On the Hadamard-Foulkes-Howe conjecture and multidimensional resultants

Abdelmalek Abdesselam
Université Paris 13

I - Resultants

II - Feynman diagrams

III - HFH conjecture

I - Resultants : (base field = $\mathbb{C}$)

$x = (x_1, \dots, x_n)$ variables

$F_1(x), \dots, F_n(x)$ n polynomials

$$\begin{cases} F_1 & \text{homogeneous degree } d_1 \\ \vdots & \vdots \\ F_n & \text{" \quad " \quad } d_n \end{cases}$$

$\mathrm{Res}(F_1, \dots, F_n)$ polynomial in coefficients of $F_1, \dots, F_n$

$$\mathrm{Res}(F_1, \dots, F_n) = 0 \iff \left\{ \begin{array}{l} \exists\, (x_1, \dots, x_n) \in \mathbb{C}^n \setminus \{(0, \dots, 0)\}, \\ \begin{cases} F_1(x_1, \dots, x_n) = 0 \\ \vdots \\ F_n(x_1, \dots, x_n) = 0 \end{cases} \end{array} \right.$$

Normalization: $\mathrm{Res}(x_1^{d_1}, \dots, x_n^{d_n}) = 1$

$\mathrm{Res}(F_1, \dots, F_n)$ homog. in F_i of degree $\prod_{j \neq i} d_j$

Invariant Theory:

$g \in GL_n(\mathbb{C})$

$$(gF)(x) = F(g^{-1}x)$$

$$\boxed{\mathrm{Res}(gF_1, \dots, gF_n) = (\det g)^{-w} \, \mathrm{Res}(F_1, \dots, F_n)}$$

$w = \text{weight} = \prod_{i=1}^{n} d_i$ ($\Rightarrow$ Bezout thm.)

Examples:

1) $d_1 = \dots = d_n = 1$: $F_i(x) = a_{i1} x_1 + \dots + a_{in} x_n$

$$\mathrm{Res}(F_1, \dots, F_n) = \det [a_{ij}]_{1 \leq i,j \leq n}$$

linear case

2) $\underline{n=2}$: two binary forms

$$F_1(x_1,x_2) = a_0 x_1^{d_1} + a_1 x_1^{d_1-1} x_2 + \cdots + a_{d_1} x_2^{d_1}$$

$$F_2(x_1,x_2) = b_0 x_1^{d_2} + b_1 x_1^{d_2-1} x_2 + \cdots + b_{d_2} x_2^{d_2}$$

$$\mathrm{Res}(F_1,F_2) = \begin{array}{|cccccc|l} a_0 & a_1 & \cdots & a_{d_1} & & 0 & \\ 0 & a_0 & a_1 & \cdots & a_{d_1} & & \\ & & \ddots & & \ddots & & \left.\right\} d_2 \\ & 0 & & a_0 & \cdots & a_{d_1} & \\ b_0 & \cdots & \cdots & b_{d_2} & & 0 & \\ & \ddots & & & \ddots & & \left.\right\} d_1 \\ 0 & & & b_0 & \cdots & b_{d_2} & \end{array}$$

*) First fundamental theorem of invariant theory for SL_n (Cayley-Clebsch)

$$\Rightarrow \mathrm{Res}(F_1,\dots,F_n) = \sum_G c_G \mathcal{A}_G$$

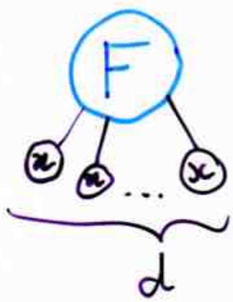
sum over Feynman diagrams

II - Feynman Diagrams :

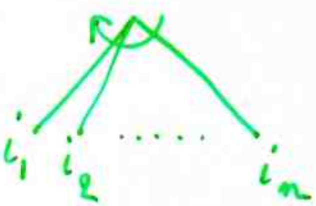
*) vector $x = (x_1, \dots, x_n)$

 $= x_i \qquad , \; 1 \leq i \leq n$

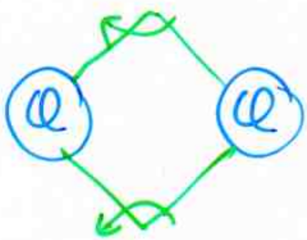
*) F homogeneous polynomial in x of degree d

$F(x) =$  $= \sum_{i_1, \dots, i_d = 1}^{n} F_{i_1 \dots i_d} \, x_{i_1} \cdots x_{i_d}$

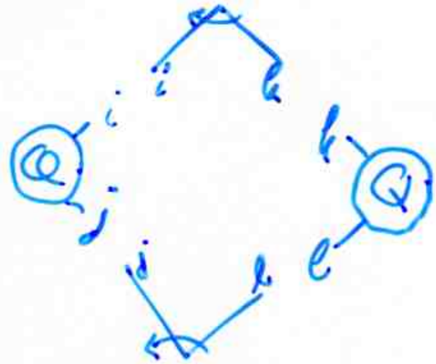
 $= F_{i_1 \dots i_d} = \frac{1}{d!} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_d}} F(x)$

