Pointwise multiplication of random Schwartz distributions with Wilson's operator product expansion

Abdelmalek Abdesselam
Mathematics Department, University of Virginia

Newton Institute, Cambridge, UK
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In QFT textbooks, one finds correlation functions given by Euclidean path integrals

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{\mathcal{Z}}\int \phi(x_1)\cdots\phi(x_n)\ e^{-S(\phi)}\ D\phi$$

where the integration variable is a "function" $\phi : \mathbb{R}^d \to \mathbb{R}$, $S(\phi)$ is a functional:

$$S(\phi) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2}(\partial\phi)^2(x) + \frac{1}{2}m^2\phi^2(x) + g\phi^4(x) \right\} d^d x$$

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One also finds correlations $\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle$ involving composite operators $\mathcal{O}(x)$ such as $\phi^2(x)$, $\phi\partial_\mu\partial_\nu\phi(x)$, etc.

Fundamental questions:

Should one take seriously the probabilistic interpretation of the elementary field ϕ as a random (possibly generalized) function? Does the same hold for composite fields $\mathcal{O}$? Are they given by deterministic local functionals of the elementary field?

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(AA2016) "A second-quantized Kolmogorov-Chentsov theorem", arXiv:1604.05259[math.PR]. The subject of this talk.

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Gaussian field example: Let C denote the continuous bilinear form on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ defined by

$$C(f,g) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \ \frac{\overline{\widehat{f}(\xi)}\widehat{g}(\xi)}{|\xi|^{d-2[\phi]}}$$

where $[\phi] \in (0,\infty)$ is a parameter called the scaling dimension of the field ϕ.

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By the Bochner-Minlos Theorem, there is a unique probability measure $\mathbb{P}$ on $\mathcal{S}'(\mathbb{R}^d)$ such that

$$\mathbb{E}\ e^{i\phi(f)} = \exp\left(-\frac{1}{2}C(f,f)\right)$$

for all test functions $f \in \mathcal{S}(\mathbb{R}^d)$.

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$$(\phi * \rho_r)(x) = \langle \phi(y), \rho_r(x-y) \rangle_y \ .$$

The result is a function in $\mathcal{O}_{\mathrm{M},x}(\mathbb{R}^d)$.

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Naive attempt

$$\phi^2(f) = \lim_{r \to -\infty} \int_{\mathbb{R}^d} dx\ \left[(\phi * \rho_r)(x)\right]^2 f(x)$$

does not work but easy to fix using Wick ordering.

The correct pointwise square $:\phi^2:$ is given by

$$:\phi^2:(f) = \lim_{r\to-\infty} \int_{\mathbb{R}^d} dx \ \left([(\phi * \rho_r)(x)]^2 - \mathbb{E}\,[(\phi * \rho_r)(x)]^2 \right) f(x)$$

with convergence in every $L^p(\mathcal{S}'(\mathbb{R}^d), \mathrm{Borel}, \mathbb{P})$, $p \geq 1$, and almost surely, provided $0 < [\phi] < \frac{d}{4}$, i.e., pointwise correlations of fields involved are $L^{1,\mathrm{loc}}$.

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Technically this is a generalized stochastic process, but one can make this a measurable map $\Omega = \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $\phi \mapsto :\phi^2:$, which produces the stronger notion of random distribution.

When $[\phi] \in \left(0, \frac{d}{2}\right)$, then one has a pointwise representation for the covariance C, i.e.,

$$C(f,g) = \int_{\mathbb{R}^{2d}} dx\ dy\ \langle \phi(x)\phi(y) \rangle\ f(x)g(y)$$

where the pointwise correlation $\langle \phi(x)\phi(y) \rangle$ is defined outside the diagonal by

$$\langle \phi(x)\phi(y) \rangle = \frac{\varkappa}{|x-y|^{2[\phi]}}$$

with

$$\varkappa = \pi^{\frac{d}{2}} \times 2^{2[\phi]} \times \frac{\Gamma\left([\phi]\right)}{\Gamma\left(\frac{d}{2} - [\phi]\right)} .$$

