MATH 8450 – LECTURE 9 – FEB 15, 2023

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Proof of the Isserlis-Wick Theorem:

For convenience, we will modify our notations and write $A(i,j)$ instead of $A_{i,j}$ for matrix elements. Likewise, a component of the vectorial integration variable x will be denoted $x(i)$ instead of x_i. Our goal is to prove

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \ x(i_1)\cdots x(i_p)\ e^{-\frac{1}{2}x^{\mathrm{T}}Ax} \ = \ \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}} \sum_P \prod_{\{a,b\}\in P} A^{-1}(i_a,i_b)\ , \qquad (1)$$

where the sum is over perfect matchings P of the set $[p]$, namely, set partitions with only blocks of size 2. We first note that the theorem holds if p is odd. Indeed, in this case, the LHS of (1) vanishes as can be seen from the change of variables $x \to -x$. The RHS also vanishes since the sum over P is empty. We now focus on the case where p is even.

Suppose $A = R^{\mathrm{T}}BR$ where $R \in O(n)$, the group of orthogonal matrices with real entries. Then B also must be real symmetric and positive definite. It should have the same eigenvalues as A, and also the same determinant. We will show that if the theorem holds for B (*for all choices of indices*) then it must also hold for A.

We do the change of variables $x = R^{-1}y$, with Jacobian equal to $|\det(R^{-1})| = 1$ because R is orthogonal. We then get, referring to the left-hand side of (1),

$$\begin{aligned}
\mathrm{LHS} &= \int_{\mathbb{R}^n} \mathrm{d}^n y \ \prod_{c=1}^{p}\left(\sum_{j_c=1}^{n} R^{-1}(i_c,j_c)y(j_c)\right)\ e^{-\frac{1}{2}y^{\mathrm{T}}By} \\
&= \sum_{j_1,\ldots,j_p=1}^{n}\left(\prod_{c=1}^{p} R^{-1}(i_c,j_c)\right)\int_{\mathbb{R}^n}\mathrm{d}^n y\ y(j_1)\cdots y(j_p)\ e^{-\frac{1}{2}y^{\mathrm{T}}By} \\
&= \sum_{j_1,\ldots,j_p=1}^{n}\left(\prod_{c=1}^{p} R^{-1}(i_c,j_c)\right)\frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(B)}}\sum_P\prod_{\{a,b\}\in P} B^{-1}(j_a,j_b) \\
&= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}}\sum_{j_1,\ldots,j_p=1}^{n}\sum_P\left(\prod_{c=1}^{p} R^{-1}(i_c,j_c)\right)\prod_{\{a,b\}\in P} B^{-1}(j_a,j_b) \\
&= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}}\sum_P\sum_{j_1,\ldots,j_p=1}^{n}\left(\prod_{c=1}^{p} R^{-1}(i_c,j_c)\right)\prod_{\{a,b\}\in P} B^{-1}(j_a,j_b)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}} \sum_{P} \sum_{j_1,\ldots,j_p=1}^{n} \prod_{\{a,b\}\in P} \left(R^{-1}(i_a,j_a)B^{-1}(j_a,j_b)R^{-1}(i_b,j_b) \right) \\
&= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}} \sum_{P} \prod_{\{a,b\}\in P} \left(\sum_{j_a,j_b=1}^{n} R^{-1}(i_a,j_a)B^{-1}(j_a,j_b)R^{-1}(i_b,j_b) \right) \\
&= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}} \sum_{P} \prod_{\{a,b\}\in P} A^{-1}(i_a,i_b)\ .
\end{aligned}
$$

In going from the 2nd to the 3rd line, we used the hypothesis that B satisfies (1) for all assignments of the indices. From the 3rd to the 4th line, we used $\det(B) = \det(A)$ and put the R^{-1} factors inside the inner sum. We then used the discrete Fubini Theorem to exchange sums over the partition P and over the j indices. From the 5th to the 6th line, we used the fact P is a set partition of $[p]$, in order to write a product over $[p]$ as a product, over blocks of P, of subproducts within these blocks. Then we undid the expansion of a product of sums. Finally, we used the matrix equation $R^{-1}B^{-1}(R^{-1})^{\mathrm{T}} = R^{\mathrm{T}}B^{-1}R = A^{-1}$ since $R^{-1} = R^{\mathrm{T}}$.