*)  $= \varepsilon_{i_1 \dots i_n} = \begin{cases} * \text{ signature if } i_1 \dots i_n \text{ permutation of } 1\,2 \dots n \\ * \; 0 \text{ otherwise} \end{cases}$

ex 1: $n = 2$, $d = 2$, Q binary quadratic



$$= \sum_{i,j,k,l=1}^{2}$$



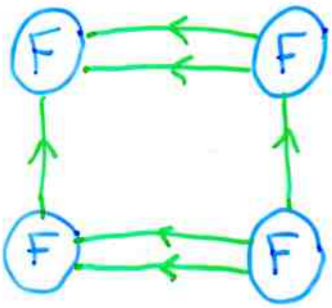
$$= \sum_{i,j,k,l=1}^{2} Q_{ij}\, \varepsilon_{ik}\, \varepsilon_{jl}\, Q_{kl}$$

$$= 2\left(Q_{11} Q_{22} - Q_{12}^{2} \right)$$

$$Q(x) = Q_{11} x_1^2 + 2 Q_{12} x_1 x_2 + Q_{22} x_2^2$$

$\mathcal{A}_G$ amplitude of graph $\sim$ discriminant ex 2: $n=2$, $d=3$, F binary cubic

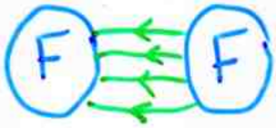
fundamental invariant



$\sim$ discriminant

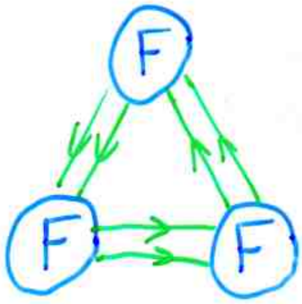
Abbreviation for binary forms: $\leftarrow$ = 

ex 3: $n=2$, $d=4$, F binary quartic

$S = \frac{1}{2}$ 

$= a_0 a_4 - 4 a_1 a_3 + 3 a_2^2$

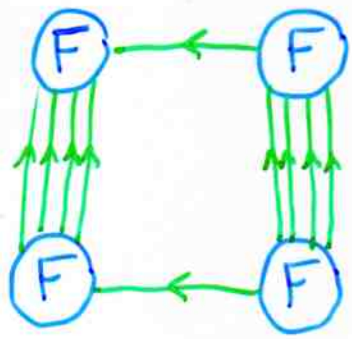
if $F(x_1,x_2) = a_0 x_1^4 + 4 a_1 x_1^3 x_2 + 6 a_2 x_1^2 x_2^2 + 4 a_3 x_1 x_2^3 + a_4 x_2^4$

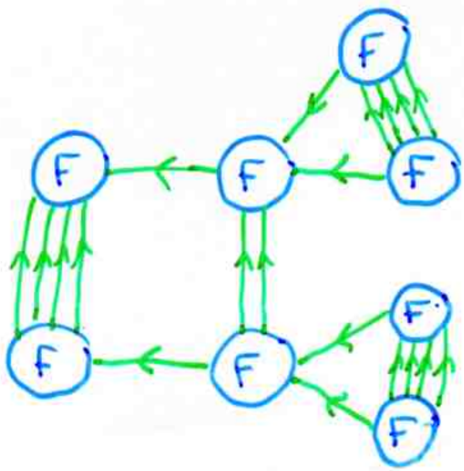
$T = \frac{1}{6}$ 

$= \underbrace{a_0 a_2 a_4}_{} + 2 a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4$

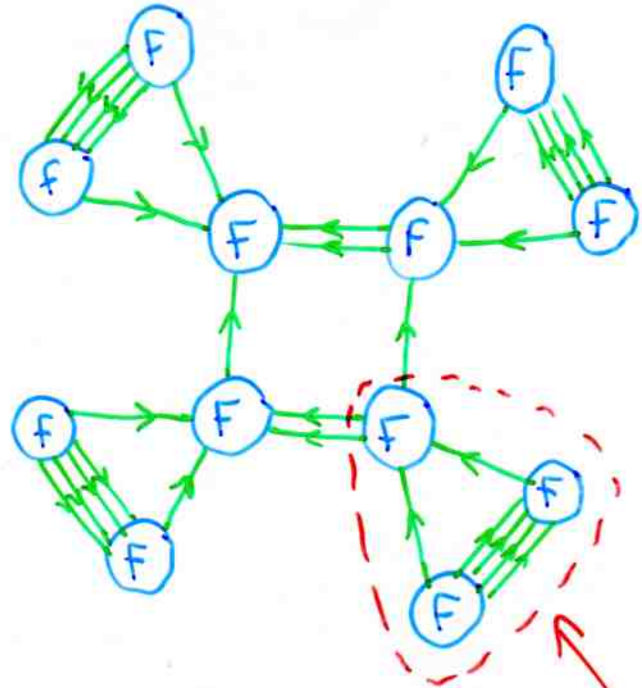
$\boxed{\text{Rk:}}$ sum indices = # $\leftarrow$ = $\boxed{\text{weight } w}$