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For $f_1, \ldots, f_4 \in \mathcal{S}(\mathbb{R}^d)$, then one also has a pointwise representation for say the fourth moment

$$\mathbb{E}\left[\phi(f_1)\phi(f_2)\phi(f_3)\phi(f_4)\right] = C(f_1,f_2)C(f_3,f_4) + C(f_1,f_3)C(f_2,f_4) + C(f_1,f_4)C(f_2,f_3)$$

Namely,

$$\mathbb{E}\left[\phi(f_1)\phi(f_2)\phi(f_3)\phi(f_4)\right] = \int_{\mathbb{R}^{4d}} dx_1\ dx_2\ dx_3\ dx_4$$
$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle\ f_1(x_1)f_2(x_2)f_3(x_3)f_4(x_4)$$

featuring the pointwise correlation

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Pointwise correlations are seen as ordinary functions on the open subset Conf_n of $\mathbb{R}^{nd}$ where the points $x_1,\ldots,x_n\in\mathbb{R}^d$ are distinct.

Define (again at non-coinciding points) the new function

$$\langle :\phi^2:(x_1)\phi(x_2)\phi(x_3)\rangle = \frac{2\varkappa^2}{|x_1-x_2|^{2[\phi]}|x_1-x_3|^{2[\phi]}} \ .$$

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Then one has the asymptotic behavior

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \\ \langle \phi(x_1)\phi(x_2)\rangle\langle \phi(x_3)\phi(x_4)\rangle + \langle :\phi^2:(x_2)\phi(x_3)\phi(x_4)\rangle + o\,(1)$$

when $x_1 \to x_2$ while the three points x_2, x_3, and x_4 are fixed.

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when $x_1 \to x_2$ while the three points x_2, x_3, and x_4 are fixed. This is the simplest instance of Wilson's Operator Product Expansion which says that, "inside correlations", one has

$$\phi(x_1)\phi(x_2) = \mathcal{C}_{\phi\phi}^{\mathbb{1}}(x_1,x_2)\times 1 + \mathcal{C}_{\phi\phi}^{\phi^2}(x_1,x_2) :\phi^2:(x_2) + o(1)$$

with OPE coefficients $\mathcal{C}_{\phi\phi}^{\mathbb{1}}(x_1,x_2) = \frac{\varkappa}{|x_1-x_2|^{2[\phi]}}$ and $\mathcal{C}_{\phi\phi}^{\phi^2}(x_1,x_2) = 1$.

Our theorem shows how in general such an OPE, with precise bounds on the remainder, allows one to establish convergence in L^p and almost surely for suitably renormalized products of random distributions.

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We do this in a general setting which can handle Gaussian and non-Gaussian measures, massive and massless fields, anomalous scaling dimensions, logarithmic corrections, finite degeneracy in the dimension spectrum, as well as lack of translation invariance (e.g., for SPDEs).

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But first we need some notation and terminology.

Assume that we have a collection $(\mathcal{O}_A)_{A\in\mathcal{B}}$ of $\mathcal{S}'(\mathbb{R}^d)$-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
Suppose they have moments of all orders, i.e., for all test function $f \in \mathcal{S}(\mathbb{R}^d)$, all $A \in \mathcal{B}$ and all $p \geq 1$, we have that the real-valued random variable $\mathcal{O}_A(f) \in L^p(\Omega, \mathcal{F}, \mathbb{P})$.

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We also define translates of the rescaled mollifier $\rho_{r,x}(y) = \rho_r(y - x)$ for each point $x \in \mathbb{R}^d$.

