# Multivariable Power Series Reversion and Lagrange-Good Inversion via Quantum Field Theory

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What is quantum field theory, really?

→ a generalization of calculus!

I - A game of symbolic integration

1) Composition

2) Reversion

3) Lagrange-Good Inversion

II - Feynman Diagrams: Like M. Jourdain physicists have been computing with combinatorial species without knowing.

# I. A Game of Symbolic Integration:

$K$ commutative field.

$F \in K[[X_1, \dots, X_n]] \iff$ function $K^n \to K$

$F = (F_i)_{1 \le i \le n}$, $F_i \in K[[X_1, \dots, X_n]] \iff$ function $K^n \to K^n$

$u$ collection of variables $u_1, \dots, u_n$

$du = du_1 \cdots du_n$

$uv = u_1 v_1 + \dots + u_n v_n$

Rule 1: $$\int du \, F(u)\, \delta(u) = F(0)$$

Rule 2: $$\int du \, e^{-uv} = \delta(v)$$

Rule 3: All rules of calculus allowed:

Integration by parts, Fubini, change of variables (no 1.1 for Jacobian) ...

## I-1) Composition:

$F, G \; : K^n \to K^n$ (no constant term)

$$\boxed{(F\circ G)_i(X) = \int d\bar{s}ds\, d\bar{t}dt\, d\bar{u}du \quad s_i \times \exp\left(-\bar{s}s - \bar{t}t - \bar{u}u + \bar{s}F(t) + \bar{t}G(u) + \bar{u}X\right)} \quad (*)$$

"Proof": $\int d\bar{u} \quad e^{-\bar{u}u + \bar{u}X} = \delta(u - X)$ (Rule 2)

$\int du \; \delta(u-X) \ldots$ = replace $u$ by $X$ in "…" (Rule 1 & 3)

$$\Rightarrow \text{RHS} = \int d\bar{s}ds\, d\bar{t}dt \quad s_i\, e^{-\bar{s}s - \bar{t}t + \bar{s}F(t) + \bar{t}G(X)}$$

$$= \int d\bar{s}ds \quad s_i\, e^{-\bar{s}s + \bar{s}F(G(X))} \qquad \text{(same steps)}$$

$$= F(G(X))_i \,.$$

Comments: • (*) is a combinatorial statement!

• Generalization to $F^1 \circ F^2 \circ \cdots \circ F^p$ immediate.

## I-2) Reversion:

Pb: $F : K^n \to K^n$ with invertible linear term and no constant term.

? $F^{-1}(Y)$ the compositional inverse

Computation:

let $\Omega : K^n \to K$ (not necessarily without constant term)

$$\int d\bar{u}\, du\ \Omega(u)\, e^{-\bar{u}F(u) + \bar{u}y} = ?$$

let $v = F(u) - y \Rightarrow u = F^{-1}(v + y)$

$$du = \det\left[\partial(F^{-1})(v+y)\right] dv$$

Notation: If $F = (F_i(X))_{1 \le i \le n}$ $X = (X_1, \dots, X_n)$

$$\partial F(Z) = \left(\frac{\partial F_i}{\partial X_j}(Z)\right)_{1 \le i,j \le n}$$ the Jacobian matrix of $F$ at the point $Z$.

Rule 3

$$\Rightarrow \int d\bar{u}\, du\, \Omega(u)\, e^{-\bar{u}F(u) + \bar{u}y} = \int d\bar{u}\, dv\ \Omega(F^{-1}(v+y)) \det[\partial F^{-1}(v+y)]\, e^{-\bar{u}v}$$

$$\overset{\text{Rule 2}}{=} \int dv\ \delta(v)\ \Omega(F^{-1}(v+y)) \det[\partial F^{-1}(v+y)]$$

$$\overset{\text{Rule 1}}{=} \Omega(F^{-1}(Y)) \det[\partial F^{-1}(Y)]$$

In particular

$$\boxed{F^{-1}(Y)_i = \frac{\int d\bar{u}\, du\ u_i\, e^{-\bar{u}F(u) + \bar{u}Y}}{\int d\bar{u}\, du\ e^{-\bar{u}F(u) + \bar{u}Y}}}$$

## I-3) Lagrange - Good inversion :

**Theorem:**

let $G = (G_i)_{1 \leq i \leq n}$ be a system of formal power series in $n$ letters

$F = (F_i)_{1 \leq i \leq n}$ in $K[[X_1, \dots, X_n]]$ without constant term

defined by $\boxed{F_i = X_i G_i(F)}$

then ...

