A Toy Model for Three-Dimensional Conformal Probability

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Colloquium at the UMD Mathematics Department
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1. Introduction
2. The Euclidean CFT model: conjectures
3. The p-adic toy model: some theorems
4. The method: space-dependent renormalization group

1) Scaling limits:

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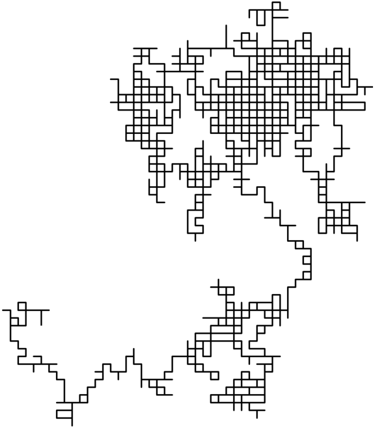
Simple random walk on a lattice



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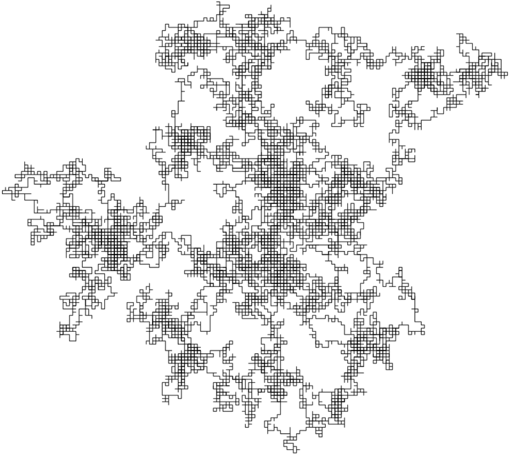
from far away. . .



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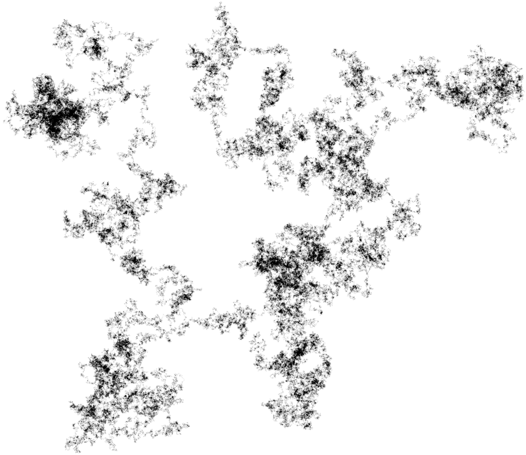
from far, far, far away...



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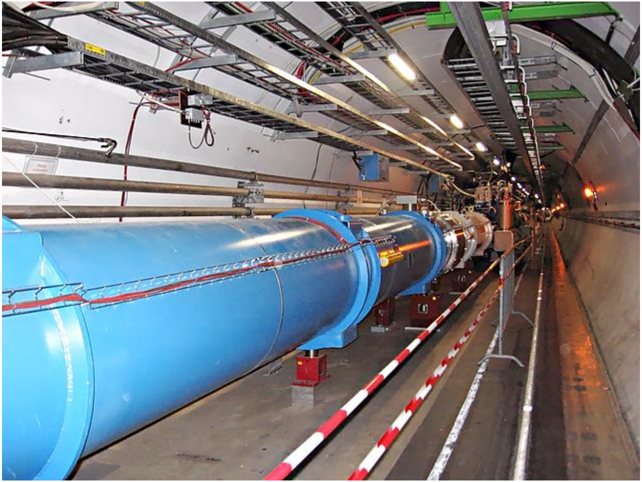
The dilation factor λ becomes $|f'(t)|$, i.e., local or space-dependent.

2) Second motivation, quantum field theory:



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(by Maximilien Brice, CERN, via Wikimedia Commons)

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$$\frac{1}{\mathcal{Z}} \exp\left(- \int_{\mathbb{R}^d} \{ \frac{1}{2} (\nabla \phi)^2(x) + \mu \phi(x)^2 + g \phi(x)^4 \} d^d x \right) \; D\phi$$

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For all test function $f \in S(\mathbb{R}^d)$ we have
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Hence $L^{rd} \sum_{x \in L^r \mathbb{Z}^d} \phi(x) \delta_x \to \phi$ in $S'(\mathbb{R}^d)$ (for weak-$*$).