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The moments of the given random variables such as

$$\mathbb{E}\left[\mathcal{O}_{A_1}(f_1)\cdots\mathcal{O}_{A_n}(f_n)\right]$$

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can be seen as continuous n-linear forms on Schwartz space and also, via the nuclear theorem, as elements of $\mathcal{S}'(\mathbb{R}^{dn})$.The pointwise correlations or moments are defined by the limit

$$\langle\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_n}(x_n)\rangle = \lim_{r\to-\infty}\mathbb{E}\left[\mathcal{O}_{A_1}(\rho_{r,x_1})\cdots\mathcal{O}_{A_n}(\rho_{r,x_n})\right]$$

if it exists.

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Furthermore, we require that moments are given by integration against such pointwise correlations. More precisely, this includes the local integrability condition

$$\int_{K^n \cap \mathrm{Conf}_n} dx_1 \dots dx_n \, |\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_n}(x_n) \rangle| \; < \infty$$

for every compact $K \subset \mathbb{R}^d$, as well as the condition

$$\mathbb{E}\left[\mathcal{O}_{A_1}(f_1) \cdots \mathcal{O}_{A_n}(f_n)\right] = \int_{\mathrm{Conf}_n} dx_1 \dots dx_n \; \langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_n}(x_n) \rangle \; f_1(x_1) \cdots f_n(x_n)$$

for all $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R}^d)$.

Given these hypotheses it is trivial to define more complicated pointwise correlations using formal multilinear expansion. For instance if $f(x, y)$ is a function on Conf_2, then

$$\langle\ (\mathcal{O}_A(x)\mathcal{O}_B(y) - f(x,y)\mathcal{O}_C(y))\ (\mathcal{O}_D(z) - \mathcal{O}_E(u))\ \mathcal{O}_F(v)\ \rangle$$

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is to be understood as

$$\begin{aligned} & \langle\mathcal{O}_A(x)\mathcal{O}_B(y)\mathcal{O}_D(z)\mathcal{O}_F(v)\rangle \\ - & \langle\mathcal{O}_A(x)\mathcal{O}_B(y)\mathcal{O}_E(u)\mathcal{O}_F(v)\rangle \\ - & f(x,y)\,\langle\mathcal{O}_C(y)\mathcal{O}_D(z)\mathcal{O}_F(v)\rangle \\ + & f(x,y)\,\langle\mathcal{O}_C(y)\mathcal{O}_E(u)\mathcal{O}_F(v)\rangle \end{aligned}$$

which is a well defined function of $(x, y, z, u, v) \in \mathrm{Conf}_5$.

We also assume that for each field $\mathcal{O}_A$ we are given a number called the scaling dimension $[A]$ which governs the short distance singularities on the big diagonal. For instance, we are requiring that covariance kernels $\langle \mathcal{O}_A(x)\mathcal{O}_A(y)\rangle$ are bounded by $|x-y|^{-2[A]}$ (modulo eventual logarithmic corrections) for $|x-y|$ small.

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We say that an abstract system of pointwise correlations (eventually, with fields indexed by a set $\mathcal{A}$ containing $\mathcal{B}$) satisfies Wilson's operator product expansion if there exists smooth functions $\mathcal{C}_{A,B}^{C}(x,y)$ on Conf_2 such that one has "inside correlations" an expansion of the form

$$\mathcal{O}_A(x)\mathcal{O}_B(y) = \sum_{[C]\le\Delta} \mathcal{C}_{A,B}^{C}(x,y)\mathcal{O}_C(y) + o(|x-y|^{\Delta-[A]-[B]})$$

for given cutoff Δ on scaling dimensions.

