A Note on Cumulants and Möbius Inversion

Let Ω be a probability space equipped with a σ-algebra $\mathcal{A}$ and a probability measure ν. For any $p \in [1, +\infty)$ we denote the corresponding real L^p space by $L^p(\Omega)$. Let $\mathcal{E}$ be the intersection of these L^p spaces. We work in this space, in order for arbitrary moments to be well defined. We will define a collection of multilinear forms E_n^T, $n \geq 1$, on this space $\mathcal{E}$:

$$\begin{array}{rccc} E_n^T : & \mathcal{E}^n & \rightarrow & \mathbb{R} \\ & (X_1, \ldots, X_n) & \mapsto & E_n^T[X_1, \ldots, X_n] \end{array}$$

This construction is by induction on n. One lets $E_1^T[X] = E[X]$, the usual expectation. Then for $n \geq 2$ one lets

$$E_n^T[X_1, \ldots, X_n] = E[X_1 \cdots X_n] - \sum_{\substack{\pi \in \mathcal{L}_n \\ \pi \neq \hat{1}}} \prod_{I \in \pi} E_{|I|}^T[(X_i)_{i \in I}]$$

For this to make sense, one must show by the same induction that the miltilinear forms E_n^T are *symmetric*. This implies that an expression such as $E_{|I|}^T[(X_i)_{i \in I}]$ is well defined, and does not need the choice of an ordering for the collection of random variables indexed by I. The goal of this note is to use Möbius inversion in order to find an explicit expression for the cumulants in terms of the moments. Note that we denoted by $\mathcal{L}_n$ the partition lattice on the set $[n] = \{1, 2, \ldots, n\}$. We also write $\hat{0}$ for the smallest element, i.e., the partition made of singletons, and we write $\hat{1}$ for the greatest element, i.e., the partition with one block equal to $[n]$ itself.

From the inductive definition of the cumulants one has the fundamental property that

$$E[X_1 \cdots X_n] = \sum_{\pi \in \mathcal{L}_n} \prod_{I \in \pi} E_{|I|}^T[(X_i)_{i \in I}]$$

For given n, and given choice of the possibly repeated random variables $X_1, \ldots, X_n$, this fundamental property also holds for subcollections. Namely for any $I \subseteq [n]$ one has

$$E[\prod_{i \in I} X_i] = \sum_{\pi_I \in \mathcal{L}_I} \prod_{J \in \pi_I} E_{|J|}^T[(X_j)_{j \in J}] \tag{1}$$

where $\mathcal{L}_I$ is the lattice of partitions of the set I. Now define, for any partition $\pi \in \mathcal{L}_n$

$$f(\pi) = \prod_{I \in \pi} E[\prod_{i \in I} X_i]$$

By (1) one has

$$f(\pi) = \prod_{I \in \pi} \sum_{\pi_I \in \mathcal{L}_I} \prod_{J \in \pi_I} E_{|J|}^T[(X_j)_{j \in J}]$$

then collecting the partitions π_I into one partition π' of $[n]$ which is finer than π, one gets

$$f(\pi) = \sum_{\pi' \preceq \pi} g(\pi')$$

with the new function

$$g(\pi) = \prod_{I \in \pi} E^T_{|I|}[(X_i)_{i \in I}]$$

Then by Möbius inversion one has

$$g(\pi) = \sum_{\pi' \preceq \pi} \mu_{\mathcal{L}_n}(\pi', \pi)\ f(\pi')$$

In particular

$$E^T_n[X_1, \ldots, X_n] = g(\hat{1}) = \sum_{\pi \in \mathcal{L}_n} \mu_{\mathcal{L}_n}(\pi, \hat{1}) \prod_{I \in \pi} E[\prod_{i \in I} X_i]$$

Finally, in today's lecture we have seen that

$$\mu_{\mathcal{L}_n}(\pi, \hat{1}) = (-1)^{|\pi|-1}(|\pi| - 1)!$$

so we have an explicit formula for the cumulants. Of course $E^T_2[X_1, X_2] = \mathrm{cov}(X_1, X_2)$ and $E^T_2[X, X] = \mathrm{var}(X)$.