A proof of Wilson's epsilon expansion for a toy model of three-dimensional conformal field theory

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1. Generalities about Ising and phi-four ferromagnets
2. Results and conjectures
3. A new method: space-dependent renormalization group

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Let $\overline{Q}_{DW}$ correspond to the double-well measures: $\rho_{a,b}$ with $a > 0$, $b < 0$ or $\frac{1}{2}(\delta_\lambda + \delta_{-\lambda})$ with $\lambda > 0$.

Let $\mathbb{L}$ be a countably infinite set (the lattice) and $J = (J_{\mathbf{xy}})_{\mathbf{x},\mathbf{y}\in\mathbb{L}}$ be an infinite matrix with $J_{\mathbf{xx}} = 0$, $J_{\mathbf{xy}} = J_{\mathbf{yx}} \geq 0$. Also assume

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Pick $\rho \in \overline{Q}_{DW}$, let $\beta, h \geq 0$, and for Λ finite subset of $\mathbb{L}$ and $\phi_\Lambda \in \mathbb{R}^\Lambda$ define

$$H_\Lambda(\phi_\Lambda) = -\sum_{\mathbf{x},\mathbf{y}\in\Lambda} J_{\mathbf{xy}}\phi_{\mathbf{x}}\phi_{\mathbf{y}} \ - h\sum_{\mathbf{x}\in\Lambda}\phi_{\mathbf{x}} \ .$$

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This gives a Borel probability measure $\nu_{\Lambda,\beta,h}$ on lattice fields $\phi \in \mathbb{R}^{\mathbb{L}}$ where $\phi|_{\mathbb{L}\backslash\Lambda} = 0$ and $\phi_\Lambda = \phi|_\Lambda$ is sampled according to the measure

$$\frac{1}{Z_{\Lambda,\beta,h}} \ e^{-\beta H_\Lambda(\phi_\Lambda)} \ \prod_{\mathbf{x}\in\Lambda} d\rho(\phi_{\mathbf{x}}) \ .$$

For a large class of models one can show existence of infinite volume limit, i.e., probability measure $\nu_{\beta,h}$ on $\mathbb{R}^{\mathbb{L}}$ such that for all sequences $\Lambda_n \nearrow \mathbb{L}$, $\nu_{\Lambda_n,\beta,h} \longrightarrow \nu_{\beta,h}$ weakly and in sense of moments.

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$\exists$ phase transition iff $\exists \beta_1, \beta_2 \in (0,\infty)$ such that $\chi(\beta_1) < \infty$ and $\chi(\beta_2) = \infty$. If so, let

$$\beta_c = \sup\{\beta \mid \chi(\beta) < \infty\} = \inf\{\beta \mid \chi(\beta) = \infty\} \ .$$

Then $\nu_c := \nu_{\beta_c,0}$ is the critical theory.

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4) Hierarchical models. $d(\mathbf{x}, \mathbf{y})$ hierarchical distance on $\mathbb{L}$ and for some constant $K > 0$,

$$J_{\mathbf{xy}} = K \frac{1}{d(\mathbf{x}, \mathbf{y})^{d+\sigma}} .$$

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Let $\mathbb{L}_k$, $k \in \mathbb{Z}$, be the set of cubes $\prod_{i=1}^{d}[a_i p^k, (a_i + 1)p^k)$ with $a_1, \ldots, a_d \in \mathbb{N}_0$. The cubes of $\mathbb{L}_k$ form a partition of the octant $[0, \infty)^d$.

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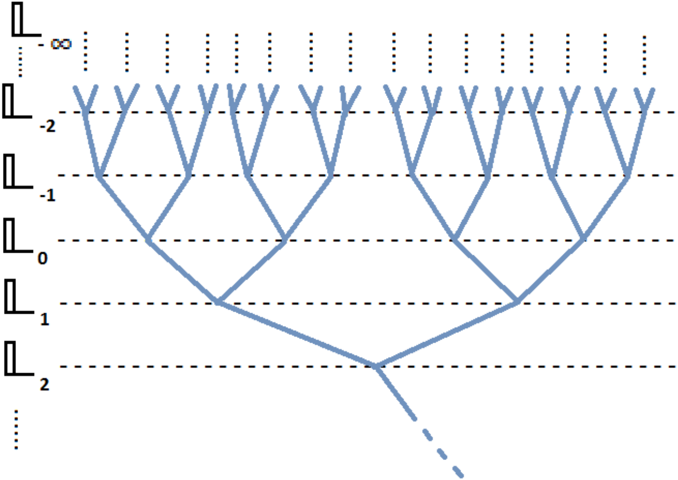
Hence $\mathbb{T} = \cup_{k\in\mathbb{Z}}\mathbb{L}_k$ naturally has the structure of a doubly infinite tree which is organized into layers or generations $\mathbb{L}_k$:



Picture for $d = 1$, $p = 2$

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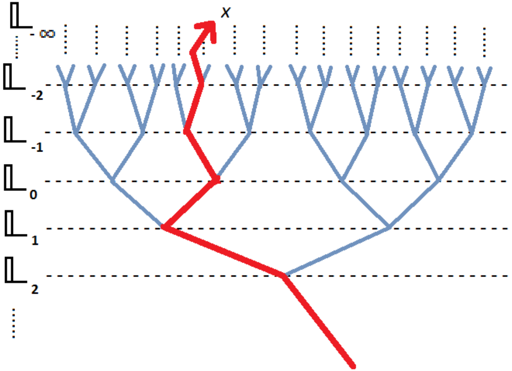
More precisely, these are the infinite bottom-up paths in the tree.



A path representing an element $x \in \mathbb{Q}_p^d$

A point $x \in \mathbb{Q}_p^d$ is encoded by a sequence $(a_n)_{n\in\mathbb{Z}}$, $a_n \in \{0, 1, \ldots, p-1\}^d$.
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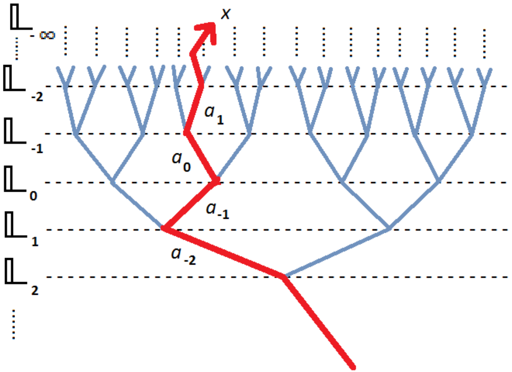

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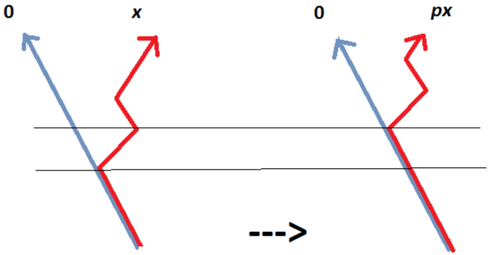

Likewise $p^{-1}x$ is downward shift, and so on for the definition of $p^k x$, $k \in \mathbb{Z}$.

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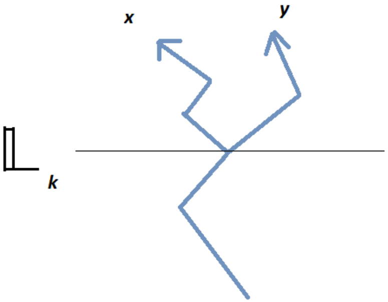

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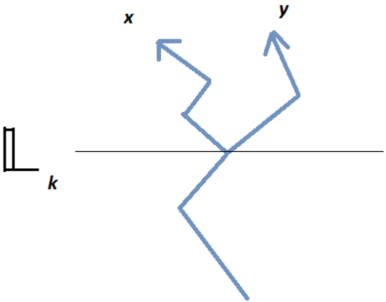

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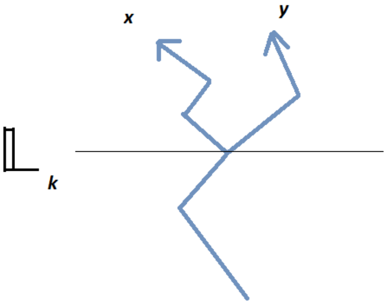

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$$|px|_p = p^{-1}|x|_p$$

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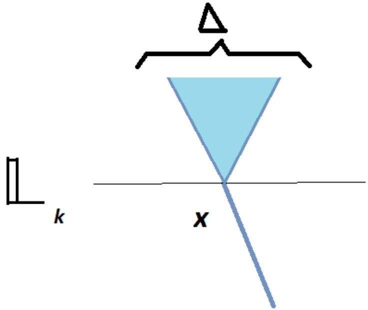