Key hypothesis: $\exists\eta>0$, $\exists\gamma>0$, $\forall\epsilon>0$, $\exists k\in\mathbb{N}$, $\exists K>0$,

$$\left|\left\langle\prod_{i=1}^{m}\mathrm{OPE}_i(y_i,x_i)\prod_{i=m+1}^{m+n}\mathrm{CZ}_i(y_i,x_i)\prod_{i=m+n+1}^{m+n+p}\mathcal{O}_{B_i}(x_i)\right\rangle\right|\times$$

$$\prod_{i=1}^{m+n}\mathbb{1}\left\{|y_i-x_i|\le\eta\min_{j\neq i}|x_i-x_j|\right\}\le K\prod_{i=1}^{m+n+p}\langle x_i\rangle^k\times\prod_{i=1}^{m+n}\langle y_i\rangle^k$$

$$\times\prod_{i=1}^{m}\left\{|y_i-x_i|^{\Delta_i+\gamma-[A_i]-[B_i]}\times\left(\min_{j\neq i}|x_i-x_j|\right)^{-\Delta_i-\gamma-\epsilon}\right\}$$

$$\times\prod_{i=m+1}^{m+n}\left\{|y_i-x_i|^{\gamma}\times\left(\min_{j\neq i}|x_i-x_j|\right)^{-[B_i]-\gamma-\epsilon}\right\}$$

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We used the notation $\langle x\rangle = \sqrt{1+|x|^2}$, as well as "$\mathrm{OPE}$" for objects of the form

$$\mathrm{OPE}_i(y_i,x_i) = \mathcal{O}_{A_i}(y_i)\mathcal{O}_{B_i}(x_i) - \sum_{[C_i]\le\Delta_i} \mathcal{C}_{A_i,B_i}^{C_i}(y_i,x_i)\mathcal{O}_{C_i}(x_i)\ ,$$

and "CZ" for objects of the form

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and "CZ" for objects of the form

$$\mathrm{CZ}_i(y_i, x_i) = \mathcal{O}_{B_i}(y_i) - \mathcal{O}_{B_i}(x_i) \ .$$

A special case of the ENNFB is the Basic Nearest Neighbor Factorized Bound (BNNFB):

$$\left|\langle \mathcal{O}_{B_1}(x_1)\cdots\mathcal{O}_{B_p}(x_p)\rangle\right| \le K \prod_{i=1}^{p} \langle x_i\rangle^k \times \prod_{i=1}^{p} \left(\min_{j\ne i} |x_i - x_j|\right)^{-[B_i]-\epsilon} .$$

The third needed hypothesis is a mild condition on the kernels $\mathcal{C}_{A,B}^{C}(x,y)$ which means that the corresponding distribution in $\mathcal{S}'_{x,y}(\mathbb{R}^{2d}) = \mathcal{S}'_x(\mathbb{R}^d) \widehat{\otimes} \mathcal{S}'_y(\mathbb{R}^d)$ in fact belongs to the smaller space $\mathcal{S}'(\mathbb{R}^d)_x \widehat{\otimes} \mathcal{O}_{M,y}(\mathbb{R}^d)$, together with a bound of the form $|x-y|^{[C]-[A]-[B]-\epsilon}$ near the diagonal. The ϵ is for eventual logarithmic corrections.

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Suppose we have a system of abstract pointwise correlations indexed by $\mathcal{A} = \mathcal{B} \cup \{C_*\}$ satisfying the previous hypotheses and a pair $A, B \in \mathcal{B}$ such that $\mathcal{C}_{A,B}^{C_*}(x,y)$ is nonzero and obeys a lower bound of the form $|x-y|^{[C_*]-[A]-[B]}$.

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Our main theorem is a construction of the a priori "virtual" field $\mathcal{O}_{C_*}$ as a Borel measurable functional of the already existing fields $(\mathcal{O}_C)_{C\in\mathcal{B}}$ on our probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Essentially, we define it as a renormalized product of the fields $\mathcal{O}_A$ and $\mathcal{O}_B$, intuitively given by the formula

$$\mathcal{O}_{C_*}(x) = \lim_{y \to x} \frac{1}{\mathcal{C}_{A,B}^{C_*}(y,x)} \left(\mathcal{O}_A(y)\mathcal{O}_B(x) - \sum_{[C] \leq [C_*], C \neq C_*} \mathcal{C}_{A,B}^{C}(y,x)\mathcal{O}_C(x) \right) .$$

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Of course, this is to be understood in the sense of distributions and needs proper smearing with the rescaled mollifier ρ_r. We also require the scaling dimensions of all the fields to belong to the interval $\left[0, \frac{d}{2}\right)$.