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Let $\sigma_{\mathbf{x}}=0$ if $\mathbf{x}\le 0$ and $\sigma_{\mathbf{x}}=\sum_{0<\mathbf{y}\le\mathbf{x}}\omega_{\mathbf{y}}$ if $\mathbf{x}>0$, where the steps ω are independent equal to ±1 with probability $\frac{1}{2}$.

Let $\phi_r = L^{r(d-[\phi])} \sum_{\mathbf{x}\in\mathbb{Z}^d} \sigma_{\mathbf{x}} \delta_{L^r\mathbf{x}}$. Then for $f \in S(\mathbb{R})$ we have

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By the Dominated Convergence Theorem

$$\langle e^{i\phi_r(f)} \rangle := \mathbb{E}\ e^{i\phi_r(f)}$$

$$= \lim_{N\to+\infty} \left\langle \exp\left(i\ L^{r(1-[\phi])} \sum_{0<\mathbf{y}\leq N} \omega_{\mathbf{y}} \left(\sum_{\mathbf{x}\geq\mathbf{y}} f(L^r \mathbf{x}) \right) \right) \right\rangle$$

$$= \lim_{N\to+\infty} \prod_{0<\mathbf{y}\leq N} \cos\left(L^{r(1-[\phi])} \left(\sum_{\mathbf{x}\geq\mathbf{y}} f(L^r\mathbf{x}) \right) \right)$$

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$$\to \exp\left[-\frac{1}{2}\int_0^{+\infty}\left(\int_y^{+\infty} f(x)\, dx\right)^2 dy\right] \quad \text{if} \quad [\phi] = -\frac{1}{2}.$$

Hence

$$\lim_{r \to -\infty} \langle e^{i\phi_r(f)} \rangle = \exp\left(-\frac{1}{2}\langle \phi(f)\phi(f) \rangle\right)$$

with

$$\langle \phi(f)\phi(f) \rangle = \int_{\mathbb{R}^2} \langle \phi(x_1)\phi(x_2) \rangle f(x_1) f(x_2) \; dx_1 dx_2$$

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Finally, use the Lévy Continuity Theorem on $S'(\mathbb{R})$. QED

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Result due to Camia-Garban-Newman (Ann. Probab. 2015) and Chelkak-Hongler-Izyurov (Ann. Math. 2015).

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Can be seen as continuous limit of spin models, like Ising, with ferromagnetic long-range interactions.

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Let $C_{-\infty}$ be the continuous bilinear form on $S(\mathbb{R}^3)$ given by

$$C_{-\infty}(f,g) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\overline{\widehat{f}(\xi)}\widehat{g}(\xi)}{|\xi|^{3-2[\phi]}} d^3\xi$$

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Again, fix zooming-out ratio $L > 1$.

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The scale invariant measure for (fractional) ϕ^4 model should be the weak limit $\nu_\phi = \lim_{r\to-\infty}\lim_{s\to\infty}\nu_{r,s}$ for a choice $(g_r, \mu_r)_{r\in\mathbb{Z}}$ that emulates the scaling limit of a fixed critical lattice random field (like for 2D Ising).

Conjecture 1:

Let $[\phi] = \frac{3-\epsilon}{4}$ with $0 < \epsilon \ll 1$.
There exists a nonempty open interval $I \subset (0, \infty)$ and a function $\mu_c : I \to \mathbb{R}$ such that for all $g \in I$, if one lets $g_r = L^{-r(3-4[\phi])}g$ and $\mu_r = L^{-r(3-2[\phi])}\mu_c(g)$, then the weak limit ν_ϕ exists, is non-Gaussian, stationary, $O(3)$-invariant, and scale invariant with exponent $[\phi]$, i.e., $\lambda^{[\phi]}\phi(\lambda\cdot) \stackrel{dd}{=} \phi(\cdot)$ for all $\lambda > 0$.
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Measure constructed on $\mathbb{T}^3$ torus by Mitter ($\sim$ 2004) using RG fixed point obtained by Brydges-Mitter-Scoppola CMP 2003.