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The hierarchical lattice:

Truncate the tree at level zero and take $\mathbb{L} := \mathbb{L}_0$. Using the identification of nodes with balls, define the hierarchical distance as

$$d(\mathbf{x}, \mathbf{y}) = \inf\{|x - y|_p \mid x \in \mathbf{x},\ y \in \mathbf{y}\} .$$

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$$\Phi_r = L^{r(d-[\phi])} \sum_{\mathbf{x}\in\mathbb{Z}^d} \phi_{\mathbf{x}} \ \delta_{L^r\mathbf{x}} \ .$$

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Here $\delta_{L^r\mathbf{x}}(y) = \delta^d(y - L^r\mathbf{x})$ translated Dirac delta on $\mathbb{R}^d$. The scaling limit is the limit in (probability) distribution of Φ_r when $r \to -\infty$. It is a Borel probability measure on $\mathcal{S}'(\mathbb{R}^d)$.

For another suitable number $[\phi^2]$ one can also consider the random distribution

$$\Phi_r^2 = L^{r(d-[\phi^2])} \sum_{\mathbf{x}\in\mathbb{Z}^d} (\phi_{\mathbf{x}}^2 - \langle \phi_{\mathbf{x}}^2 \rangle_c)\ \delta_{L^r \mathbf{x}}\ .$$

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$[\phi]$ is the scaling dimension of the spin field. $[\phi^2]$ is the scaling dimension of the energy field. Can be read from long distance asymptotics

$$\langle \phi_{\mathbf{x}} \phi_{\mathbf{y}} \rangle_c \approx \frac{1}{d(\mathbf{x}, \mathbf{y})^{2[\phi]}} \text{ and } \langle \phi_{\mathbf{x}}^2, \phi_{\mathbf{y}}^2 \rangle_c^T \approx \frac{1}{d(\mathbf{x}, \mathbf{y})^{2[\phi^2]}}$$

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$[\phi]$ is the scaling dimension of the spin field. $[\phi^2]$ is the scaling dimension of the energy field. Can be read from long distance asymptotics

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$$[\phi^2] - 2[\phi] = 0.376327\ldots$$

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Hierarchical 3D phi-four: A.A., Chandra and Guadagni (arXiv 2013) showed that also with $d = 3$, $\sigma = \frac{3+\epsilon}{2}$

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where $[\phi] = [\phi]_{\text{Gauss}} = \frac{3-\epsilon}{4}$ (this part was already done by Gawędzki and Kupiainen JSP 1984) and $[\phi^2] > 2[\phi]$.

More precisely, we showed

$$[\phi^2] - 2[\phi] = \frac{\epsilon}{3} + o(\epsilon)$$

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A.A. in progress: derivation of OPE $\Phi \times \Phi = \mathbb{1} + \Phi^2 + \cdots$ (fusion rule notation).

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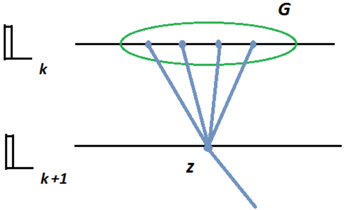

To every set G of offsprings of a node $\mathbf{z} \in \mathbb{L}_{k+1}$ associate a centered Gaussian random vector $(\zeta_{\mathbf{x}})_{\mathbf{x} \in G}$ with $p^d \times p^d$ covariance matrix made of $1 - p^{-d}$'s on the diagonal and $-p^{-d}$'s everywhere else. We impose that Gaussian vectors corresponding to different layers or different litters are independent.

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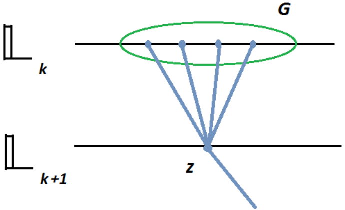

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This is heuristic since ϕ is not well-defined in a pointwise manner. We need random Schwartz(-Bruhat) distributions.

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where for all $t_- \leq t_+$, $S_{t_-,t_+}(\mathbb{Q}_p^d)$ denotes the space of functions which are constant in each of the closed balls of radius p^{t_-} and with support inside $\overline{B}(0, p^{t_+})$.