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Consider a critical ferromagnetic spin field $(\sigma_{\mathbf{x}})_{\mathbf{x}\in\mathbb{Z}^d}$, of Ising or ϕ^4 type. The formal Hamiltonian is of the usual form

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Form the pair of random distributions (ϕ_r, ϕ_r^2):

$$\phi_r = L^{r(d-[\phi])}\sum_{\mathbf{x}\in\mathbb{Z}^d} \sigma_{\mathbf{x}}\ \delta_{L^r\mathbf{x}} \ ,$$

$$\phi_r^2 = L^{r(d-[\phi^2])}\sum_{\mathbf{x}\in\mathbb{Z}^d} (\sigma_{\mathbf{x}}^2 - \langle\sigma_{\mathbf{x}}^2\rangle)\ \delta_{L^r\mathbf{x}} \ .$$

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In many cases, one expects the pair of random distributions to converge in joint law when $r \to -\infty$ to a pair of continuum fields (ϕ, ϕ^2).

This gives rise to a system of pointwise correlations corresponding to $(\mathcal{O}_A)_{A\in\mathcal{A}}$ with $\mathcal{A} = \{\mathbb{1}, \phi, \phi^2\}$.
This system should satisfy the OPE

$$\phi(x_1)\phi(x_2) = \mathcal{C}^{\mathbb{1}}_{\phi\phi}(x_1, x_2) \times 1 + \mathcal{C}^{\phi^2}_{\phi\phi}(x_1, x_2)\phi^2(x_2) + \dots$$

with OPE coefficients

$$\mathcal{C}^{\mathbb{1}}_{\phi\phi}(x_1, x_2) = \frac{c^{\mathbb{1}}_{\phi\phi}}{|x_1 - x_2|^{2[\phi]}} \text{ and } \mathcal{C}^{\phi^2}_{\phi\phi}(x_1, x_2) = c^{\phi^2}_{\phi\phi}|x_1 - x_2|^{[\phi^2]-2[\phi]}$$

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The scaling dimensions should be given, already at the lattice level, by the long distance asymptotics

$$\langle\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}\rangle \approx \frac{1}{d(\mathbf{x}, \mathbf{y})^{2[\phi]}} \text{ and } \langle\sigma^2_{\mathbf{x}}, \sigma^2_{\mathbf{y}}\rangle^T \approx \frac{1}{d(\mathbf{x}, \mathbf{y})^{2[\phi^2]}} \ .$$

(again for Ising $\sigma^2_{\mathbf{x}} \longrightarrow \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}}$).

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4) Hierarchical models. $d(\mathbf{x}, \mathbf{y}) =$ hierarchical distance on $\mathbb{Z}^d$ yet still with

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Euclidean 2D SR Ising: Scaling limit ϕ with $[\phi] = \frac{1}{8}$ constructed and shown to be a conformal field theory. Dubedat (arXiv 2011), Chelkak, Hongler and Izyurov (AM 2015), Camia, Garban and Newman (AP 2015).

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$$[\phi^2] - 2[\phi] = 0.376327\ldots$$

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Hierarchical 3D phi-four: A.A., Chandra and Guadagni (arXiv 2013) showed that also with $d = 3$, $\sigma = \frac{3+\epsilon}{2}$

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where $[\phi] = [\phi]_{\mathrm{Gauss}} = \frac{3-\epsilon}{4}$ (this part was known, e.g., Gawędzki-Kupiainen JSP 1984) and $[\phi^2] > 2[\phi]$ (this is new).

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A.A. in progress: derivation from first principles of OPE
$\phi \times \phi = \mathbb{1} + \phi^2 + \cdots$ (fusion rule notation).