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Conj. 3 is a precise formulation of predictions made in "Conformal invariance in the long-range Ising model" by Paulos, Rychkov, van Rees and Zan, Nucl. Phys. B 2016 $- >$ Higher dimensional conformal bootstrap program.

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Correpondence: $f \in \mathcal{M}(\mathbb{R}^3)$ $\leftrightarrow$ hyperbolic isometry of the interior $\mathbb{B}^4$ or $\mathbb{H}^4$.

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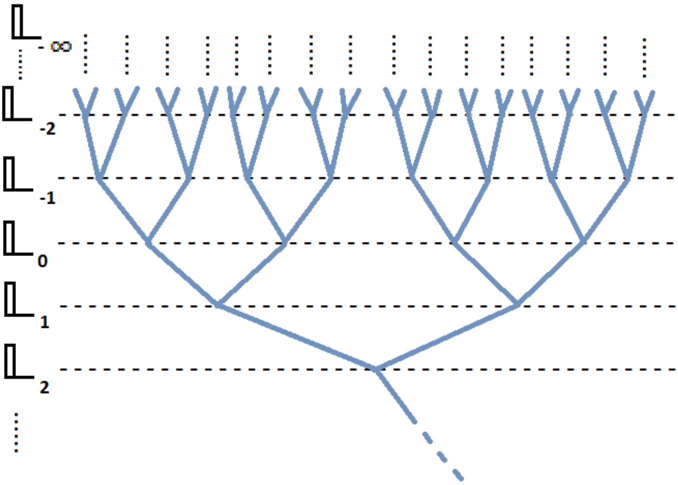
Hence $\mathbb{T} = \cup_{k \in \mathbb{Z}} \mathbb{L}_k$ naturally has the structure of a doubly infinite tree which is organized into layers or generations $\mathbb{L}_k$:



Picture for $d = 1$, $p = 2$

Forget $[0, \infty)^d$ and $\mathbb{R}^d$ and just keep the tree.
Define the substitute for the continuum $\mathbb{Q}_p^d$:= leafs at infinity "$\mathbb{L}_{-\infty}$".

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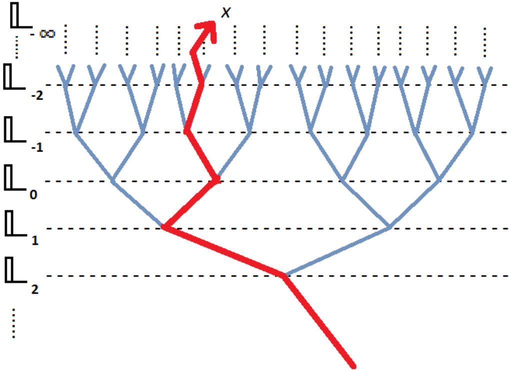
Define the substitute for the continuum $\mathbb{Q}_p^d$:= leafs at infinity "$\mathbb{L}_{-\infty}$".
More precisely, these are the infinite bottom-up paths in the tree.



A path representing an element $x \in \mathbb{Q}_p^d$

A point $x \in \mathbb{Q}_p^d$ is encoded by a sequence $(a_n)_{n\in\mathbb{Z}}$, $a_n \in \{0, 1, \ldots, p-1\}^d$.
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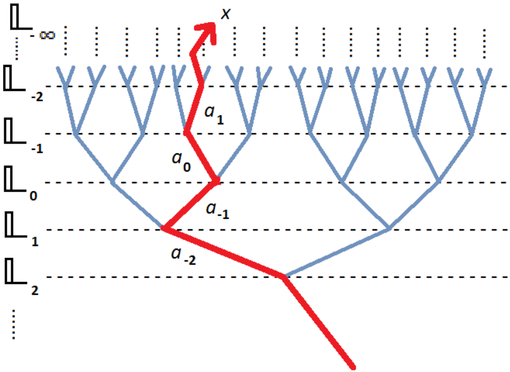

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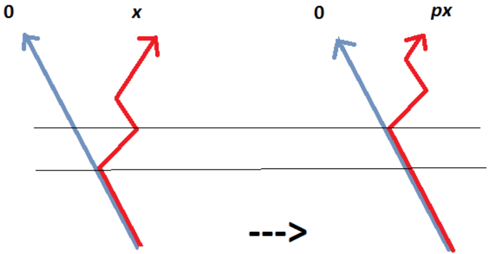

Likewise $p^{-1}x$ is downward shift, and so on for the definition of $p^k x$, $k \in \mathbb{Z}$.