<u>ex 4:</u> $n=2$, $d=5$, F binary quintic

$J = -\frac{1}{2}$ 

$K = \frac{1}{8}$ 

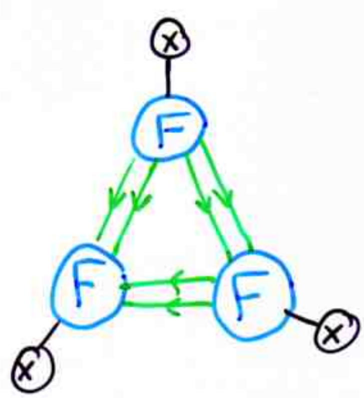
$$L = \frac{1}{96}$$



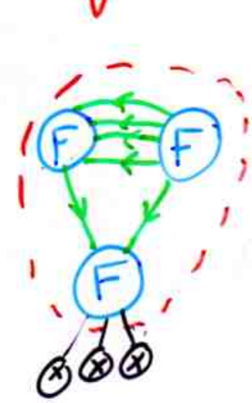
$\boxed{\text{grafting :}}$

$$L \sim \mathrm{Disc}(\underbrace{\mathrm{Can}(F)}_{\text{cubic } C})$$

$C(x) =$



$= -$



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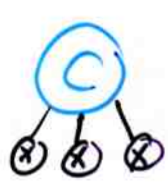
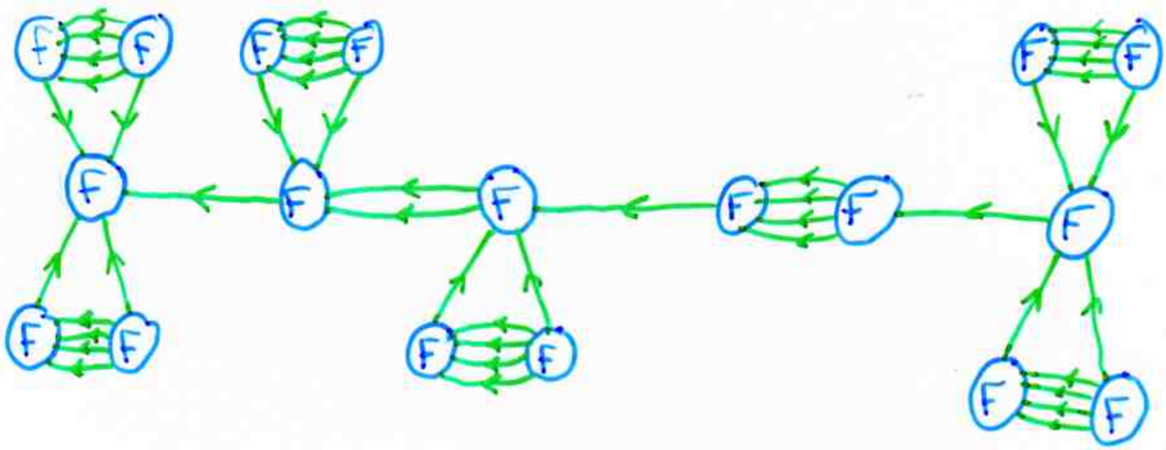


