remains is to show the inclusion $\widetilde{T} \subset \cup_{i=1}^{k} T_i$. Let $e = \{v, w\} \in \widetilde{T}$. Since $e \notin S$, we have $v \neq v_*$ and $w \neq v_*$. By (3), there exist $i, j \in [k]$ such that $v \in V_i$ and $w \in V_j$. Since $\{v, w\} \in \widetilde{T}$, the vertices v, w are connected by the graph $\widetilde{T}$ on V, and they must then be in the same connected component. So we must have $V_i = V_j$ and thus $i = j$ by item (2). Since both v, w are in V_i and $\{v, w\} \in \widetilde{T} \subset T$, we also have $\{v, w\} \in T_i$. So the needed inclusion is proved and therefore so is item (5) and the lemma. $\square$

Proof of Proposition 1 from Lecture 6: We use induction on $|V|$. If $|V| = 1$, then $T \subset V^{(2)} = \varnothing$ so the equality trivially holds. Now suppose $|V| \geq 2$ and pick some element v_* in V, in order to invoke Lemma 1. From item (5) of the lemma and the induction hypothesis, we immediately get

$$\begin{aligned} |T| &= k + |T_1| + \cdots + |T_k| \ , \\ &= k + (|V_1| - 1) + \cdots + (|V_k| - 1) \ , \end{aligned}$$

and the RHS of the last equation clearly reduces to $|V \backslash \{v_*\}| = |V| - 1$. $\square$

For the purpose of tree counting, we will need a converse to Lemma 1.

Lemma 2. *Let V be a finite set with $|V| \geq 2$ and let v_* be an element of V. Let $k \in \mathbb{Z}_{>0}$, and let $V_1, \ldots, V_k$ be nonempty, pairwise disjoint subsets of V whose union is equal to $V \backslash \{v_*\}$. Let $v_1, \ldots, v_k$ be elements of V such that $\forall i \in [k], v_i \in V_i$. Let $T_1, \ldots, T_k$ be trees on $V_1, \ldots, V_k$ respectively. Then,*

$$T := \{\{v_*, v_i\} \mid i \in [k]\} \cup \left(\cup_{i=1}^{k} T_i\right)$$

is a tree on V.

Exercise 1. *Prove the last lemma.*

The following lemma is not earth-shattering, but does not cost much to state and prove. It says: trees have leaves.

Lemma 3. *Let T be a tree on a set V with $|V| \geq 2$. Then there exists at least two vertices v in V with $\delta_T(v) = 1$.*

Proof: Let $V_{\mathrm{L}} := \{v \in V \mid \delta_T(v) = 1\}$. By Lemma 1 of Lecture 6, all vertices have degree at least 1. So if $v \in V \backslash V_{\mathrm{L}}$, we must have $\delta_T(v) \geq 2$. This remark, with the help of Proposition 2 of Lecture 6, implies

$$2|V| - 2 = \sum_{v \in V} \delta_T(v) \geq |V_{\mathrm{L}}| + 2(|V| - |V_{\mathrm{L}}|) \ ,$$

from which $|V_{\mathrm{L}}| \geq 2$ trivially follows. □

We also have the following purely numerical variant of the previous lemma.

Lemma 4. *Let V be a finite set with $|V| \geq 2$ and let $(d_v)_{v \in V} \mathbb{Z}_{>0}^{V}$ be such that $\sum_{v \in V} d_v = 2|V| - 2$. Then there exists at least two elements v in V with $d_v = 1$.*

Proof: Define $V_{\mathrm{L}} := \{v \in V \mid d_v = 1\}$ and repeat the previous proof. □

Of course, it would have been quicker to prove Lemma 3 as a corollary of Lemma 4, but one could argue this order of presentation is more intuitive/inevitable.

Proof of Theorem 1 of Lecture 6: First suppose $n = 2$. From the hypothesis $\forall v \in V, d_v \geq 1$ and $\sum_{v \in V} d_v = 2n - 2 = 2$ we get $\forall v \in V, d_v = 1$. So the RHS of Cayley's formula reduces in this particular case to