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Topology generated by the set of all possible semi-norms.

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Probability Theory on $S'(\mathbb{Q}_p^d)$ is super!

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Sample fields are true functions that are locally constant on scale L^r. These measures are scaled copies of each other. If the law of $\phi(\cdot)$ is μ_{C_0}, then that of $L^{-r[\phi]}\phi(L^r\cdot)$ is μ_{C_r}.

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Let

$$V_{r,s}(\phi) = \int_{\Lambda_s} \{g_r : \phi^4 :_{C_r} (x) + \mu_r : \phi^2 :_{C_r} (x)\} d^3x$$

and define the probability measure

$$d\nu_{r,s}(\phi) = \frac{1}{\mathcal{Z}_{r,s}} e^{-V_{r,s}(\phi)} d\mu_{C_r}(\phi)$$

Let $\phi_{r,s}$ be the random distribution in $S'(\mathbb{Q}_p^3)$ sampled according to $\nu_{r,s}$ and define the squared field $N_r[\phi_{r,s}^2]$ which is a deterministic function(al) of $\phi_{r,s}$, with values in $S'(\mathbb{Q}_p^3)$, given by

$$N_r[\phi_{r,s}^2](j) = Z_2^r \int_{\mathbb{Q}_p^3} \{Y_2 :\phi_{r,s}^2:_{C_r}(x) - Y_0 L^{-2r[\phi]}\}\, j(x)\, d^3x$$

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The main result concerns the limit law of the pair $(\phi_{r,s}, N_r[\phi_{r,s}^2])$ in $S'(\mathbb{Q}_p^3) \times S'(\mathbb{Q}_p^3)$ when $r \to -\infty$, $s \to \infty$ (in any order).
For the precise statement we need the approximate fixed point value

$$\bar{g}_* = \frac{p^\epsilon - 1}{36 L^\epsilon (1 - p^{-3})}$$

Theorems:

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Theorem 1: A.A.-Chandra-Guadagni 2013

$\exists\rho > 0$, $\exists L_0$, $\forall L \geq L_0$, $\exists\epsilon_0 > 0$, $\forall\epsilon \in (0, \epsilon_0]$, $\exists[\phi^2] > 2[\phi]$, $\exists$ fonctions $\mu(g)$, $Y_0(g)$, $Y_2(g)$ on $(\bar{g}_* - \rho\epsilon^{\frac{3}{2}}, \bar{g}_* + \rho\epsilon^{\frac{3}{2}})$ such that if one lets $\mu = \mu(g)$, $Y_0 = Y_0(g)$, $Y_2 = Y_2(g)$ and $Z_2 = L^{-([\phi^2]-2[\phi])}$ then the joint law of $(\phi_{r,s}, N_r[\phi^2_{r,s}])$ converge weakly and in the sense of moments to that of a pair $(\phi, N[\phi^2])$ such that:

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1. $\forall k \in \mathbb{Z}$, $(L^{-k[\phi]}\phi(L^k\cdot), L^{-k[\phi^2]}N[\phi^2](L^k\cdot)) \stackrel{d}{=} (\phi, N[\phi^2])$.

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2. $\langle\phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} < 0$ i.e., ϕ is non-Gaussian. Here, $\mathbf{1}_{\mathbb{Z}_p^3}$ denotes the indicator function of $\overline{B}(0,1)$.

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3. $\langle N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3}), N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} = 1$.

The mixed correlation functions satisfy, in the sense of distributions,

$$\langle \phi(L^{-k}x_1)\cdots\phi(L^{-k}x_n)N[\phi^2](L^{-k}y_1)\cdots N[\phi^2](L^{-k}y_m)\rangle$$

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The law $\nu_{\phi\times\phi^2}$ of $(\phi, N[\phi^2])$ is independent of g: universality.

Theorem 2: A.A.-Chandra-Guadagni 2013

$\nu_{\phi\times\phi^2}$ is fully scale invariant, i.e., invariant under the action of the scaling group $p^{\mathbb{Z}}$ instead of the subgroup $L^{\mathbb{Z}}$. Moreover, $\mu(g)$ and $[\phi^2]$ are independent of the arbitrary factor L.

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Note that $2[\phi^2] = 3 - \frac{1}{3}\epsilon + o(\epsilon) \to$ still $L^{1,\mathrm{loc}}$!