By the spectral theorem for real symmetric matrices and the claim just proved, the theorem reduces to the case where A is a diagonal matrix with diagonal entries given by say $\lambda_1, \ldots, \lambda_n > 0$. In this case,

$$
\begin{aligned}
\mathrm{LHS} &= \int_{\mathbb{R}^n} \mathrm{d}^n x\ x(i_1)\cdots x(i_p)\ e^{-\frac{1}{2}\sum_{j=1}^{n}\lambda_j x(j)^2} \\
&= \int_{\mathbb{R}^n} \mathrm{d}^n x\ x(1)^{\mu_1}\cdots x(n)^{\mu_n}\ e^{-\frac{1}{2}\sum_{j=1}^{n}\lambda_j x(j)^2} \\
&= \prod_{j=1}^{n} \left(\int_{\mathbb{R}} \mathrm{d}x(j)\ x(j)^{\mu_j}\ e^{-\frac{1}{2}\lambda_j x(j)^2} \right)\ ,
\end{aligned}
$$

where $\mu = (\mu_1, \ldots, \mu_n)$ denotes the multiplicity multiindex $\mu(i_1, \ldots, i_p)$, and we used Fubini's Theorem.

One the other hand, the right-hand side of (1) is now given by

$$
\mathrm{RHS} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\lambda_1\cdots\lambda_n}} \sum_{P} \prod_{\{a,b\}\in P} (\lambda_{i_a}^{-1} \mathbb{1}\{i_a = i_b\})\ ,
$$

by an easy inversion of a diagonal matrix. It is more advantageous to use the condition $i_a = i_b$ enforced by the indicator, in order to split the factor $\lambda_{i_a}^{-1}$ evenly as $\lambda_{i_a}^{-\frac{1}{2}}\lambda_{i_b}^{-\frac{1}{2}}$. Hence,

$$
\begin{aligned}
\mathrm{RHS} &= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\lambda_1\cdots\lambda_n}} \sum_{P} \prod_{\{a,b\}\in P} (\lambda_{i_a}^{-\frac{1}{2}}\lambda_{i_b}^{-\frac{1}{2}} \mathbb{1}\{i_a = i_b\}) \\
&= (2\pi)^{\frac{n}{2}} \times \left(\prod_{j=1}^{n} \lambda_j^{-\frac{(\mu_j+1)}{2}} \right) \times \sum_{P} \prod_{\{a,b\}\in P} \mathbb{1}\{i_a = i_b\})\ ,
\end{aligned}
$$

because

$$\prod_{\{a,b\}\in P}(\lambda_{i_a}^{-\frac{1}{2}}\lambda_{i_b}^{-\frac{1}{2}}) = \prod_{c\in[p]}\lambda_{i_c}^{-\frac{1}{2}} = \lambda_1^{-\frac{\mu_1}{2}}\cdots\lambda_n^{-\frac{\mu_n}{2}}\ .$$

For $1\le j\le n$, let $V_j := \{c\in[p]\ |\ i_c=j\}$ which has cardinality μ_j. The sum

$$\sum_P\prod_{\{a,b\}\in P}\mathbb{1}\{i_a=i_b\}$$

is the number of perfect matchings on $[p]$ with the additional constraint of respecting the $V_1,\ldots,V_n$ division into compartments, i.e., P's which satisfy

$$\forall\{a,b\}\in P,\ \exists j\in[n],\ \{a,b\}\subset V_j\ .$$

Clearly, choosing such P's amounts to choosing perfect matchings in each V_j, independently. Therefore,

$$\sum_P\prod_{\{a,b\}\in P}\mathbb{1}\{i_a=i_b\}\ =\ \mathbb{1}\{\forall j\in[n],\mu_j\in 2\mathbb{N}\}\times\prod_{j=1}^{n}\frac{\mu_j!}{2^{\frac{\mu_j}{2}}\left(\frac{\mu_j}{2}\right)!}\ .$$