$$\frac{(n-2)!}{\prod_{v \in V} (d_v - 1)!} = 1 \ .$$

A tree T on V must be such that $|T| = n - 1 = 1$ while T is contained in $V^{(2)}$ which is a singleton. Therefore there is exactly only one tree on V, namely $T = V^{(2)}$. So the theorem holds for $n = 2$. We then proceed by induction on n, and let us now assume $n \geq 3$. By Lemma 4, one can pick some $v_* \in V$ such that $d_{v_*} = 1$. A tree T satisfying the vertex-degree conditions must be so that $\delta_T(v_*) = 1$ and therefore, when invoking Lemma 1 and its notations, the following should happen. We must have $k = 1$. There is only one vertex v_1, set V_1 and tree T_1. We therefore condition the count, i.e., summing the term 1 over trees T, according to the value of $v_1 \in V_1 = V \backslash \{v_*\}$. For given fixed v_1, the tree $T_1 = T \cap \mathscr{P}(V_1)$ on V_1 must have the following vertex degrees:

$$\forall w \in V_1 \backslash \{v_1\}, \ \delta_{T_1}(w) = \delta_T(w) = d_w \ ,$$

and

$$\delta_{T_1}(v_1) = \delta_T(v_1) - 1 = d_{v_1} - 1 \ ,$$

since the edge of T joining v_1 back to the root v_* is not included in T_1. Note that, thanks to Lemmas 1 and 2, there is a bijection between trees T with fixed v_1 and trees T_1 on V_1, with degrees specified by d_w for $w \neq v_1$ and $d_{v_1} - 1$ when $w = v_1$. We indeed have $T_1 = T \cap \mathscr{P}(V \backslash \{v_*\})$ while the inverse map is given by $T = T_1 \cup \{v_*, v_1\}$. Also note that we must have $d_{v_1} \geq 2$. Otherwise, v_* and v_1 would be disconnected from the rest which is nonempty since $n \geq 3$. Putting all this together, and using the induction hypothesis, we have

$$\begin{aligned}
&\sum_T \mathbb{1}\{\forall v \in V, \delta_T(v) = d_v\} \\
&= \sum_{v_1 \in V \backslash \{v_*\}, d_{v_1} \geq 2} \ \sum_T \mathbb{1}\{\forall v \in V, \delta_T(v) = d_v\} \times \mathbb{1}\{\{v_*, v_1\} \in T\} \\
&= \sum_{v_1 \in V \backslash \{v_*\}, d_{v_1} \geq 2} \ \sum_{T_1} \mathbb{1}\{\delta_{T_1}(v_1) = d_{v_1} - 1, \text{ and } \forall v \in V \backslash \{v_*, v_1\}, \delta_{T_1}(v) = d_v\} \\
&= \sum_{v_1 \in V \backslash \{v_*\}, d_{v_1} \geq 2} \frac{(n-3)!}{(d_{v_1} - 2)! \times \prod_{v \in V \backslash \{v_*, v_1\}} (d_v - 1)!} \\
&= \frac{(n-3)!}{(d_{v_*} - 1)! \times (d_{v_1} - 1)! \times \prod_{v \in V \backslash \{v_*, v_1\}} (d_v - 1)!} \times \sum_{v_1 \in V \backslash \{v_*\}, d_{v_1} \geq 2} (d_{v_1} - 1) \\
&= \frac{(n-3)!}{\prod_{v \in V} (d_v - 1)!} \times \sum_{v_1 \in V \backslash \{v_*\}, d_{v_1} \geq 2} (d_{v_1} - 1) \\
&= \frac{(n-3)!}{\prod_{v \in V} (d_v - 1)!} \times \sum_{v \in V} (d_{v_1} - 1) \\
&= \frac{(n-3)!}{\prod_{v \in V} (d_v - 1)!} \times [(2n - 2) - n] \\
&= \frac{(n-2)!}{\prod_{v \in V} (d_v - 1)!} \ ,
\end{aligned}$$

which completes the inductive proof. □

Discrete geodesics:

As in differential geometry, where geodesics (locally) correspond to shortest paths, we have a similar notion in trees. The following proposition is intuitively obvious, and its proof will ring a bell for anyone who had to fix a separated zipper.