$$H = -\frac{1}{384} \times$$



$\boxed{Rk:}$ If $F(x) = u x_1^5 + v x_2^5 - w (x_1+x_2)^5$

$$J(F) = (uv+uw+vw)^2 - 4uvw(u+v+w)$$

$$K(F) = u^2v^2w^2(uv+uw+vw)$$

$$L(F) = u^4v^4w^4$$

$$H(F) = u^5v^5w^5(u-v)(u-w)(v-w)$$

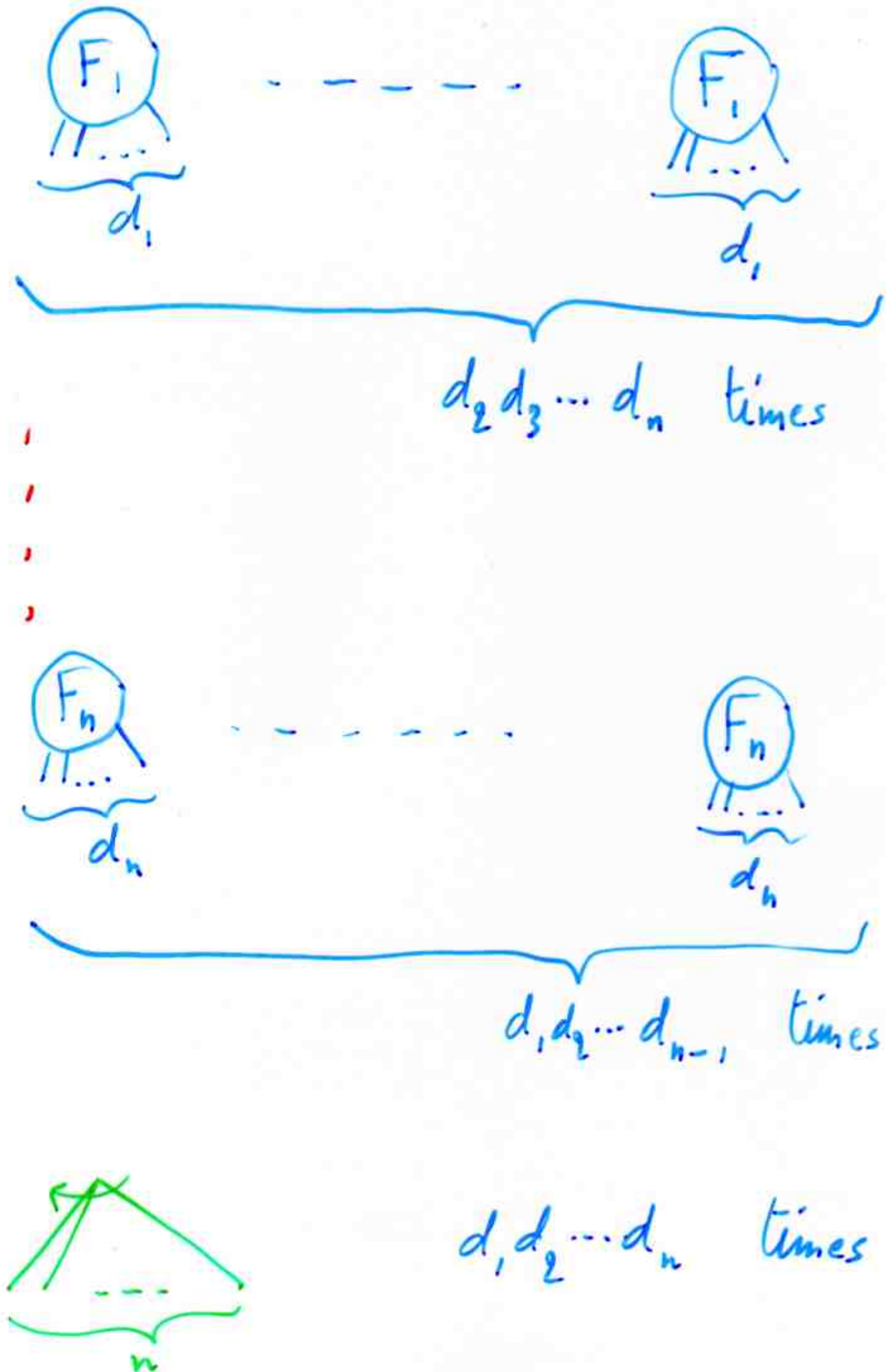
Resultants: for format $(n; d_1, \ldots, d_n)$

$$\mathrm{Res}(F_1, \ldots, F_n) = \sum_{\text{graph } G} c_G \, \mathcal{A}_G$$

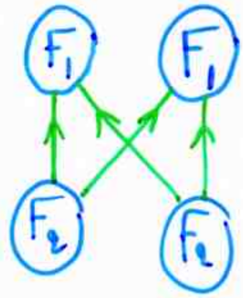
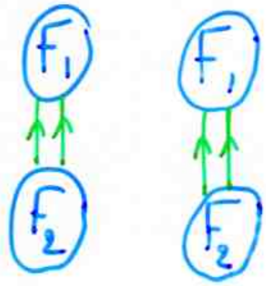
c_G: numerical coefficient

$\mathcal{A}_G$: amplitude

G made of:

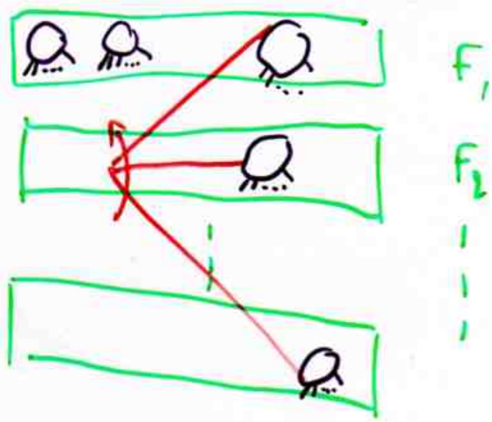


Ex: F_1, F_2 binary quadratics

$\mathrm{Res}(F_1, F_2) = 2$  $-$ 

Hadamard Form:

if $\forall\, G$ in formula for $\mathrm{Res}(F_1, \dots, F_n)$

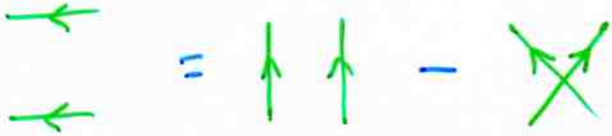
$\forall$  $\rightsquigarrow$ 

Pb: For which formats $\exists$ Hadamard form for Resultant

???

