On the Ground State Energy for Massless Nelson Model~~s~~

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Partial support: NSF

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arXiv: 1008.4628v1 [math-ph]

I - Introduction
II - Theorem
III - Idea of the proof

I- Introduction :

Massless translation invariant Nelson model $(d \geq 3)$

$$\mathfrak{h} = L^2(\mathbb{R}^d)$$

Fock space $\quad \mathcal{F}(\mathfrak{h}) = \underset{\underset{\Omega}{\cup}}{\mathbb{C}} \bigoplus_{n=1}^{\infty} \mathfrak{h}^{\otimes_s^n}$ bosons

free vacuum

$a(k)$, $a^*(k)$

CCR : $\quad [a(k), a(k')] = [a^*(k), a^*(k')] = 0$

$$[a(k), a^*(k')] = \delta(k-k')$$

$$a(k)\Omega = 0$$

Dispersion relation $\quad \omega(k) = |k|$

$$H_f = \int_{\mathbb{R}^d} \omega(k)\, a^*(k) a(k)\, d^d k$$

$H_{-\frac{1}{2}}$ completion of $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^d)$ for $\quad (f,g)_{-\frac{1}{2}} = \int_{\mathbb{R}^d} \frac{\overline{\hat{f}(k)}\,\hat{g}(k)}{2\omega(k)}\, d^d k$

$$\phi(f) = \int_{\mathbb{R}^d} \frac{d^d k}{\sqrt{2\omega(k)}} \left(\hat{f}(k) a(k) + \hat{f}(-k) a^*(k) \right) \quad \text{for } f \in H_{-\frac{1}{2}}$$

Form factor $\rho \in H_{-\frac{1}{2}}$ such that $\frac{\hat{\rho}}{\omega} \in L^2$.

$\rho_x(\cdot) = \rho(\cdot + x)$

Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}(h)$

Hamiltonian $H_\lambda = -\frac{1}{2}\Delta_x + H_f + \lambda\, \phi(\rho_x)$

coupling

Translation invariance:

H_λ commutes with total momentum $P = -i\nabla_x + P_f$

$$P_f = \int_{\mathbb{R}^d} k\, a^*(k) a(k)\, d^d k$$

$$\rightarrow \quad H_\lambda \simeq \int_{\mathbb{R}^d}^{\oplus} H_\lambda(P)\, d^d P$$

c-number

fiber Hamiltonian

$$H_\lambda(P) = \frac{1}{2}(P - P_f)^2 + \lambda\, \phi(\rho_0) + H_f$$

on $\mathcal{F}(h)$.

s.a. for λ real, bounded below

$$\rightarrow \quad \underline{E_\lambda(P) = \inf \sigma(H_\lambda(P))}$$

Problem: Regularity of $E_\lambda(P)$ as a function of λ and P.

regularity in P $\rightarrow$ $$\frac{1}{m_{ren}} = \frac{\partial^2}{\partial P^2} E_\lambda(P) \Big|_{P=0}$$

renormalized mass of dressed particle
$\rightarrow$ construction of scattering states (Pizzo 2005)

Previous results:

- Fröhlich 1973 : a.e. differentiability in P
  $\nabla_P E_\lambda(P)$ locally Lipschitz

- Bach, Chen, Fröhlich, Sigal 2007
- Chen 2008
- Fröhlich, Pizzo 2010

  $\Big\}$: C^2 regularity in P for QED

- Griesemer, Hasler 2009 : analyticity in λ & P for less IR singular models

II - Theorem: (A.A. & D. Hasler 2010)

For ρ such that $\frac{\hat{\rho}}{\sqrt{\omega}}, \frac{\hat{\rho}}{\omega} \in L^2$ and $\hat{\rho} \geq 0$ a.e.