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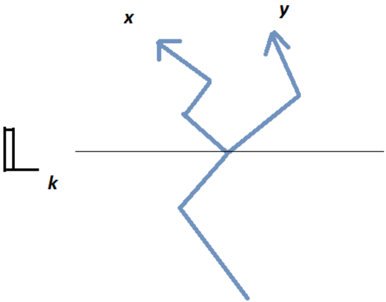

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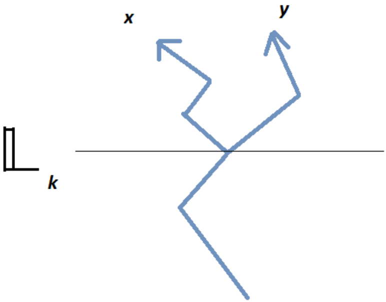

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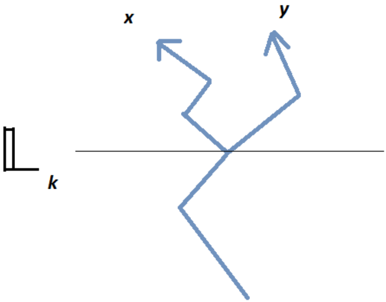

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$$|px| = p^{-1}|x|$$

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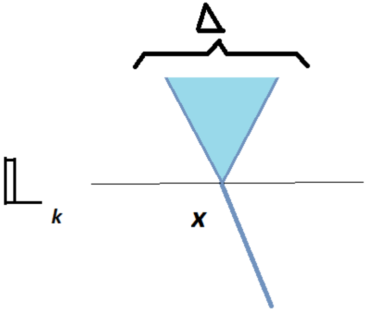

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Construction: take product of uniform probability measures on $(\{0, 1, \ldots, p-1\}^d)^{\mathbb{N}_0}$ for $\overline{B}(0, 1)$. Do the same for the other closed unit balls, and collate.

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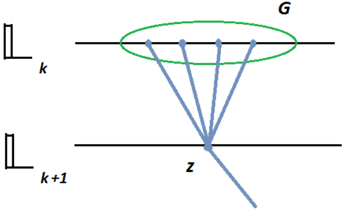

To every litter G of Mama Cat $\mathbf{z} \in \mathbb{L}_{k+1}$ associate a centered Gaussian random vector $(\zeta_{\mathbf{x}})_{\mathbf{x}\in G}$ with $p^d \times p^d$ covariance matrix made of $1 - p^{-d}$'s on the diagonal and $-p^{-d}$'s everywhere else. We impose that Gaussian vectors corresponding to different layers or different litters are independent.

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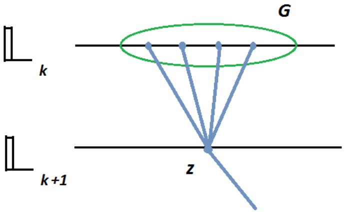

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This is heuristic since ϕ is not well-defined in a pointwise manner. We need random Schwartz(-Bruhat) distributions.

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where for all $t_- \leq t_+$, $S_{t_-,t_+}(\mathbb{Q}_p^d)$ denotes the space of functions which are constant in each of the closed balls of radius p^{t_-} and with support inside $\overline{B}(0, p^{t_+})$.

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Topology generated by the set of all possible semi-norms.

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Probability Theory on $S'(\mathbb{Q}_p^d)$ is super!

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6. $S'(\mathbb{Q}_p^d) \times S'(\mathbb{Q}_p^d) \simeq S'(\mathbb{Q}_p^d)$ the machinery also works for join laws of pairs of random distributions, e.g., $(\phi, N[\phi^2])$ in following slides.

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Sample fields are true fonctions that are locally constant on scale L^r.
These measures are scaled copies of each other.
If the law of $\phi(\cdot)$ is μ_{C_0}, then that of $L^{-r[\phi]}\phi(L^r \cdot)$ is μ_{C_r}.