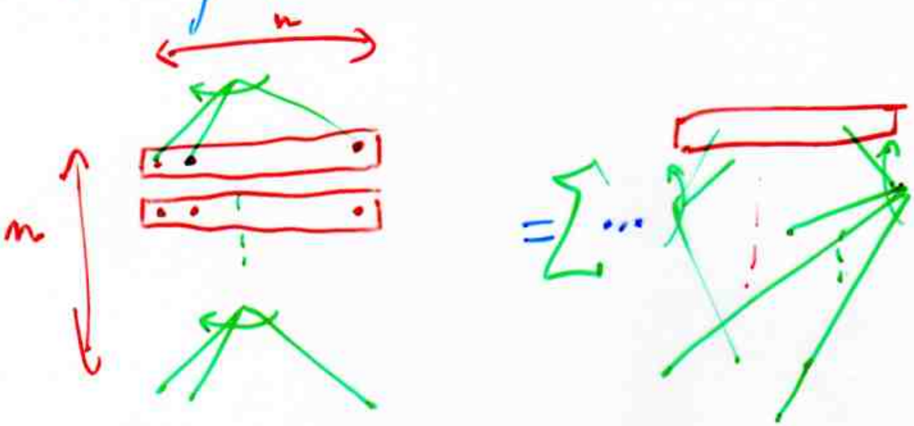
*) For $n=2$: Yes $\forall\, d_1, d_2$

<u>Proof</u>:



*) Same argument for $n>2$

→ Toy pb



$\rightsquigarrow$ Alon-Tarsi cj

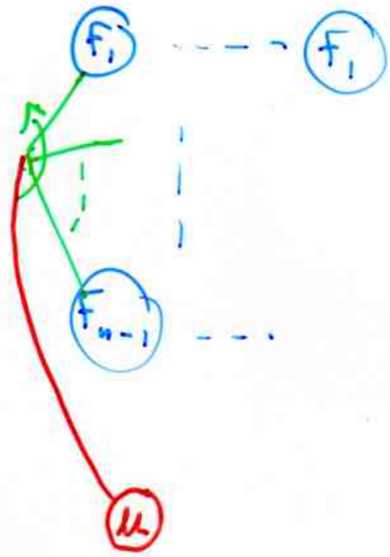
&

Rota basis cj

Conjectural construction of Res (Hadamard 1896)

$\underbrace{F_1, \dots, F_{n-1}}, F_n$

$$\mathrm{Res}_{d_1,\dots,d_{n-1}} = \sum_{\text{graphs } \gamma} c_\gamma$$



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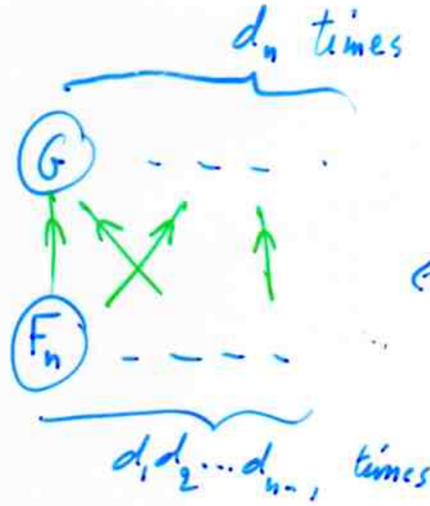
$$\mathrm{Res}_{d_1 d_2 \cdots d_{n-1}, d_n} = \sum_{\Gamma} \mathcal{C}_\Gamma$$



← in Hadamard form

$$\mathrm{Res}_{d_1,\dots,d_n} \overset{???}{=} \sum_{\Gamma, \gamma_1, \dots, \gamma_{d_n}} \mathcal{C}_\Gamma \, c_{\gamma_1} \cdots c_{\gamma_{d_n}} \left\langle \underbrace{\gamma_1, \dots, \gamma_{d_n}}_{\text{inserted inside the } G\text{'s of } \Gamma} \to \Gamma \right\rangle$$

$\boxed{\text{Rk:}}$ Makes sense also for

$$\mathrm{Res}(d_1,\dots,d_{\alpha_1})$$

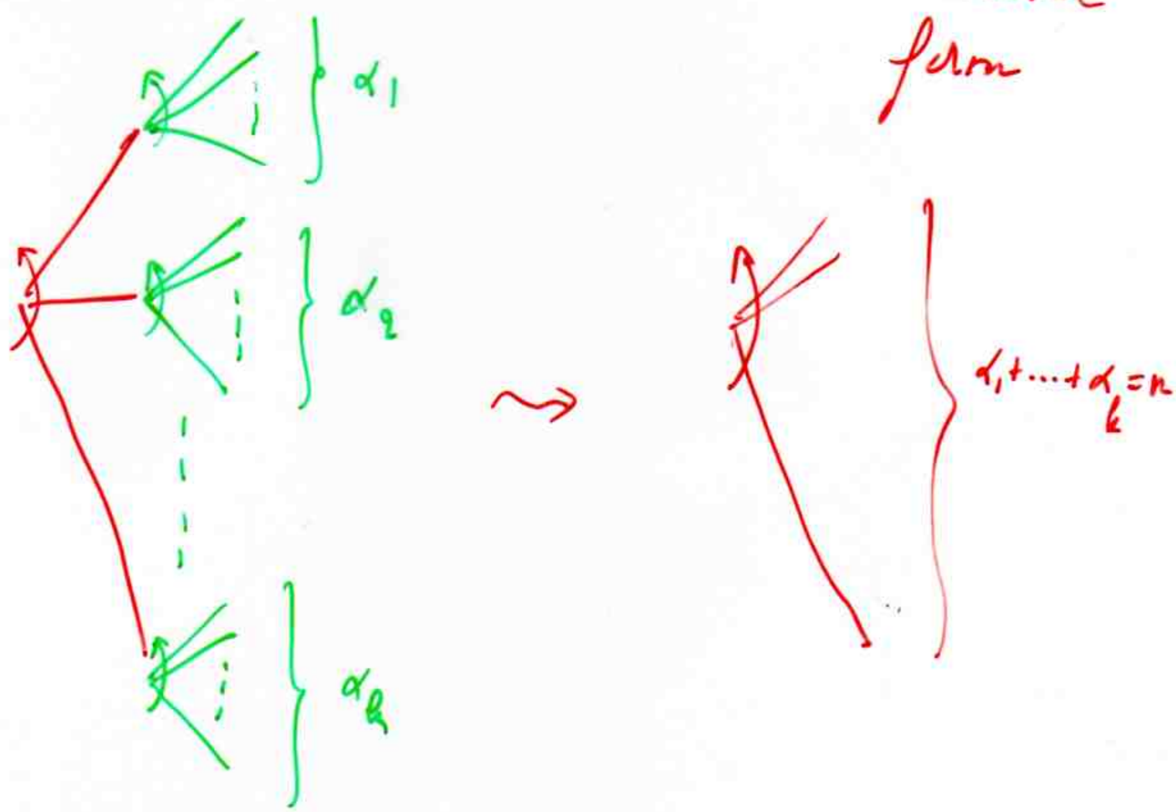
$$\mathrm{Res}(d_{\alpha_1+1},\dots,d_{\alpha_1+\alpha_2})$$