Proposition 1. *Let T be a tree on a set V and let $a, b \in V$. Then, there exists a unique path of minimal length from a to b.*

Proof: Let $k = d_T(a, b)$, then there exists such a path $a = v_0, v_1, \ldots, v_k = b$ because T is a connected graph. We also noted earlier that such a minimal path must be such that $v_0, \ldots, v_k$ are all distinct. If $a = b$, then $k = 0$ and the minimal path is clearly unique given by the vertex sequence (v_0) with $v_0 = a = b$. Suppose now that $a \neq b$, and suppose that

$a = v_0, v_1, \ldots, v_k = b$ and $a = w_0, v_1, \ldots, w_k = b$ are two different minimal paths from a to b in the tree T. Since the paths are different, we have $k \geq 2$ and $\exists i, 1 < i < k$, such that $v_i \neq w_i$. Pick the smallest such i, which will give us the additional information $v_{i-1} = w_{i-1}$. Let $R := \{v_{i+1}, v_{i+2}, \ldots, v_k\}$. Since $k > i$ satisfies $v_k = w_k = b$, the set of j's such that $i < j \leq k$ and $w_j \in R$ is nonempty. We then pick the smallest such j. We then define ℓ, $i < \ell \leq k$ as the unique index such that $v_\ell = w_j$. It is then easy to see that the vertices $v_{i-1}, v_i, \ldots, v_\ell, w_{j-1}, w_{j-2}, \ldots, w_i$ are all distinct and form, in this order, a circuit. Since T is a tree, this is a contradiction. Hence, there cannot be two different minimal paths. $\square$

The above makes a bit more precise the notion of shortest path between two vertices in a tree, with respect to the graph distance. These are discrete analogues of geodesics.

Remark 1. *(for those who took differential and Riemannian geometry) A tree is analogous to a hyperbolic space. If a, b, c are three distinct points, the two geodesics $a \to b$, $a \to c$ stay close to each other at first (in fact coincide) and then diverge from each other. This is similar to the behavior of geodesics in hyperbolic spaces (with negative curvature) like the Lobachevsky/Poincaré disc $\{z \in \mathbb{C} \mid |z| < 1\}$ or upper half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.*

The parent function:

The following proposition defines the parent function on the tree.

Proposition 2. *Let T be a tree on a set V and let $v_* \in V$ which we will call the root. Then there exists a unique function $P : V\backslash\{v_*\} \to V$, such that:*

(1) $\forall v \in V\backslash\{v_*\}$, $\{v, P(v)\} \in T$,
(2) $\forall v \in V\backslash\{v_*\}$, $d_T(v_*, P(v)) = d_T(v_*, v) - 1$.

The proof is easy. The existence can be done by letting $P(v) = v_1$ if $v = v_0, v_1, \ldots, v_k = v_*$ is the minimal length path from v to v_*.

The formal definition of the tree amplitudes:

We can now give a more formal definition of the tree amplitudes $\mathscr{A}_i(T)$ from Lecture 7. We reuse the notations from that lecture. Firstly, we generalize the evaluation $Q_{i,j,k}$ of the Q tensors. Let A be a finite set with $|A| = 2$, and let $J : A \to [n]$. Finally, let $i \in [n]$. Then, by definition, we let

$$Q_{i,J} := Q_{i,J(a_1),J(a_2)} \ ,$$

if $A = \{a_1, a_2\}$. This is not ambiguous because the tensor is symmetric in its last two indices. Namely, if we took the other numbering of the elements of A which exchanges a_1 and a_2, we would still get the same result

$$Q_{i,J(a_2),J(a_1)} = Q_{i,J(a_1),J(a_2)} \ .$$

Given a tree T on $V_{m,\ell}$ with the vertex degree conditions imposed in Lecture 7, we let $v_* = 1 \in V_{m,\ell}^{\mathrm{R}}$ and denote the corresponding parent function by P. We can now state

$$\mathscr{A}_i(T) := \sum_{J \in [n]^{V_{m,\ell}^{\mathrm{R}} \cup V_{m,\ell}^{\mathrm{L}}}} \delta_{i,J(P^{-1}(1))} \times \left(\prod_{a \in V_{m,\ell}^{\mathrm{I}}} Q_{J(a),J|_{P^{-1}(\{a\})}} \right) \times \left(\prod_{b \in V_{m,\ell}^{\mathrm{L}}} y_{J(b)} \right) \ ,$$

where $J|_A$ means the restriction to a set A .