Note that, as in measure theory when dealing with functions defined almost everywhere, the writing in the last equation could appear sloppy, but it is not really a problem. While the factorial of $\frac{\mu_j}{2}$ could be problematic if μ_j is odd, we don't have to bother defining/assigning a value for it because the preceding indicator makes the whole thing zero anyways. As a result, we have

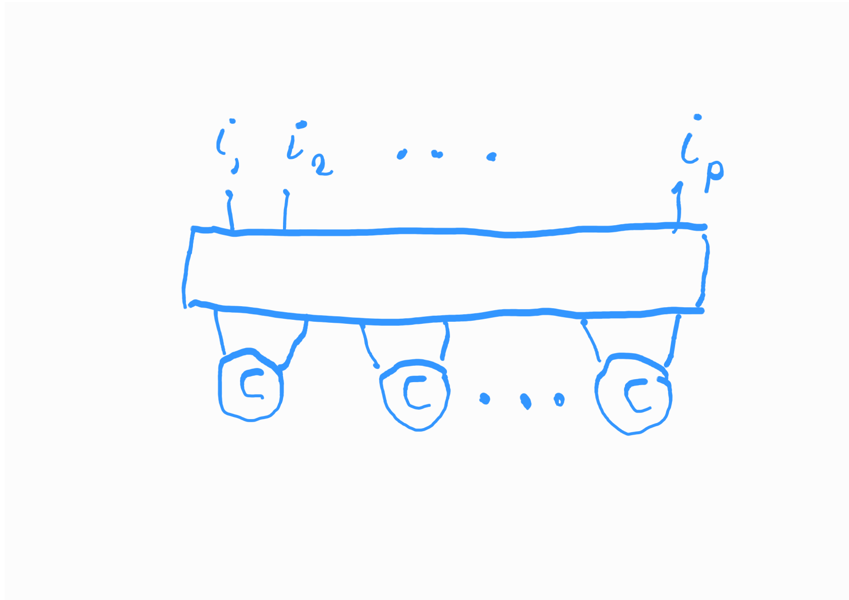
$$\mathrm{RHS} = \prod_{j=1}^{n}\left(\mathbb{1}\{\mu_j\in 2\mathbb{N}\}\times(2\pi)^{\frac{1}{2}}\times\lambda_j^{-\frac{(\mu_j+1)}{2}}\times\frac{\mu_j!}{2^{\frac{\mu_j}{2}}\left(\frac{\mu_j}{2}\right)!}\right)\ .$$

So the the equality LHS = RHS reduces to showing that, for all $j\in[n]$, we have

$$\int_{\mathbb{R}}\mathrm{d}x(j)\ x(j)^{\mu_j}\ e^{-\frac{1}{2}\lambda_j x(j)^2} = \mathbb{1}\{\mu_j\in 2\mathbb{N}\}\times(2\pi)^{\frac{1}{2}}\times\lambda_j^{-\frac{(\mu_j+1)}{2}}\times\frac{\mu_j!}{2^{\frac{\mu_j}{2}}\left(\frac{\mu_j}{2}\right)!}$$

which is the one-dimensional case of the theorem. The latter is left as an easy exercise on calculations with the Euler Gamma function. □

We will typically denote the inverse matrix A^{-1}, which plays the lead role in the Isserlis-Wick Theorem, by C and will call it the (free Euclidean) propagator. Note that when p is even, and with the help of Eq. (3) of Lecture 2 and the graphical calculus developed in Lectures 4 and 5, the result of the evaluation of the Gaussian integral delivered by the theorem can be written as



(2)

where the box is a symmetrizer of size p and we represented the symmetric tensor with two indices C with a round blob.