$$\vdots$$

$$\mathrm{Res}(d_{\alpha_1+\dots+\alpha_{k-1}+1},\dots,d_{\alpha_1+\dots+\alpha_k})$$

& $\mathrm{Res}(d_1\cdots d_{\alpha_1},\dots,d_{\alpha_1+\dots+\alpha_{k-1}+1}\cdots d_{\alpha_1+\dots+\alpha_k})$

if known in Hadamard form



Hadamard's construction $\longrightarrow$ HFH conj.

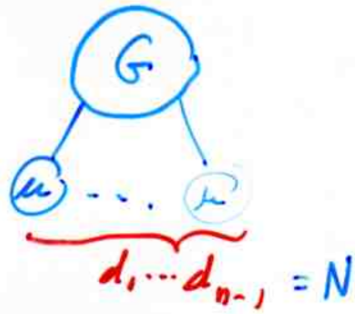
III The Hadamard-Foulkes-Howe conjecture:

Why? Poisson product formula

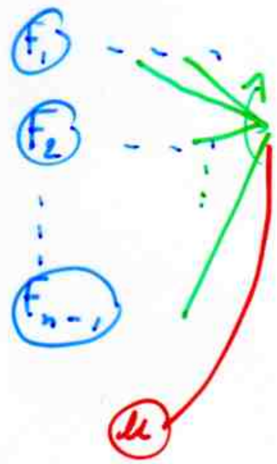
$F_1,\dots,F_n$ $\quad(d_1,\dots,d_n)\quad$ in n variables

u linear form, $\quad u(x)=u_1x_1+\dots+u_nx_n$

$$\mathrm{Res}(F_1,\dots,F_{n-1},u) = G(u) =$$



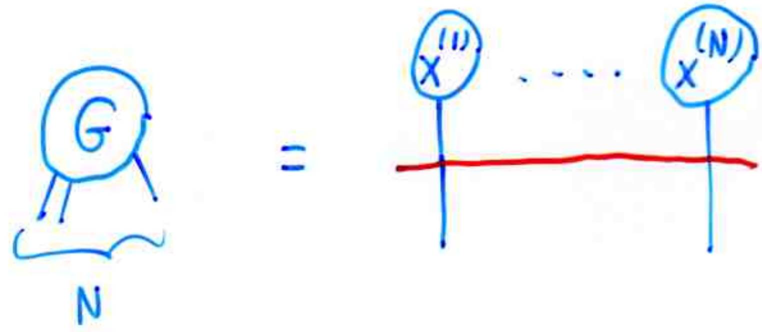
$$\overset{\text{Thm}}{=} \sum_{\substack{G \text{ format} \\ (d_1,\dots,d_{n-1})}} c_G$$