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Let

$$V_{r,s}(\phi) = \int_{\Lambda_s} \{g_r : \phi^4 :_{C_r} (x) + \mu_r : \phi^2 :_{C_r} (x)\} d^3x$$

and define the probability measure

$$d\nu_{r,s}(\phi) = \frac{1}{\mathcal{Z}_{r,s}} e^{-V_{r,s}(\phi)} d\mu_{C_r}(\phi)$$

Let $\phi_{r,s}$ be the random distribution in $S'(\mathbb{Q}_p^3)$ sampled according to $\nu_{r,s}$ and define the squared field $N_r[\phi_{r,s}^2]$ which is a deterministic function(al) of $\phi_{r,s}$, with values in $S'(\mathbb{Q}_p^3)$, given by

$$N_r[\phi_{r,s}^2](j) = Z_2^r \int_{\mathbb{Q}_p^3} \{ Y_2 : \phi_{r,s}^2 :_{C_r} (x) - Y_0 L^{-2r[\phi]} \} \, j(x) \, d^3x$$

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The main result concerns the limit law of the pair $(\phi_{r,s}, N_r[\phi_{r,s}^2])$ in $S'(\mathbb{Q}_p^3) \times S'(\mathbb{Q}_p^3)$ when $r \to -\infty$, $s \to \infty$ (in any order).

For the precise statement we need the approximate fixed point value

$$\bar{g}_* = \frac{p^\epsilon - 1}{36 L^\epsilon (1 - p^{-3})}$$

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$\exists\rho > 0$, $\exists L_0$, $\forall L \geq L_0$, $\exists\epsilon_0 > 0$, $\forall\epsilon \in (0, \epsilon_0]$, $\exists[\phi^2] > 2[\phi]$, $\exists$ fonctions $\mu(g)$, $Y_0(g)$, $Y_2(g)$ on $(\bar{g}_* - \rho\epsilon^{\frac{3}{2}}, \bar{g}_* + \rho\epsilon^{\frac{3}{2}})$ such that if one lets $\mu = \mu(g)$, $Y_0 = Y_0(g)$, $Y_2 = Y_2(g)$ and $Z_2 = L^{-([\phi^2]-2[\phi])}$ then the joint law of $(\phi_{r,s}, N_r[\phi^2_{r,s}])$ converge weakly and in the sense of moments to that of a pair $(\phi, N[\phi^2])$ such that:

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3. $\langle N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3}), N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} = 1$.

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The law $\nu_{\phi\times\phi^2}$ of $(\phi, N[\phi^2])$ is independent of g: universality.

Theorem 2: A.A.-Chandra-Guadagni 2013

$\nu_{\phi\times\phi^2}$ is fully scale invariant, i.e., invariant under the action of the scaling group $p^{\mathbb{Z}}$ instead of the subgroup $L^{\mathbb{Z}}$. Moreover, $\mu(g)$ and $[\phi^2]$ are independent of the arbitrary factor L.

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Note that $2[\phi^2] = 3 - \frac{1}{3}\epsilon + o(\epsilon) \to$ still $L^{1,\mathrm{loc}}$!

Theorem 3: A.A., May 2015

Use ψ_i to denote ϕ or $N[\phi^2]$. Then, for all mixed correlation $\exists$ a smooth fonction $\langle\psi_1(z_1)\cdots\psi_n(z_n)\rangle$ on $(\mathbb{Q}_p^3)^n\backslash\mathrm{Diag}$ which is locally integrable (on the diagonal Diag and such that

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In other words, $\nu_{\phi\times\phi^2}$ is DPC (this is the toy model version of Conj. 2).

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Can also define the absolute cross-ratio for the ultrametric distance. $\mathcal{M}(\mathbb{Q}_p^3)$ is also the group of transformations of $\widehat{\mathbb{Q}_p^3} = \mathbb{Q}_p^3 \cup \{\infty\}$ which preserve this cross-ratio.

9) Other work in preparation:

Progress towards proof of p-adic analogue of Conj. 3.

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The AdS bulk (interior) is the tree $\mathbb{T}$ with the graph distance. Analogue of hyperbolic metric.

Mumford-Manin-Drinfeld Lemma

$$CR(x_1, x_2, x_3, x_4) := \frac{|x_1 - x_3|\ |x_2 - x_4|}{|x_1 - x_4|\ |x_2 - x_3|} = p^{-\delta(x_1 \to x_2; x_3 \to x_4)} ,$$

where $\delta(x_1 \to x_2; x_3 \to x_4)$ is the number of common edges for the two bi-infinite paths $x_1 \to x_2$ and $x_3 \to x_4$, counted positively if orientations agree and negatively otherwise.