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Define the new matrix $A = (A_{\mathbf{xy}})_{\mathbf{xy}\in\mathbb{L}}$ by

$$A_{\mathbf{xy}} = \lim_{s\to\infty} (C_0|_{\Lambda_s})^{-1}_{\mathbf{xy}} \ .$$

Then

$$A_{\mathbf{xy}} = - \frac{p^{3-2[\phi]} - 1}{1 - p^{-(6-2[\phi])}} \times \frac{1}{d(\mathbf{x},\mathbf{y})^{3+\sigma}}$$

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Here again $\sigma = 3 - 2[\phi] = \frac{3+\epsilon}{2}$.
We proved that $\lim_{s\to\infty} \nu_{0,s}$ is the same infinite volume lattice measure as previous ν_c for suitable a, b, β_c, K related to $g, \mu(g)$.

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Take $RG(a, b) = \left(\frac{a+b}{2}, \sqrt{ab}\right)$.

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Rigorous nonperturbative version of the local RG:
Wilson-Kogut PR 1974, Drummond-Shore PRD 1979,
Jack-Osborn NPB 1990,...
used for generalizations of Zamolodchikov's c-"Theorem",
study of scale versus conformal invariance, AdS/CFT,...

1st step: switch to unit lattice/cut-off

$$\mathcal{S}^{\mathrm{T}}_{r,s}(f) := \log \mathbb{E}_{\nu_{r,s}} e^{i\phi(f)} = \log$$

$$\frac{\int d\mu_{C_r}(\phi) \exp\left(- \int_{\Lambda_s} \{ g_r : \phi^4 :_r (x) + \mu_r : \phi^2 :_r \} dx + \int \phi(x) f(x) dx \right)}{\int d\mu_{C_r}(\phi) \exp\left(- \int_{\Lambda_s} \{ g_r : \phi^4 :_r (x) + \mu_r : \phi^2 :_r \} dx \right)}$$

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with

$$\mathcal{I}^{(r,r)}[f](\phi) = \exp\left(-\int_{\Lambda_{s-r}}\{g : \phi^4 :_0 (x) + \mu : \phi^2 :_0\}d^3x\right.$$

$$\left. + L^{(3-[\phi])r} \int \phi(x) f(L^{-r}x) d^3x\right)$$

2nd step: define inhomogeneous RG

Fluctuation covariance $\Gamma := C_0 - C_1$.

Associated Gaussian measure is the law of the fluctuation field

$$\zeta(x) = \sum_{0 \leq k < \ell} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

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$$\int \mathcal{I}^{(r,r)}[f](\phi)\, d\mu_{C_0}(\phi) = \int\int \mathcal{I}^{(r,r)}[f](\zeta + \psi)\, d\mu_{\Gamma}(\zeta) d\mu_{C_1}(\psi)$$

$$= \int \mathcal{I}^{(r,r+1)}[f](\phi)\, d\mu_{C_0}(\phi)$$

with new integrand

$$\mathcal{I}^{(r,r+1)}[f](\phi) = \int \mathcal{I}^{(r,r)}[f](\zeta + L^{-[\phi]}\phi(L\cdot))\, d\mu_{\Gamma}(\zeta)$$

Need to extract vacuum renormalization $\rightarrow$ better definition is

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One must control

$$\mathcal{S}^{\mathrm{T}}(f) = \lim_{\substack{r\rightarrow-\infty \\ s\rightarrow\infty}} \sum_{r\leq q<s} \left(\delta b(\mathcal{I}^{(r,q)}[f]) - \delta b(\mathcal{I}^{(r,q)}[0])\right)$$

limit of logarithms of characteristic functions.

Use a Brydges-Yau lift

$$\begin{array}{ccc} \vec{V}^{(r,q)} & \xrightarrow{RG_{\mathrm{inhom}}} & \vec{V}^{(r,q+1)} \\ \downarrow & & \downarrow \\ \mathcal{I}^{(r,q)} & \longrightarrow & \mathcal{I}^{(r,q+1)} \end{array}$$