Oops I forgot !

Computation :

let $\Omega : K^n \to K$.

$$\int d\bar{u}\, du\, \Omega(u)\, e^{-\bar{u}\,\underbrace{(u - XG(u))}} = ?$$

equation satisfied by F $(\ldots = 0)$

$$\bar{u}XG(u) := \sum_{i=1}^{n} \bar{u}_i X_i G_i(u_1, \ldots, u_n)$$

We let $H(u) = u - XG(u) \Rightarrow F = H^{-1}(0)$

change of variable $v = H(u) \Rightarrow u = H^{-1}(v)$

and $du = \det(\partial H^{-1}(v))\, dv$

$$\Rightarrow \int d\bar{u}\, du\, \Omega(u)\, e^{-\bar{u}u + \bar{u}XG(u)} = \int d\bar{u}\, dv\, \Omega(H^{-1}(v)) \det(\partial H^{-1}(v))\, e^{-\bar{u}v}$$

(Rule 3)

$$= \int dv\, \delta(v)\, \Omega(H^{-1}(v)) \det(\partial H^{-1}(v))$$

(Rule 2)

$$= \Omega(H^{-1}(0)) \det(\partial H^{-1}(0))$$

$$= \Omega(F)\, \frac{1}{\det(\partial H(H^{-1}(0)))}$$

or

$$\boxed{\int d\bar{u}\, du\, \Omega(u)\, e^{-\bar{u}u + \bar{u}XG(u)} = \Omega(F)\, \frac{1}{\det(Id - X\partial G(F))}}$$

## Second way to do the computation :

$$\int d\bar{u}du\, \Omega(u)\, e^{-\bar{u}u+\bar{u}XG(u)} = \int d\bar{u}du\, \Omega(u)\, e^{-\bar{u}u} \sum_{M_1,\dots,M_n \geq 0} \frac{1}{M_1!\dots M_n!} (\bar{u}_1 X_1 G_1(u))^{M_1} \dots (\bar{u}_n X_n G_n(u))^{M_n}$$

Multiindex notation

$$= \sum_{M\in\mathbb{N}^n} \frac{X^M}{M!} \int d\bar{u}du\, \underbrace{\bar{u}^M e^{-\bar{u}u}}_{= \left(-\frac{\partial}{\partial u}\right)^M e^{-\bar{u}u}} \Omega(u)\, G(u)^M$$

Integration by parts

$$= \sum_{M\in\mathbb{N}^n} \frac{X^M}{M!} \int d\bar{u}du\, e^{-\bar{u}u} \left(\frac{\partial}{\partial u}\right)^M \left[\Omega(u)\, G(u)^M\right]$$

$$= \sum_{M\in\mathbb{N}^n} \frac{X^M}{M!} \left(\frac{\partial}{\partial u}\right)^M_{u=0} \left[\Omega(u)\, G(u)^M\right]$$

## Lagrange - Good inversion (implicit form)

$$\Omega(F) \cdot \frac{1}{\det\left(\delta_{ij} - X_i\, \partial_j G_i(F)\right)} = \sum_{M\in\mathbb{N}^n} \frac{X^M}{M!} \left(\frac{\partial}{\partial u}\right)^M\Bigg|_{u=0} \Omega(u)\, G(u)^M$$

Remarks:

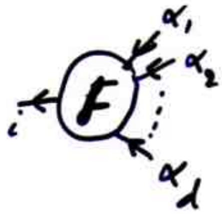
- $$F_i(X) = \frac{\int d\bar{u}du\, u_i\, e^{-\bar{u}u+\bar{u}XG(u)}}{\int d\bar{u}du\, e^{-\bar{u}u+\bar{u}XG(u)}}$$ Combinatorial
- Generalization to $F_i = \sum_{j=1}^{n} X_{ij}\, G_j(F)$ easy.