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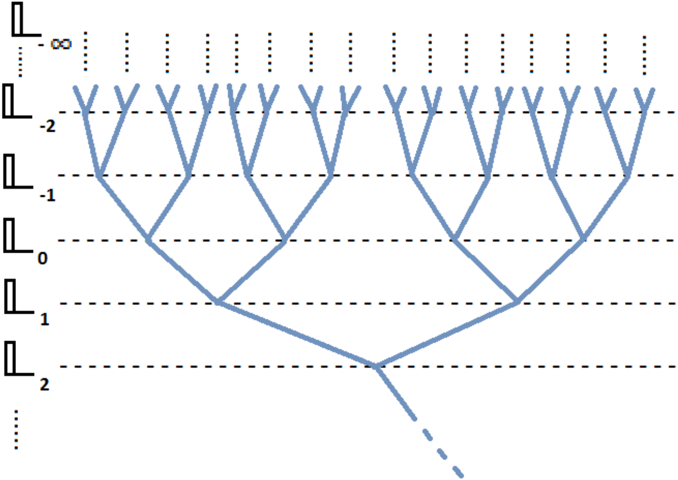
The space-dependent RG of ACG 2013 $\rightarrow$ space-dependent UV cut-off $\rightarrow$ Conj. 3 by showing the equivalence between usual flat (in half-space) cut-off hypersurface and the spherical one in conformal ball model.



The tree, once again.

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Take $RG(a, b) = \left(\frac{a+b}{2}, \sqrt{ab}\right)$.

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$$\int \{g : \phi^4 : (x) + \mu : \phi^2 : (x)\} d^d x$$

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Rigorous nonperturbative version of the local RG:
Wilson-Kogut PR 1974, Drummond-Shore PRD 1979,
Jack-Osborn NPB 1990,...
used for generalizations of Zamolodchikov's c-"Theorem",
study of scale versus conformal invariance, AdS/CFT,...

1st step: switch to unit lattice/cut-off

$$\mathcal{S}^{\mathrm{T}}_{r,s}(f) := \log \mathbb{E}_{\nu_{r,s}} e^{i\phi(f)} = \log$$

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with

$$\mathcal{I}^{(r,r)}[f](\phi) = \exp\left(-\int_{\Lambda_{s-r}}\{g : \phi^4 :_0 (x) + \mu : \phi^2 :_0\}d^3x\right.$$

$$\left. + L^{(3-[\phi])r} \int \phi(x) f(L^{-r}x) d^3x\right)$$

2nd step: define inhomogeneous RG

Fluctuation covariance $\Gamma := C_0 - C_1$.

Associated Gaussian measure is the law of the fluctuation field

$$\zeta(x) = \sum_{0 \le k < \ell} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

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L-blocks (closed balls of radius L) are independent. Hence

$$\int \mathcal{I}^{(r,r)}[f](\phi)\ d\mu_{C_0}(\phi) = \int \int \mathcal{I}^{(r,r)}[f](\zeta + \psi)\ d\mu_\Gamma(\zeta) d\mu_{C_1}(\psi)$$

$$= \int \mathcal{I}^{(r,r+1)}[f](\phi)\ d\mu_{C_0}(\phi)$$

with new integrand

$$\mathcal{I}^{(r,r+1)}[f](\phi) = \int \mathcal{I}^{(r,r)}[f](\zeta + L^{-[\phi]}\phi(L\cdot))\ d\mu_\Gamma(\zeta)$$

Need to extract vacuum renormalization $\rightarrow$ better definition is

$$\mathcal{I}^{(r,r+1)}[f](\phi) = e^{-\delta b(\mathcal{I}^{(r,r)}[f])} \int \mathcal{I}^{(r,r)}[f](\zeta + L^{-[\phi]}\phi(L\cdot))\, d\mu_{\Gamma}(\zeta)$$

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One must control

$$\mathcal{S}^{\mathrm{T}}(f) = \lim_{\substack{r\rightarrow-\infty \\ s\rightarrow\infty}} \sum_{r\le q<s} \left(\delta b(\mathcal{I}^{(r,q)}[f]) - \delta b(\mathcal{I}^{(r,q)}[0])\right)$$

limit of logarithms of characteristic functions.