The intended application to QFT perturbation theory:
Recall from Lecture 2, that we want to study the limits

$$C_n(f_1, \ldots, f_n) = \lim_{r \to -\infty} \lim_{s \to \infty} \frac{C^{\mathrm{U}}_{n,r,s}(f_1, \ldots, f_n)}{C^{\mathrm{U}}_{0,r,s}}$$

where $f_1, \ldots, f_n$ are test functions in Schwartz space $\mathscr{S}(\mathbb{R}^d)$. The unnormalized correlations are

$$C^{\mathrm{U}}_{n,r,s}(f_1, \ldots, f_n) := \int_{\mathbb{R}^{\Lambda_{r,s}}} \prod_{x \in \Lambda_{r,s}} \mathrm{d}\phi(x) \quad \prod_{j=1}^{n} \left(\sum_{y_j \in \Lambda_{r,s}} L^{dr} f_j(y_j) \phi(y_j) \right) \times \exp\left(-S_{r,s}(\phi)\right) \ ,$$

where the full Euclidean action functional is

$$S_{r,s}(\phi) := \sum_{x \in \Lambda_{r,s}} L^{dr} \left[\frac{a_{r,s}}{2} \sum_{i=1}^{d} \left(\frac{\phi(x + L^r e_i) - \phi(x)}{L^r} \right)^2 + \frac{m^2_{r,s}}{2} \phi(x)^2 + \frac{\lambda_{r,s}}{24} \phi(x)^4 \right] \ .$$

Again recall that $\Lambda_{r,s}$ is the finite set $(L^r\mathbb{Z})^d \cap \left[-\frac{L^s}{2}, \frac{L^s}{2}\right]^d$ seen as a subset of $\mathbb{R}^d$, but also identified with the additive group $(L^r\mathbb{Z})^d/(L^s\mathbb{Z})^d$ in order to make sense of expressions such as $x + L^r e_i$, since we are imposing *periodic boundary conditions*.

In order to proceed, we split the parameters as a free part plus and interaction or perturbation part:

$$
\begin{aligned}
m_{r,s}^2 &= m^2 + \delta m_{r,s}^2 \ , \\
a_{r,s} &= 1 + \delta a_{r,s} \ , \\
\lambda_{r,s} &= 0 + \lambda_{r,s} \ .
\end{aligned}
$$

This results in a splitting $S_{r,s}(\phi) = S_{r,s}^{\mathrm{G}}(\phi) + \delta S_{r,s}(\phi)$ of the action into a free or Gaussian part

$$
S_{r,s}^{\mathrm{G}}(\phi) := \sum_{x\in\Lambda_{r,s}} L^{dr} \left[\frac{1}{2} \sum_{i=1}^{d} \left(\frac{\phi(x+L^r e_i) - \phi(x)}{L^r} \right)^2 + \frac{m^2}{2}\phi(x)^2 \right] \ ,
$$

and an interaction or perturbation part

$$
\delta S_{r,s}(\phi) := \sum_{x\in\Lambda_{r,s}} L^{dr} \left[\frac{\delta a_{r,s}}{2} \sum_{i=1}^{d} \left(\frac{\phi(x+L^r e_i) - \phi(x)}{L^r} \right)^2 + \frac{\delta m_{r,s}^2}{2}\phi(x)^2 + \frac{\lambda_{r,s}}{24}\phi(x)^4 \right] \ .
$$

The idea is to keep $\exp(-S_{r,s}^{\mathrm{G}}(\phi))$ while completely expanding the exponential of $-\delta S_{r,s}(\phi)$. The latter is a sum of homogeneous polynomials which can be represented graphically with appropriate blobs as in Lecture 5. One will then have an infinite sum of Gaussian integrals which can be computed using the Isserlis-Wick Theorem. This series expansion method applied to the unnormalized correlations is called perturbation theory in QFT. As can be seen from (2), the result will be an infinite sum of evaluations of tensor diagrams. The proper handling of such sums, and in particular the issue of accurate computation of so-called symmetry factors of Feynman diagrams, will require a higher level of sophistication and abstraction, as compared to our previous treatment of the tree diagrams featuring in the problem of power series inversion from Lectures 6 and 7. Namely, we will need the theory of combinatorial species introduced by André Joyal [3, 1]), formulated using category theory.