$$= \prod_{i=1}^{N} u(x^{(i)})$$

$x^{(i)}=(x_1^{(i)},\dots,x_n^{(i)})\quad,\quad 1\le i\le N$

hom. coords. N intersection pts of $F_1=0,\dots,F_{n-1}=0$ in $\mathbb{P}^{n-1}$

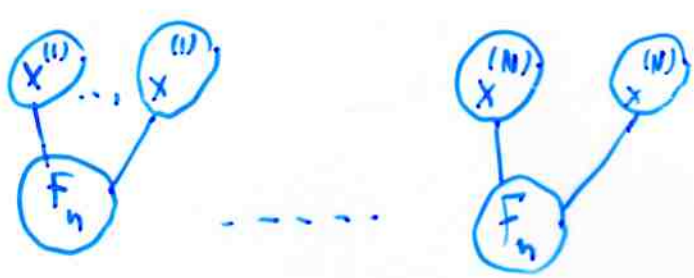


symmetrizer

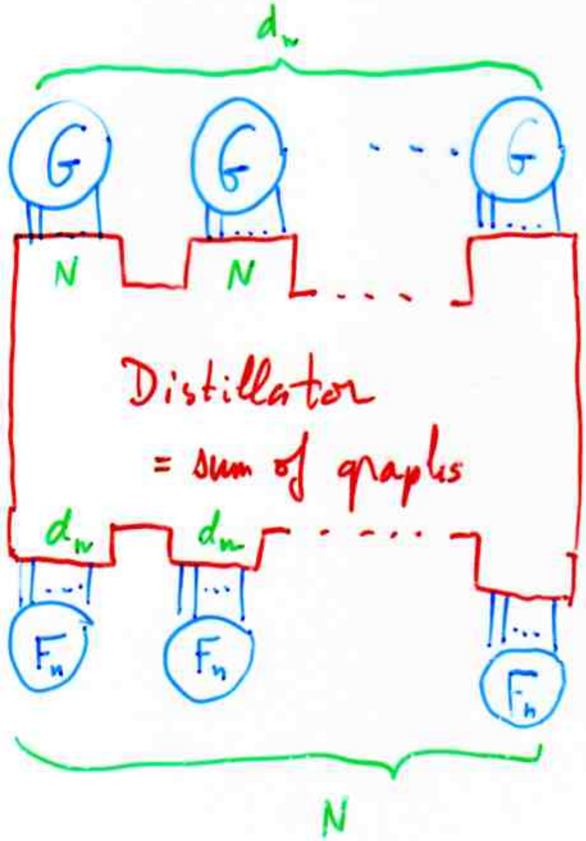
$\frac{1}{N!}\sum_{\sigma \in \mathfrak{S}_N}$ σ

$$\boxed{\mathrm{Res}(F_1,\ldots,F_n) = F_n(x^{(1)})\cdots F_n(x^{(N)})}$$

$=$

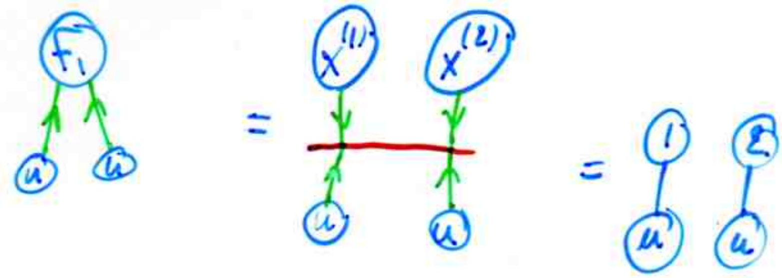


$\overset{?}{=}$

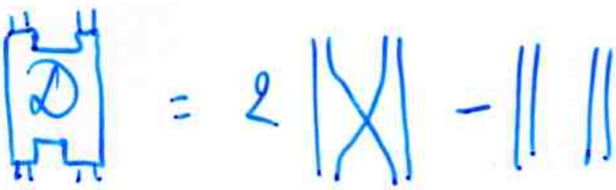


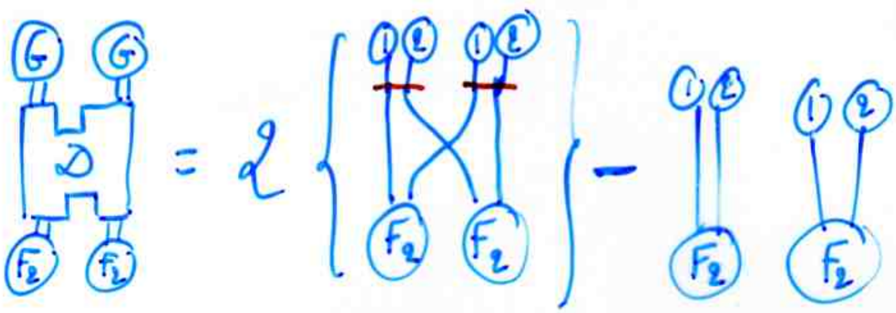
ex: $n=2$, $d_1=d_2=2$.