Exercise 2. *Prove that the series over trees $g_i(Y)$ from Eq. (2) in Lecture 7 is indeed the solution of the inversion problem given by Eq. (1) of the same lecture. Namely, show that $g_i(Y) = Y_i + \frac{1}{2}Q_i(g(Y))$, by separating the $m = 0$ term which gives Y_i from the rest of the sum. See* [1] *for more details.*

Exercise 3. *Write an explicit series expansion in terms of trees (this time without having the vertex degree 3 or 1 restriction) for the inversion problem considered in Lecture 4, Theorem 1, in the case where $h_k(Y) := Y_k$, and $L = I$. The key is to abandon the sum of monomials point of view and work with tensors. Namely, the formal power series*

$$f_k = \sum_{\alpha \in \mathbb{N}^n} f_{k,\alpha} X^\alpha$$

should be rewritten

$$f_k = X_k + \sum_{d=2}^{\infty} \frac{1}{d!} \sum_{j_1,\ldots,j_d \in [n]} F^{(d)}_{k;j_1,\ldots,j_d} X_{j_1} X_{j_2} \cdots X_{j_d} \ .$$

For the relation between the $f_{k,\alpha}$ monomial coefficients and the $F^{(d)}_{k;j_1,\ldots,j_d}$ tensor elements for the degree d homogeneous part of the power series, see Lecture 5. Finaly, see what changes if one allows a linear part given by a general invertible matrix L. What happens also if the h_k are more general. For more details, see [2].

GAUSSIAN INTEGRALS

A important tool we will need in order to introduce perturbation theory and its organization in terms of Feynman diagrams, is a general formula for integrating monomials with respect to a Gaussian measure. Let A be a $n \times n$ real symmetric positive definite matrix. Let $p \in \mathbb{N}$ and let $i_1, \ldots, i_p$ be in $[n]$. We want to compute the integral

$$\int_{\mathbb{R}^n} x_{i_1} \cdots x_{i_p} \ e^{-\frac{1}{2}x^{\mathrm{T}}Ax} \ \mathrm{d}^n x \ .$$

Note that even if we loosely write $x = (x_1, \ldots, x_n)$ we still think of x as a column vector so that the matrix product $x^{\mathrm{T}}Ax$ is meaningful and gives a number. We first show that the integral converges (Exercise 2 from Lecture 1). Let $\mu \in \mathbb{N}^n$ be the multiplicity multiindex of the sequence of indices $i_1, \ldots, i_p$. After putting absolute values, the integral becomes

$$B := \int_{\mathbb{R}^n} |x_1|^{\mu_1} \cdots |x_n|^{\mu_n} \ e^{-\frac{1}{2}x^{\mathrm{T}}Ax} \ \mathrm{d}^n x \ .$$

Note that for any real number $u \geq 0$, we have the (elementary yet quite powerful) bound

$$\frac{u^n}{n!} \leq e^u \ ,$$

because a term is bounded by the series, if all terms are nonnegative. Pick some $\epsilon > 0$. Then, for any $j \in [n]$, letting $u = \epsilon |x_j|$ and $n = \mu_j$, we get

$$|x_j|^{\mu_j} \leq \epsilon^{-\mu_j} \mu_j! e^{\epsilon |x_j|} \ .$$

Since A is positive definite, there exists $\eta > 0$ such that all eigenvalues of A are bounded below by η. Therefore $x^{\mathrm{T}}Ax \geq \eta x^{\mathrm{T}}x$ for all $x \in \mathbb{R}^n$. As a result, using our previous notations for multiindices, we have the bound

$$\begin{aligned} B &\leq \epsilon^{-|\mu|} \mu! \int_{\mathbb{R}^n} \mathrm{d}^n x \ \exp\left(-\frac{1}{2} \eta x^{\mathrm{T}} x + \sum_{i=1}^{n} \epsilon |x_j| \right) \\ &= \epsilon^{-|\mu|} \mu! \left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2} + \epsilon |y|} \mathrm{d}y \right)^n \ . \end{aligned}$$

The one dimensional integrals over y are obviously finite, so $B < \infty$, and the original integral converges.

Theorem 1. *(Isserlis-Wick) We have the identity*

$$\int_{\mathbb{R}^n} x_{i_1} \cdots x_{i_p} \ e^{-\frac{1}{2}x^{\mathrm{T}}Ax} \ \mathrm{d}^n x = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}} \sum_{P} \prod_{\{a,b\} \in P} A^{-1}_{i_a, i_b}$$

where the sum is over all perfect matchings of the set $[p]$ *(see Lecture 2 for the definition and notations).*

For the proof see the next lecture.

Remark 2. *The above theorem about Gaussian integrals is due to Leon Isserlis* [4]*. Note that in the physics literature, this is called Wick's Theorem and is a workhorse of perturbation theory of QFT, in the path integral formalism developed by Richard Feynman. The* *related version due to Gian Carlo Wick was the analogous version in the canonical quantization framework* [5]. *Later, Feynman realized the importance of Gaussian integrals in QFT perturbation theory computations* [3, Appendix C].

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