A BRIEF INTRODUCTION TO CATEGORY THEORY

Category theory is an important area of mathematics. In some situations, it provides the very objects of study, while in others it also serves as a language and way to organize one's thoughts. Even in apparently remote areas, e.g., in analysis, when building a theory, introducing new concepts and definitions, it is helpful to do so while wearing category-theoretic pairs of glasses, in order for the theory one is buiding to be most harmonious and effective. For the physics students in this course, the use of categories may seem intimidating and perhaps too much leaning towards abstraction. It is however useful in developing physical theories too, e.g., for the study [2] of the $O(N)$ model of statistical mechanics and N-component ϕ^4 models for exotic values of N which are not in $\mathbb{Z}_{>0}$. We will only explore the bare minimum of category theory: the notions of category, (covariant) functor, and natural transformation or morphism of functors.

By definition, a category C is made of the following data:

(1) a collection of objects $\mathrm{Ob}(\mathsf{C})$,
(2) for any objects $X, Y \in \mathrm{Ob}(\mathsf{C})$, a set $\mathrm{Hom}(X, Y)$ called the set of morphisms or arrows from X to Y,
(3) for any objects $X, Y, Z \in \mathrm{Ob}(\mathsf{C})$, a map

$$\circ : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(X, Z)$$

called a composition operation,
(4) for each object $X \in \mathrm{Ob}(\mathsf{C})$, a choice of an element $\mathbf{1}_X$ in $\mathrm{Hom}(X, X)$ called the identity morphism of X.

If needed, like when there are several categories around in the discussion, we will write $\mathrm{Hom}_{\mathsf{A}}(X, Y)$ instead of just $\mathrm{Hom}(X, Y)$, in order to avoid any ambiguity. As part of the definition of a category, the previous data must satisfy the following properties or axioms.

(1) The composition is associative, namely, for all objects $X, Y, Z, U \in \mathrm{Ob}(\mathsf{C})$, and for all morphisms $f \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}(Y, Z)$, and $h \in \mathrm{Hom}(Z, U)$, we have

$$(h \circ g) \circ f = h \circ (g \circ f) \ .$$

(2) The identity morphisms behave like neutral elements, namely, for all objects $X, Y \in \mathrm{Ob}(\mathsf{C})$, and for all morphism $f \in \mathrm{Hom}(X, Y)$, we have

$$\mathbf{1}_Y \circ f = f \quad \text{and} \quad f \circ \mathbf{1}_X = f \ .$$

Example 1. *The category* Sets *whose objects are sets and whose morphisms are just maps. Namely, for two sets* X, Y*, the set of morphisms is* $\mathrm{Hom}(X, Y) := Y^X$*. The composition is the usual composition of maps, and identity morphisms are identity maps.*

Example 2. *The category* Vect *of vector spaces over some fixed field like* $\mathbb{R}$ *or* $\mathbb{C}$*. The objects are vector spaces, and morphisms are given by linear maps.*

Example 3. *The category* Top *of topological spaces. The objects are given by topological spaces* $(X, \mathscr{T})$ *where* X *is set and* $\mathscr{T} \in \mathscr{P}(\mathscr{P}(X))$ *is a topology on* X*. Morphisms are given by continuous maps, i.e., maps* $f : (X, \mathscr{T}) \to (X', \mathscr{T}')$ *such that* $\forall U' \in \mathscr{T}', f^{-1}(U') \in \mathscr{T}$*.*

Example 4. *The category* $\mathsf{Measurable}$ *of measurable spaces. The objects are given by measurable spaces* $(X, \mathscr{M})$ *made of a set* X *and a* σ*-algebra* $\mathscr{M}$ *on* X*. Morphisms are given by measurable maps, i.e., maps* $f : (X, \mathscr{M}) \to (X', \mathscr{M}')$ *such that* $\forall A' \in \mathscr{M}', f^{-1}(A') \in \mathscr{M}$*.*

Example 5. *The category* Grp *of groups with objects given by groups and morphisms given by group homomorphisms, i.e., maps* $\varphi : G \to H$ *between groups, such that* $\forall g_1, g_2 \in G$*,* $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$*.*

Example 6. *The category* $\mathsf{MetCont}$ *of metric spaces with continuous maps. Objects are metric spaces* (X, d) *where* X *is a set and* d *is a metric or distance on* X*. Continuity is with respect to the topologies defined by these distance functions.*