$E_\lambda(P)$ is jointly analytic in λ & P in domain

$$|\lambda| < \frac{1}{2}(1-|P|)^{-3/2} \times \left(\int_{\mathbb{R}^d} d^d k \frac{|\hat{\rho}(k)|^2}{|k|^2} \right)^{-1/2}$$

One has explicit convergent expression:

$$E_\lambda(P) = \frac{P^2}{2} - \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\lambda^2}{4}\right)^n \sum_{\substack{\tau \text{ tree} \\ \text{on } \{1,\dots,n\}}} \int_{[0,1]^\tau} \prod_{\{i,j\}\in\tau} dh_{\{ij\}}$$

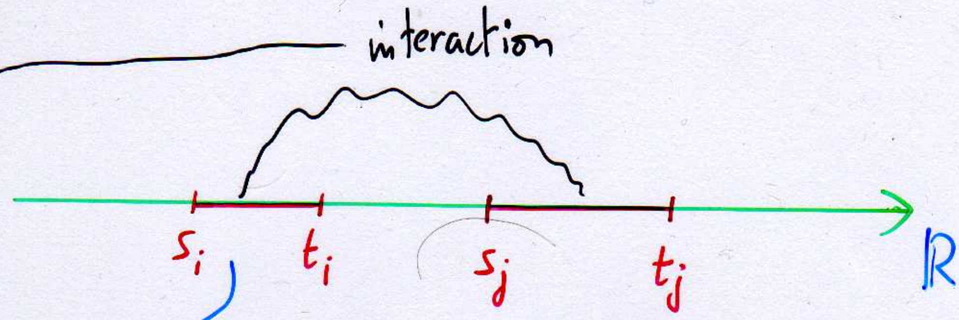
$$\int_{\mathbb{R}^{2n-1}} \prod_{j=2}^{n} ds_j \prod_{j=1}^{n} dt_j \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} d^d k_j \prod_{j=1}^{n} \left(e^{-|k_j||s_j - t_j| - (k_j \cdot P)(s_j - t_j)} \frac{|\hat{\rho}(k_j)|^2}{|k_j|} \right)$$

$$\times \prod_{\{ij\}\in\tau} \left(-k_i \cdot k_j \, A_{ij}(s,t)\right)$$

$$\times \exp\left[-\frac{1}{2} \sum_{i=1}^{n} k_i^2 A_{ii}(s,t) - \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} k_i \cdot k_j \, A_{ij}(s,t) \, u(\tau,h)_{\{ij\}} \right].$$

$s_1 := 0$ (Stat Mech Cluster Expansion)

Interacting gas of intervals $[s_j, t_j]$ on real line



$$\int_{\mathbb{R}^d} d^d k_i \; \frac{|\hat{\rho}(k_i)|^2}{|k_i|} \; e^{-|k_i||s_i - t_i| - (k_i \cdot P)(s_i - t_i)}$$

$$k_i \cdot k_j \;\; A_{ij}(s,t)$$

$$A_{ij}(s,t) = C(s_i, s_j) - C(s_i, t_j) - C(t_i, s_j) + C(t_i, t_j)$$

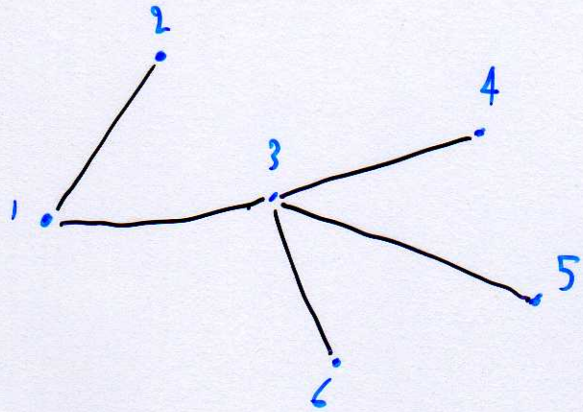
for $C(u,v) = \min(u,v)$ = covariance of 1d Brownian motion starting at 0.

matrix $\underline{A(s,t) \text{ is sym } \geq 0}$.