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$$\mathcal{I}^{(r,q)}(\phi) = \prod_{\substack{\Delta\in\mathbb{L}_0 \\ \Delta\subset\Lambda_{s-q}}} \Big[e^{f_\Delta \phi_\Delta} \times$$

$$\Big\{ \exp\Big(-\beta_{4,\Delta} :\phi_\Delta^4:_{C_0} -\beta_{3,\Delta} :\phi_\Delta^3:_{C_0} -\beta_{2,\Delta} :\phi_\Delta^2:_{C_0} -\beta_{1,\Delta} :\phi_\Delta^1:_{C_0} \Big)$$

$$\times \Big(1 + W_{5,\Delta} :\phi_\Delta^5:_{C_0} + W_{6,\Delta} :\phi_\Delta^6:_{C_0} \Big)$$

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Dynamical variable is $\vec{V} = (V_\Delta)_{\Delta\in\mathbb{L}_0}$ with

$$V_\Delta = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_\Delta, R_\Delta)$$

RG_{inhom} acts on $\mathcal{E}_{\mathrm{inhom}}$, essentially,

$$\prod_{\Delta \in \mathbb{L}_0} \left\{ \mathbb{C}^7 \times C^9(\mathbb{R}, \mathbb{C}) \right\}$$

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Stable subspaces

$\mathcal{E}_{\mathrm{hom}} \subset \mathcal{E}_{\mathrm{inhom}}$: spatially constant data.
$\mathcal{E} \subset \mathcal{E}_{\mathrm{hom}}$: even potential, i.e., g, μ's only and R even function.
Let RG be induced action of RG_{inhom} on $\mathcal{E}$.

3rd step: stabilize bulk (homogeneous) evolution

Show that $\forall q \in \mathbb{Z}$, $\lim_{r\to-\infty} \vec{V}^{(r,q)}[0]$ exists, i.e.,

$$\lim_{r\to-\infty} RG^{q-r}\left(\vec{V}^{(r,r)}[0]\right)$$

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$$RG \left\{ \begin{array}{lllllllll} g' & = & L^{\epsilon} g & - & A_1 g^2 & + & \cdots & & \\ \mu' & = & L^{\frac{3+\epsilon}{2}} \mu & - & A_2 g^2 & - & A_3 g \mu & + & \cdots \\ R' & = & \mathcal{L}^{(g,\mu)}(R) & + & \cdots & & & & \end{array} \right.$$

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$$RG \begin{cases} g' = L^{\epsilon} g - A_1 g^2 + \cdots \\ \mu' = L^{\frac{3+\epsilon}{2}} \mu - A_2 g^2 - A_3 g\mu + \cdots \\ R' = \mathcal{L}^{(g,\mu)}(R) + \cdots \end{cases}$$

Tadpole graph with mass insertion

$$A_3 = 12 L^{3-2[\phi]} \int_{\mathbb{Q}_p^3} \Gamma(0,x)^2 \ d^3x$$

is main culprit for anomalous scaling dimension $[\phi^2] - 2[\phi] > 0$.

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Thus

$$\forall q \in \mathbb{Z}, \qquad \lim_{r \to -\infty} \vec{V}^{(r,q)}[0] = v_*$$

Tangent spaces at fixed point: E^{s} and E^{u}.

$E^{\mathrm{u}} = \mathbb{C} e_{\mathrm{u}}$, with e_{u} eigenvector of $D_{v_*} RG$ for eigenvalue $\alpha_{\mathrm{u}} = L^{3-2[\phi]} \times Z_2 =: L^{3-[\phi^2]}$.

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For source term with ϕ^2 add

$$Y_2 Z_2^r \int :\phi^2:_{C_r}(x)j(x)d^3x$$

to potential. $\mathcal{S}^{\mathrm{T}}_{r,s}(f,j)$ now involves two test functions. After rescaling to unit lattice/cut-off

$$Y_2\alpha_{\mathrm{u}}^r \int :\phi^2:_{C_0}(x)j(L^{-r}x)d^3x$$

to be combined with μ into $(\beta_{2,\Delta})_{\Delta\in\mathbb{L}_0}$ space-dependent mass.

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$\Psi(v,w)$ is holomorphic in v and w.
Essential for probabilistic interpretation of $(\phi, N[\phi^2])$ as pair of random variables in $S'(\mathbb{Q}_p^3)$.

References:

A.A., A. Chandra, G. Guadagni, "Rigorous quantum field theory functional integrals over the p-adics I: anomalous dimensions", arXiv 2013.

A.A., "QFT, RG, and all that, for mathematicians, in eleven pages", arXiv 2013.

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Thank you for your attention.