Example 7. *The category* MetLip *of metric spaces with morphisms given by maps which are Lipschitz. Namely, a map* $f : (X, d) \to (X', d')$ *qualifies as a morphism iff* $\exists K \geq 0$*,* $\forall x_1, x_2 \in X$*,* $d'(f(x_1), f(x_2)) \leq K\ d(x_1, x_2)$*.*

Example 8. *The category* $\mathsf{MetUnif}$ *of metric spaces with morphisms given by uniformly continuous maps. Namely, these are maps* $f : (X, d) \to (X', d')$ *such that*

$$\forall \eta > 0, \exists \varepsilon > 0, \forall x_1, x_2 \in X, (d(x_1, x_2) < \varepsilon \Longrightarrow d'(f(x_1), f(x_2)) < \eta)\ .$$

The above should give some intuition for what a category is, but, of course, the list is endless. Note that one can also mix and match.

Example 9. *The category* $\mathsf{TopVect}_{\mathbb{R}}$ *of topological vector spaces over* $\mathbb{R}$*. The objects are topological vector spaces over* $\mathbb{R}$ *(see notes on distributions). The morphisms are given by maps which are, at the same time, linear and continuous.*

Example 10. *The category* TopGrp *of topological groups. Objects are topological groups and morphisms are given by continuous group homomorphisms. A topological group* G *is a group equipped with a topology, such that the maps* $G \times G \to G$*,* $(g, h) \mapsto gh$ *and* $G \to G$*,* $g \mapsto g^{-1}$ *are continuous. It is implicit in the last statement that* $G \times G$ *is given the product topology of the given topology of* G *on each of the two factors.*

For our main concern which is perturbation theory based on Feynman diagrams, the main categories we will use are the ones in the next two examples.

Example 11. *The category* Fin *of finite sets and all maps. Objects are finite sets. Morphisms are just arbitrary maps.*

Example 12. *The category* FinBij *of finite sets with bijections. Object are arbitrary finite sets, but morphisms are given by bijective maps only.*

The next notion of we will need is that of (covariant) functor between two categories. If A and B are categories, a covariant functor or simply functor $\mathscr{F}$ from A to B is given by the following data. To each object A in category A, we associate an object $\mathscr{F}(A)$ in category B. We also have, for each objects A, A' in category A, a map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathsf{A}}(A, A') & \longrightarrow & \mathrm{Hom}_{\mathsf{B}}(\mathscr{F}(A), \mathscr{F}(A')) \\ f & \longmapsto & \mathscr{F}[f]\ . \end{array}$$

These must satisfy the following axioms.

(1) For all objects $A, A', A'' \in \mathrm{Ob}(\mathsf{A})$, and for all morphisms $f \in \mathrm{Hom}_{\mathsf{A}}(A, A')$ and $g \in \mathrm{Hom}_{\mathsf{A}}(A', A'')$, we have

$$\mathscr{F}[g \circ f] = \mathscr{F}[g] \circ \mathscr{F}[f] \ ,$$

with composition understood in the context of category B.

(2) For all object $A \in \mathrm{Ob}(\mathsf{A})$, we have

$$\mathscr{F}[\mathbf{1}_A] = \mathbf{1}_{\mathscr{F}(A)} \ .$$

Note that if, in the first axiom, we had the reversed composition $\mathscr{F}[f] \circ \mathscr{F}[g]$ on the right-hand side, then this would define a contravariant functor. In that case, for $f \in \mathrm{Hom}_{\mathsf{A}}(A, A')$, its counterpart $\mathscr{F}[f]$ would, by definition, be in $\mathrm{Hom}_{\mathsf{A}}(\mathscr{F}(A'), \mathscr{F}(A))$. We will not use these contravariant functors, so from now on our functors will be covariant and we will just say "functor".