Abstract combinatorics:

$\{1, \dots, n\}$ labels for intervals

$n=6$



τ tree

$\sum_{\substack{\tau \text{ tree} \\ \text{on } \{1,\dots,n\}}} \cdots$ has n^{n-2} terms (Cayley)

$$\tau \to \int_{[0,1]^5} dh_{12}\, dh_{13}\, dh_{34}\, dh_{35}\, dh_{36}$$

(h's: interpolation parameters)

$$\boxed{\mu(\tau, h)_{\{i,j\}} = \text{min of } h \text{ parameters along } i \leftrightarrow j \text{ path in } \tau}$$

e.g. $\mu(\tau,h)_{\{1,2\}} = h_{12}$

$\mu(\tau,h)_{\{2,5\}} = \min(h_{12}, h_{13}, h_{35})$

$\mu(\tau,h)_{\{4,5\}} = \min(h_{34}, h_{35})$

III - Idea of the Proof:

① $$E_\lambda(P) = \lim_{T\to\infty} -\frac{1}{T}\log \underbrace{\left(\Omega, e^{-TH_\lambda(P)}\Omega\right)}_{\overset{!!}{Z_T(P)}}$$

$(\hat{\rho} \geq 0)$

② Path integral representation

$$Z_T(P) = \mathbb{E}\left[\exp\left(-\lambda\int_0^T \xi_s(\rho_{b_s})\,ds + iP\cdot b_T\right)\right]$$

$t \longrightarrow b_t \in \mathbb{R}^d$ Brownian motion in d dimensions

$$\mathbb{E}(b_{t,\alpha}\, b_{s,\beta}) = \delta_{\alpha\beta}\min(s,t)$$

$t \longrightarrow \xi_s \in S'(\mathbb{R}^d)$ Infinite dimensional oscillator process.

$\xi_t(f) = \xi(f\otimes\delta_t)$ in $H_{-\frac{1}{2}}$

ξ gen. Gaussian random process on $S'_{\mathbb{R}}(\mathbb{R}^{d+1})$

$$\mathbb{E}\left(\xi_s(f)\,\xi_t(g)\right) = \left(\Omega, \phi(f)\, e^{-|s-t|H_f}\phi(g)\Omega\right)$$

$$= \int_{\mathbb{R}^d} \frac{\overline{\hat{f}(k)}\,\hat{g}(k)}{2\omega(k)}\, e^{-\omega(k)|t-s|}\, d^dk$$

③ Integrate bosons out

$$Z_T(P) = \mathbb{E}\left(\exp\left(i P \cdot b_T + \lambda^2 \int_0^T \int_0^T W(b_s - b_t, s-t)\, ds\, dt \right) \right)$$

$$W(q,t) = \frac{1}{4} \int \frac{|\hat{\rho}(k)|^2}{|k|} e^{ikq - |k||t|} d^d k \qquad \text{(bounded)}$$

④ expand exponential & commute $\sum$ with $\mathbb{E}$

$$Z_T(P) = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!} \int_{[0,T]^{2n}} ds\, dt\, \mathbb{E}\left[e^{iP\cdot b_T} W(b_{s_1} - b_{t_1}, s_1 - t_1) \cdots W(b_{s_n} - b_{t_n}, s_n - t_n) \right]$$

⑤ use Fourier expression of W & integrate over b.

$$Z_T(P) = e^{-\frac{TP^2}{2}} \times \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\lambda^2}{4} \right)^n \int_{[0,T]^{2n}} ds\, dt \int_{\mathbb{R}^{nd}} dk$$

$$\prod_{j=1}^{n} \left(e^{-|k_j||s_j - t_j| - (k_j P)(s_j - t_j)} \frac{|\hat{\rho}(k_j)|^2}{|k_j|} \right) \underbrace{\exp\left[-\frac{1}{2} \sum_{i,j=1}^{n} k_i \cdot k_j\, A_{ij}(s,t) \right]}_{\text{2-body interactions}}$$