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ENNFB: $\exists\eta>0$, $\exists\gamma>0$, $\forall\epsilon>0$, $\exists k\in\mathbb{N}$, $\exists K>0$, $\forall$ points,

$$\left|\left\langle\prod_{i=1}^{m}\mathrm{OPE}_i(y_i,x_i)\prod_{i=m+1}^{m+n}\mathrm{CZ}_i(y_i,x_i)\prod_{i=m+n+1}^{m+n+p}\mathcal{O}_{B_i}(x_i)\right\rangle\right|\times$$

$$\prod_{i=1}^{m+n}\mathbb{1}\left\{|y_i-x_i|\le\eta\min_{j\neq i}|x_i-x_j|\right\}\le K\prod_{i=1}^{m+n+p}\langle x_i\rangle^k\times\prod_{i=1}^{m+n}\langle y_i\rangle^k$$

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For 2d Ising, the pure ϕ BNNFB (with $\epsilon = 0$, $k = 0$)

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The fractional free field on $\mathbb{R}^d$ with $[\phi] > 0$. Covariance:

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Equivalence classes denoted by

$$A = [\nu(1), \ldots, \nu(r)] \longleftrightarrow \mathcal{O}_A(x) =: \partial^{\nu(1)}\phi(x) \cdots \partial^{\nu(r)}\phi(x) :$$

An actual example:

The fractional free field on $\mathbb{R}^d$ with $[\phi] > 0$. Covariance:

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More generally for any finite set F and function $\nu : F \to \mathbb{N}^d$ we associate a field label $A = [\nu(a_1), \ldots, \nu(a_r)] = \{F; \nu\}$.

Abstract system of pointwise correlations given by

$$\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_n}(x_n)\rangle = \sum_{\mathcal{W}} \prod_{\{a,b\}\in\mathcal{W}} C^{\nu(a),\nu(b)}(x_{\iota(a)}, x_{\iota(b)})$$

with $C^{\alpha,\beta}(x,y) = \partial_x^{\alpha}\partial_y^{\beta}\langle\phi(x)\phi(y)\rangle$.

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Most importantly, $\mathcal{W}$ runs over pair partitions (perfect matchings) of F with no block-diagonal pair, i.e., no pair $\{a,b\} \subset F_i$ for some i. Just standard combinatorics of Wick monomial correlations.

We checked that this system satisfes OPEs

$$\mathcal{O}_A(x)\mathcal{O}_B(y) = \sum_{C,\ [C]\leq\Delta} \mathcal{C}_{AB}^{C}(x,y)\mathcal{O}_C(y) + o(|x-y|^{\Delta-[A]-[B]})$$

together with the precise estimates embodied in the ENNFB, as well as the other hypotheses of our theorem on renormalized products.

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The OPE coefficients $\mathcal{C}_{AB}^{C}(x,y)$ can be given explicitly, but with somewhat complicated combinatorial formulas.

References:

A.A., A. Chandra, G. Guadagni, "Rigorous quantum field theory functional integrals over the p-adics I: anomalous dimensions", arXiv 2013.

A.A., "QFT, RG, and all that, for mathematicians, in eleven pages", arXiv 2013.

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Thank you for your attention.

Supplementary technical slides:

$$\mathcal{C}_{AB}^{C}(x_1,x_2)=\sum_{\mathcal{V}}\sum_{(\beta(a))_{a\in G_1\setminus\cup\mathcal{V}}\in(\mathbb{N}^d)^{G_1\setminus\cup\mathcal{V}}}\mathbb{1}\{\text{condition}\}$$

$$\prod_{\substack{\{a,b\}\in\mathcal{V}\\(a,b)\in G_1\times G_2}} C^{\nu(a),\nu(b)}(x_1,x_2)\times\prod_{a\in G_1\setminus\mathcal{V}}\frac{(x_1-x_2)^{\beta(a)}}{\beta(a)!}$$