$= \mathrm{Res}[F_1, u] =$

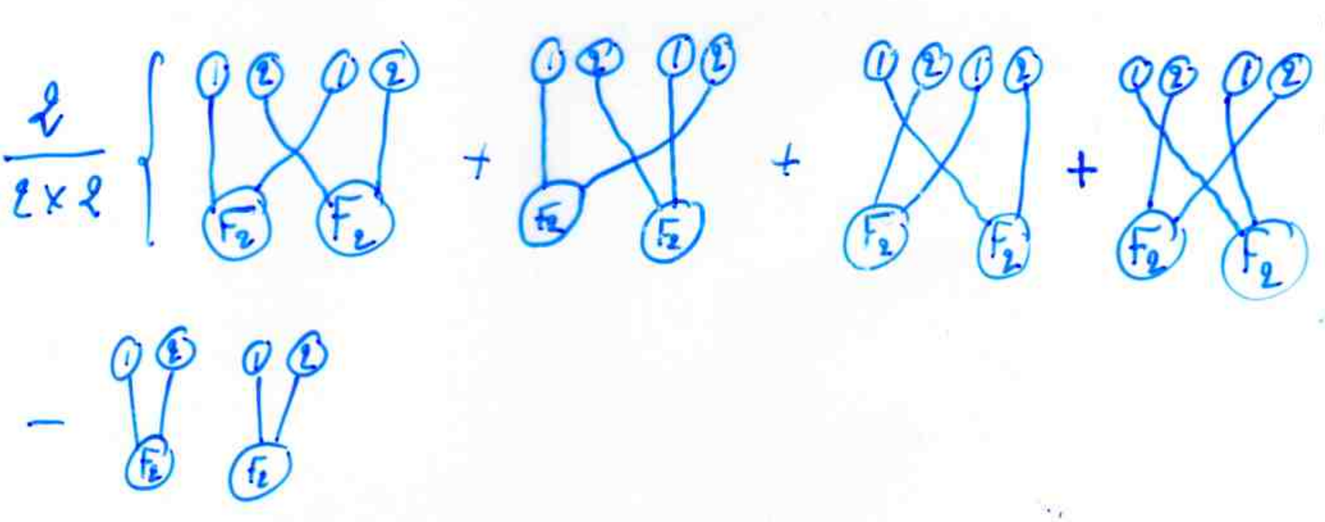


let





$= \dfrac{2}{2\times 2}$



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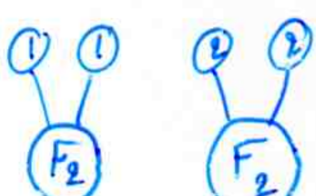


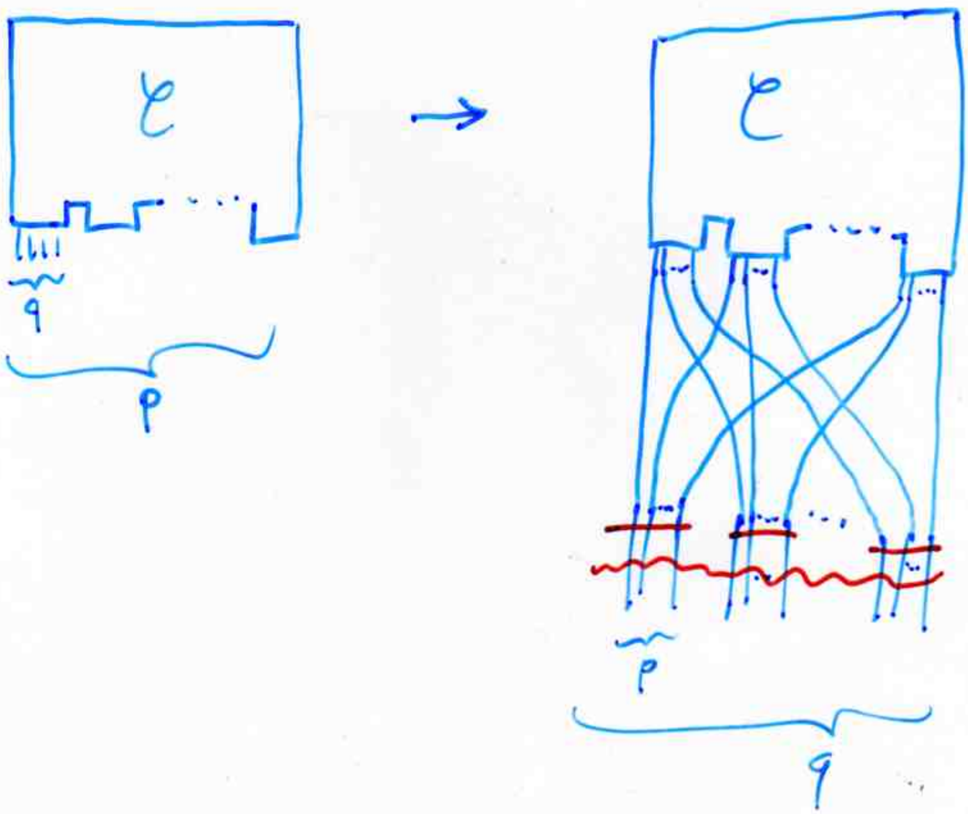
$= F_2(x^{(1)})\, F_2(x^{(2)})$.

$$\exists \mathcal{D} \Leftrightarrow S_{d_n}(S_N(\mathbb{C}^n)) \xrightarrow[\mathcal{H}_{d_n,N,n}]{} S_N(S_{d_n}(\mathbb{C}^n))$$

is <u>surjective</u>

In general :

$$\mathcal{H}_{p,q,n} : S_p(S_q(\mathbb{C}^n)) \xrightarrow{GL_n\text{-equivariant}} S_q(S_p(\mathbb{C}^n))$$



HFH conj :

$$\boxed{\begin{array}{l} p \geq q \Rightarrow \mathcal{H}_{p,q,n} \text{ surjective} \\ p \leq q \Rightarrow \mathcal{H}_{p,q,n} \text{ injective} \end{array}}$$