Grand canonical partition function of gas of intervals in 1d, in finite volume $[0,T]$ $\rightsquigarrow$? free energy/volume

→ Cluster / Mayer expansion writes $Z_T(P)$ as exponential key 1 : $A(s,t) \geq 0$ (stability)

key 2 : $A_{ij}(s,t) \neq 0 \Rightarrow [s_i, t_i] \cap [s_j, t_j] \neq \emptyset$

(Independence of increments)

⑥ Use BKAR formula to do expansion

Brydges, Kennedy 1987

A.A., Rivasseau 1995

$t = (t_{\{ij\}}) \in \mathbb{R}^{\binom{n}{2}}$ "couplings"

$$f(1,1,\dots) = \sum_{\substack{F \text{ forest} \\ \text{on } \{1,\dots,n\}}} \int_{[0,1]^F} \prod_{\{i,j\}\in F} dh_{\{ij\}} \frac{\partial^{|F|} f}{\prod_{\{i,j\}\in F} \partial t_{\{i,j\}}} (\mu(F,h))$$

$\mu(F,h)_{\{ij\}}$ as before + set $= 0$ if i & j not connected

$$f(t) = \exp\left[-\frac{1}{2}\sum_{i=1}^{n} k_i^2 A_{ii}(s,t) - \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} t_{\{ij\}}\, k_i \cdot k_j\, A_{ij}(s,t)\right]$$

collect according to connected components

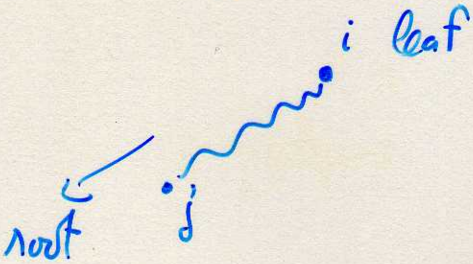
$\rightsquigarrow$ $\log Z_T(\varphi)$ as sum over trees τ

(7) convergence, uniform estimates in T

*) $u(\tau, h)$ preserves positivity

$\rightarrow \quad f(u(\tau,h)) \leq 1.$

*) integrate over s,t interval by interval from leafs of τ to root $1 \in \{1,\dots,n\}$.



$A_{ij}(s,t)$

length of intersection

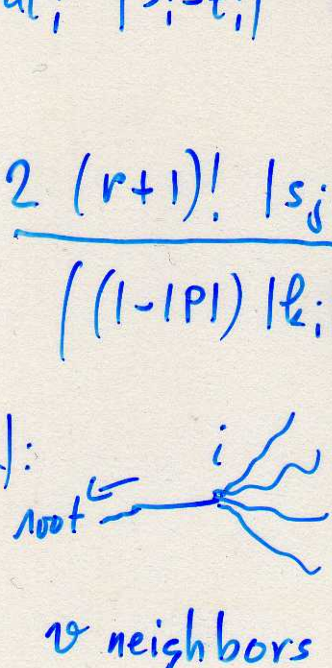
$$\int_{\mathbb{R}^2} ds_i\, dt_i\; e^{-(1-|p|)|k_i||s_i-t_i|}\, \big|[s_j,t_j]\cap[s_i,t_i]\big| = \frac{2\,|s_j-t_j|}{\big((1-|p|)|k_i|\big)^2}$$

in general

$$\int_{\mathbb{R}^2} ds_i\, dt_i\; |s_i-t_i|^r\, e^{-(1-|p|)|k_i||s_i-t_i|}\, \big|[s_j,t_j]\cap[s_i,t_i]\big|$$

$$= \frac{2\,(r+1)!\,|s_j-t_j|}{\big((1-|p|)\,|k_i|\big)^{r+2}}$$

powers of $|k|$:



$$-1 + v - \big((v-1)+2\big) = -2$$

ok if $d \geq 3$