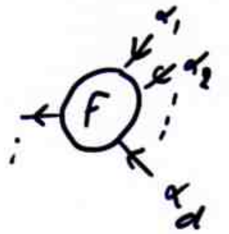
# II Feynman (-Clifford-Sylvester) diagrams:

- Tensor calculus + generating functions

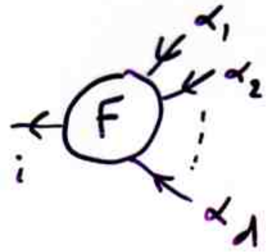
$F : K^n \to K^n$

component of degree $d$ $\in K^n \otimes \mathrm{Sym}^d((K^n)^*)$

$\to$ "matrix" element $f_{i,\underbrace{\alpha_1 \cdots \alpha_d}_{\text{symmetric}}} =$ 

$$F_i(x) = \sum_{d \geq 1} \frac{1}{d!} \sum_{\alpha_1, \ldots, \alpha_d = 1}^{n}$$  $$x_{\alpha_1} \cdots x_{\alpha_d}$$

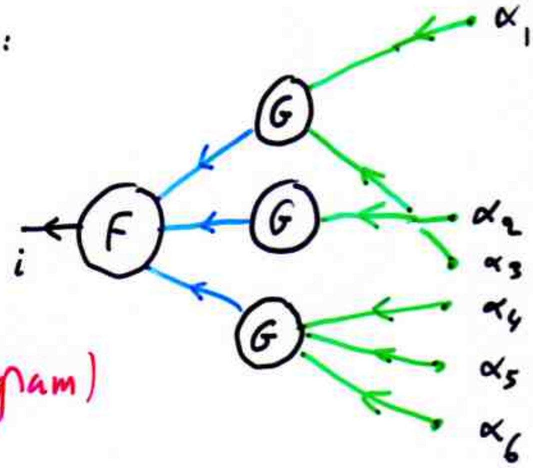
NO MULTIINDICES

"Blob" of F  $= \left. \frac{\partial}{\partial x_{\alpha_1}} \cdots \frac{\partial}{\partial x_{\alpha_d}} \right|_{x=0} F_i(x)$

Knowing F = knowing the Blob of F.

Pb: Given  and  compute 

Example:




(Diagram)

$(i, \alpha_1, \dots, \alpha_6 \in \{1, \dots, n\})$

$$\to \sum_{j_1, j_2, j_3 = 1}^{n} f_{i; j_1 j_2 j_3} \; g_{j_1, \alpha_1 \alpha_3} \, g_{j_2, \alpha_2} \, g_{j_3, \alpha_4 \alpha_5 \alpha_6}$$

(Amplitude)

*) Need:

1) Encoding of the picture:

E finite set (of half lines) ($\#(E) = 20$ above)

Feynman diagram structure on E = information to build the picture (except $i, \alpha_1, \dots, \alpha_6$)

e.g: • specification of the type of half line :

$s$
$\bar{s}$
$t$
$\bar{t}$
$u$
or $\bar{u}$

• labeling of external legs
• blob structure

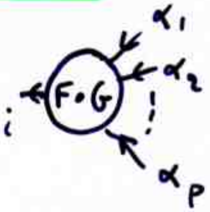
& Contraction

2) A rule associating an amplitude (in K) to a Feynman diagram structure on E.

*) Remark: $E \longrightarrow \mathcal{M}(E) = \{\text{Feyn. structure on } E\}$

is a functor combinatorial species.

*) Result:




$$= \sum_{[E,\mathcal{F}]} \frac{1}{\#(\mathrm{Aut}(E,\mathcal{F}))} \overbrace{\mathcal{A}(E,\mathcal{F}, \underbrace{i, \alpha_1, \dots, \alpha_p}_{\text{indices}})}^{\text{Amplitude}}$$

$\mathcal{F} \in \mathcal{M}(E)$

Classes

indices = external data not included in $E$, $\mathcal{F}$.

$$= \int d\bar{s}\,ds\,d\bar{t}\,dt\,d\bar{u}\,du \;\; s_i \bar{u}_{\alpha_1} \cdots \bar{u}_{\alpha_p} \exp\big(-\bar{s}s - \bar{t}t - \bar{u}u + \bar{s}F(t) + \bar{t}G(u)\big)$$

*) The same idea works for the composition of more than two "functions", Reversion, Lagrange-Good and probably more.

## Conclusion:

* Use Wick's theorem as a <u>definition</u> of formal Gaussian integration.

* Here complex bosonic fields (permanents)

* But also
  - real bosonic fields (hafnians)
  - complex fermionic fields (determinants)
  - real fermionic fields (Pfaffians)

&

mixed situations → supersymmetry

? bijective Berezin change of variable formula

...