Condition is:

$$:\prod_{a\in G_1\setminus\mathcal{V}}\partial^{\nu(a)+\beta(a)}\phi\times\prod_{b\in G_2\setminus\mathcal{V}}\partial^{\nu(b)}\phi:=\mathcal{O}_C$$

Setting: G is disjoint union of G_1, G_2. $\mathcal{V}$ is a (partial) matching of G only made of pairs $\{a,b\}$ with $a\in G_1$ and $b\in G_2$. $\{G_1;\nu|_{G_1}\}=A$, $\{G_2;\nu|_{G_2}\}=B$.

Precise statement of the main theorem:

$$Z_r(x) = \left\{ \int_{\mathrm{Conf}_2} dy\ dz\ \rho_r(x-y)\rho_r(x-z)\ \mathcal{C}_{AB}^{C_*}(y,z) \right\}^{-1}$$

is in $\mathcal{O}_{\mathrm{M},x}(\mathbb{R}^d)$.
Then let $M_r(x)$ in $\mathcal{O}_{\mathrm{M},x}(\mathbb{R}^d)$ be given by

$$M_r(x) = Z_r(x) \left[\mathcal{O}_{A,r}(x)\mathcal{O}_{B,r}(x) - \sum_{[C]\le[C_*], C\neq C_*} \widetilde{\mathcal{O}}_{C,r}(x) \right]$$

where $\mathcal{O}_{A,r}(x) = (\mathcal{O}_A * \rho_r)(x) = \langle \mathcal{O}_A(y), \rho_r(x-y)\rangle_y$ and similarly $\mathcal{O}_{B,r}(x) = (\mathcal{O}_B * \rho_r)(x) = \langle \mathcal{O}_B(z), \rho_r(x-z)\rangle_z$ while $\widetilde{\mathcal{O}}_{C,r}(x) = \langle \mathcal{O}_C(z), g_r(x,z)\rangle_z$ with

$$g_r(x,z) = \rho_r(x-z) \times \int_{\mathbb{R}^d\setminus\{z\}} dy\ \rho_r(x-y)\ \mathcal{C}_{AB}^{C}(y,z)\ .$$

Note that the dependence on the sample $\omega \in \Omega$ has been suppressed from the notation and that $\mathcal{O}_A$, $\mathcal{O}_B$, $\mathcal{O}_C$ designate the distribution-valued random variables provided by the probabilistic incarnation for $\mathcal{B}$. We view $M_r(x)$ as the random Schwartz distribution whose action on a test function $f \in \mathcal{S}(\mathbb{R}^d)$ is of course given by

$$M_r(f) = \int_{\mathbb{R}^d} dx \; M_r(x) f(x) = \langle M_r(x), f(x) \rangle_x \; .$$

One can show that $M_r(f)$ is indeed well defined, $\mathcal{F}$-measurable, and in every $L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \geq 1$.

Main Theorem:

1. For any test function f, and when taking $r \to -\infty$, the random variable $M_r(f)$ converges in every $L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \geq 1$, and $\mathbb{P}$-almost surely to a random variable which we will denote by $\mathcal{O}_{C_*}(f)$.
2. The limit is independent from the choice of mollifier ρ.
3. There exists a Borel-measurable map $\mathcal{P} : \prod_{C\in\mathcal{B}} \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$, $\mathcal{O}_{C_*}(f) = [\mathcal{P}((\mathcal{O}_C)_{C\in\mathcal{B}})](f)$, $\mathbb{P}$-almost surely.
4. If one extends the probabilistic incarnation to $\mathcal{B} \cup \{C_*\}$ by adding the $\mathcal{S}'(\mathbb{R}^d)$-valued random variable $\mathcal{P}((\mathcal{O}_C)_{C\in\mathcal{B}})$, then the result is a probabilistic incarnation of the system of pointwise correlations corresponding to the new set of labels $\mathcal{B} \cup \{C_*\}$.