Use a Brydges-Yau lift

$$\begin{array}{ccc} \vec{V}^{(r,q)} & \xrightarrow{RG_{\mathrm{inhom}}} & \vec{V}^{(r,q+1)} \\ \downarrow & & \downarrow \\ \mathcal{I}^{(r,q)} & \longrightarrow & \mathcal{I}^{(r,q+1)} \end{array}$$

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$$

$$
\Big\{ \exp\left(-\beta_{4,\Delta} : \phi_\Delta^4 :_{C_0} -\beta_{3,\Delta} : \phi_\Delta^3 :_{C_0} -\beta_{2,\Delta} : \phi_\Delta^2 :_{C_0} -\beta_{1,\Delta} : \phi_\Delta^1 :_{C_0} \right)
$$

$$
\times \left(1 + W_{5,\Delta} : \phi_\Delta^5 :_{C_0} + W_{6,\Delta} : \phi_\Delta^6 :_{C_0} \right)
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Dynamical variable is $\vec{V} = (V_\Delta)_{\Delta\in\mathbb{L}_0}$ with

$$V_\Delta = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_\Delta, R_\Delta)$$

RG_{inhom} acts on $\mathcal{E}_{\mathrm{inhom}}$, essentially,

$$\prod_{\Delta \in \mathbb{L}_0} \{\mathbb{C}^7 \times C^9(\mathbb{R}, \mathbb{C})\}$$

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Stable subspaces

$\mathcal{E}_{\mathrm{hom}} \subset \mathcal{E}_{\mathrm{inhom}}$: spatially constant data.
$\mathcal{E} \subset \mathcal{E}_{\mathrm{hom}}$: even potential, i.e., g, μ's only and R even function.
Let RG be induced action of RG_{inhom} on $\mathcal{E}$.

3rd step: stabilize bulk (homogeneous) evolution

Show that $\forall q \in \mathbb{Z}$, $\lim_{r \to -\infty} \vec{V}^{(r,q)}[0]$ exists, i.e.,

$$\lim_{r \to -\infty} RG^{q-r}\left(\vec{V}^{(r,r)}[0]\right)$$

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$$RG \begin{cases} g' &= L^{\epsilon} g - A_1 g^2 + \cdots \\ \mu' &= L^{\frac{3+\epsilon}{2}}\mu - A_2 g^2 - A_3 g\mu + \cdots \\ R' &= \mathcal{L}^{(g,\mu)}(R) + \cdots \end{cases}$$

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Tadpole graph with mass insertion

$$A_3 = 12L^{3-2[\phi]}\int_{\mathbb{Q}_p^3} \Gamma(0,x)^2\ d^3x$$

is main culprit for anomalous scaling dimension $[\phi^2] - 2[\phi] > 0$.

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Thus

$$\forall q \in \mathbb{Z}, \qquad \lim_{r \to -\infty} \vec{V}^{(r,q)}[0] = v_*$$

Tangent spaces at fixed point: $E^{\rm s}$ and $E^{\rm u}$.

$E^{\rm u} = \mathbb{C} e_{\rm u}$, with $e_{\rm u}$ eigenvector of $D_{v_*} RG$ for eigenvalue $\alpha_{\rm u} = L^{3-2[\phi]} \times Z_2 =: L^{3-[\phi^2]}$.

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For source term with ϕ^2 add

$$Y_2 Z_2^r \int :\phi^2:_{C_r}(x) j(x) d^3x$$

to potential. $\mathcal{S}^{\mathrm{T}}_{r,s}(f,j)$ now involves two test functions. After rescaling to unit lattice/cut-off

$$Y_2 \alpha_{\mathrm{u}}^r \int :\phi^2:_{C_0}(x) j(L^{-r}x) d^3x$$

to be combined with μ into $(\beta_{2,\Delta})_{\Delta\in\mathbb{L}_0}$ space-dependent mass.

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$\Psi(v,w)$ is holomorphic in v and w.
Essential for probabilistic interpretation of $(\phi, N[\phi^2])$ as pair of random variables in $S'(\mathbb{Q}_p^3)$.

Thank you for your attention.