Example 13. *One has a natural forgetful functor from* Grp *to* Sets *which to given a group G simply associates the underlying set G. To a group homomorphism, we associate itself simply seen as a map. Of course, one can construct tons of similar forgetful functors, by deciding to drop some of the structure at our disposal.*

Example 14. *(for those who took measure theory) One has a functor $\mathscr{B}$ from* Top *to* $\mathsf{Measurable}$. *If $(X, \mathscr{T})$ is a topological space, we associate to it the measurable space given by X together with the Borel σ-algebra $\mathscr{B}(\mathscr{T})$ defined by the topology $\mathscr{T}$, namely, the smallest σ-algebra on X which contains $\mathscr{T}$. To a continuous map f we associate $\mathscr{B}[f] := f$ itself. We can do that because continuous maps are measurable, between the respective Borel σ-algebras. It is common in measure theory to work with $\mathscr{L}_{\mathbb{R}}$ the set of Lebesgue-measurable sets in $\mathbb{R}$, which is much bigger than $\mathscr{B}_{\mathbb{R}}$ the set of Borel subsets of $\mathbb{R}$. This is done using the notion of completion of measures. One can argue that this is counterproductive and creates more problems than it solves, despite the apparent gain in generality, i.e., being able to integrate more (Lebesgue-measurable) functions. Even composition by a continuous function does not preserve being a Lebesgue-measurable map. So this procedure destroys the "functorial" properties which make $\mathscr{B}$ into a nice functor from topological spaces to measurable spaces.*

Example 15. *(for those who took some topology) Let* Top_* *denote the category of topological spaces with choice of base point. Objects are pairs (X, x_0) where X is a topological space and x_0 is a point in/element of X, called a base point. Morphisms from (X, x_0) to (Y, y_0) are continuous maps $f : X \to Y$ such that $f(x_0) = y_0$. By taking continuous loops based at x_0, i.e., continuous maps $\gamma : [0, 1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$, up to homotopy, i.e., continuous deformation, we get a set of equivalence classes $\pi_1(X, x_0)$. One can compose such loops by concatenation or going through them in succession. This give a group structure, so $\pi_1(X, x_0)$ is called the fundamental group. This construction provides a functor (less trivial than the previous examples) from the category* Top_* *to the category* Grp.

Remark 1. *(Thanks to Walker for information relevant to this remark, and the pointer to the reference by Shulman) Category theory poses some set-theoretic issues. The famous Russell Paradox about the set of all sets which do not contain themselves (modeled after the liar's paradox) shows that there is no set of all sets and therefore* $\mathrm{Ob}(\mathsf{Sets})$ *is not a set. For most categories that are used in mathematics, the collection of objects is not a set. When we* *wrote statements like "$A \in \mathrm{Ob}(\mathsf{A})$", this is not to be taken literally as membership in a set, but just as a quick way of saying that A is an object of category A. When discussing functors, it is all right to think intuitively of the assignment $A \mapsto \mathscr{F}(A)$ as a "map" from $\mathrm{Ob}(\mathsf{A})$ to $\mathrm{Ob}(\mathsf{B})$. However, strictly speaking, maps are between sets. Finally, note our generous use of the word "collection", a rather vague term, when talking about things like $\mathrm{Ob}(\mathsf{A})$, because of these set-theoretic issues. There are essentially two ways of dealing with these issues with Bourbaki-grade mathematical rigor, and both imply abandoning ZFC (the Zermelo-Fraenkel axioms plus AC, the axiom of choice), the standard axiomatic framework for most of the mathematical literature. One can switch to NBG or the von Neumann-Bernays-Gödel system of axioms which, in addition to sets, considers classes which are not sets. Such classes can then be used to talk about things like $\mathrm{Ob}(\mathsf{A})$. Another possibility is to work in ZFC+U, i.e., adding the axiom about the existence of a Grothendieck Universe (sometimes one needs two of those). In this course, we will ignore such set-theoretic technicalities and refer to [4] for more details about them. The sets we will ultimately/effectively use in the following lectures are quite small: they will be at most countable. One can read the theorems and proofs while ignoring set-theoretic issues, and then later go over these proofs and see how to modify the categories involved so all the discussion can take place in a manageable context like the set of all finite subsets of a fixed set (an alphabet) that is rich enough for our purposes, like say $\mathbb{N}$. Finally, note that the chosen philosophy in this course is to only use ZF+DC, i.e., we only need the more concrete/intuitive axiom of (countable) dependent choice rather than the full-fledged axiom of choice. So going to ZFC+U is not an option for us.*

References

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