Notes on the cluster expansion for the polymer gas,
a.k.a., the Mayer expansion

Let Λ be the finite set of lattice sites in $\mathbb{Z}^d$. Any nonempty subset $Y \subset \Lambda$ is called a polymer. We denote by $\mathbf{P}(\Lambda)$ the set of all such polymers. On the complex space $\mathbb{C}^{\mathbf{P}(\Lambda)}$ we consider the polynomial (in fact multiaffine) function $\mathcal{Z}_\Lambda$ defined by

$$\mathcal{Z}_\Lambda(z) = \sum_{p\geq 0} \frac{1}{p!} \sum_{Y_1,\ldots,Y_p \in \mathbf{P}(\Lambda)} \mathbb{1}\left\{ \begin{array}{c} \text{the } Y_i \text{ are} \\ \text{disjoint} \end{array} \right\} z(Y_1)\ldots z(Y_p)$$

for any $z = (z(Y))_{Y\in\mathbf{P}(\Lambda)}$ in $\mathbb{C}^{\mathbf{P}(\Lambda)}$. The variable $z(Y)$ is called the activity of the the polymer Y. The purpose of this section is to give an explicit formula for the logarithm of $\mathcal{Z}_\Lambda$. This is called the cluster expansion for the polymer gas in the statistical mechanics literature, whereas in the constructive field theory literature it is rather called the Mayer expansion.

Given a collection $Y_1, \ldots, Y_p$ of polymers, we use the notation $\phi^{\mathrm{T}}(Y_1, \ldots, Y_p)$ for the corresponding Mayer coefficient a.k.a Ursell function. It is defined by

$$\phi^{\mathrm{T}}(Y_1, \ldots, Y_p) = \sum_{\substack{H \rightsquigarrow [p] \\ H \subset G}} (-1)^{|H|}$$

where G is the graph with vertex set $[p] = \{1, \ldots, p\}$ and edges corresponding to the pairs $\{i, j\}$, $i \neq j$, such that $Y_i \cap Y_j \neq \emptyset$. The sum is over all spanning connecting subgraphs H which are identified with their edge sets. We use the notation $H \rightsquigarrow [p]$ to express this condition. Clearly, the Mayer coefficient is a symmetric function of its arguments $Y_1, \ldots, Y_p$.

Remark 1 *One can show that $\phi^T(Y_1, \ldots, Y_p)$ is none other than the Möbius function $\mu_G(\hat{0}, \hat{1})$ of L_G the lattice of contractions of the graph G, i.e., the lattice of partitions of the vertex set $[p]$ whose blocks are internally connected. By the latter we mean that any pair of points in the block can be joined by a path made of edges of G which are contained in the block. This lattice is well known in combinatorics, especially in relation to the chromatic polynomial of such a graph G (see e.g. [8, Exercise 44, p. 162]). We will see that in a forthcoming lecture.*

On the space $\mathbb{C}^{\mathbf{P}(\Lambda)}$ of polymer activities we will use the norm

$$||z|| = \sup_{\mathbf{x}\in\Lambda} \sum_{Y\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{x} \in Y\}\ |z(Y)|\ e^{|Y|}\ .$$

We can now state the following classical result (see, e.g., [3, 1, 4] as well as [7, Ch. V]).

Theorem 1 *The series*

$$f_\Lambda(z) = \sum_{p\geq 1} \frac{1}{p!} \sum_{Y_1,\ldots,Y_p\in\mathbf{P}(\Lambda)} \phi^{\mathrm{T}}(Y_1, \ldots, Y_p)\ z(Y_1)\ldots z(Y_p)$$

is absolutely convergent and defines an analytic function $f_\Lambda(z)$ on the open ball $||z|| < \frac{1}{4}$. On this domain one has

$$\exp f_\Lambda(z) = \mathcal{Z}_\Lambda(z)\ .$$

The following is a classical result in rigorous statistical mechanics (see e.g. [7, Thm. V.7A.6]) which we will need in order to prove Theorem 1.

Lemma 1 *[6, 5] One has the inequality*

$$|\phi^{\mathrm{T}}(Y_1,\ldots,Y_p)| \le \sum_{\substack{\mathcal{T}\ \text{tree} \\ \mathcal{T}\rightsquigarrow[p]}} \prod_{\{i,j\}\in\mathcal{T}} 1\!\!1\{Y_i\cap Y_j\neq\emptyset\}\ .$$

Moreover, the sign of this Mayer coefficient is $(-1)^{p-1}$.

We will see the proof of this lemma in a forthcoming lecture.

Proof of Theorem 1: Let

$$\Gamma = \sum_{p\ge 1}\frac{1}{p!}\sum_{Y_1,\ldots,Y_p\in\mathbf{P}(\Lambda)} |\phi^{\mathrm{T}}(Y_1,\ldots,Y_p)| \times |z(Y_1)|\ldots|z(Y_p)|\ .$$

From Lemma 1 one obtains

$$\begin{aligned}
\Gamma &\le \sum_{p\ge 1}\frac{1}{p!}\sum_{Y_1,\ldots,Y_p\in\mathbf{P}(\Lambda)}\sum_{\substack{\mathcal{T}\ \text{tree} \\ \mathcal{T}\rightsquigarrow[p]}}\left(\prod_{\{i,j\}\in\mathcal{T}} 1\!\!1\{Y_i\cap Y_j\neq\emptyset\}\right)\left(\prod_{i=1}^{p}|z(Y_i)|\right) \\
&\le \sum_{Y\in\mathbf{P}(\Lambda)}|z(Y)| + \sum_{p\ge 2}\frac{1}{p!}\sum_{\substack{d_1,\ldots,d_p\ge 1 \\ \Sigma d_i = 2p-2}}\sum_{\substack{\mathcal{T}\ \text{tree} \\ \mathcal{T}\rightsquigarrow[p]}} 1\!\!1\left\{\begin{array}{c}\forall i,\ \text{degree of} \\ i\ \text{in}\ \mathcal{T}\ \text{is}\ d_i\end{array}\right\} M(\mathcal{T})
\end{aligned}$$

where

$$M(\mathcal{T}) = \sum_{Y_1,\ldots,Y_p\in\mathbf{P}(\Lambda)}\left(\prod_{\{i,j\}\in\mathcal{T}} 1\!\!1\{Y_i\cap Y_j\neq\emptyset\}\right)\left(\prod_{i=1}^{p}|z(Y_i)|\right)\ .$$

Now one has the following 'pin and sum' inequality:

$$M(\mathcal{T}) \le |\Lambda| \times ||z||^p \times d_1! \times \prod_{i=2}^{p}(d_i-1)!\ . \qquad (1)$$

This is shown by choosing a root for the tree, say the vertex numbered 1, then summing recursively over the Y_i for i a leaf of the tree, and progressing towards the root. If j is the parent of the leaf i in the tree one has to bound

$$\begin{aligned}
\sum_{Y_i\in\mathbf{P}(\Lambda)} 1\!\!1\{Y_i\cap Y_j\neq\emptyset\}|z(Y_i)| &\le \sum_{Y_i\in P(\Lambda)}\sum_{\mathbf{x}\in Y_j} 1\!\!1\{\mathbf{x}\in Y_i\}\ |z(Y_i)| \\
&\le |Y_j| \times ||z||
\end{aligned}$$

at which point one erases i and continues the process. However, when one considers a vertex j which is deeper in the initial tree, such a vertex has received an additional factor $|Y_j|$ from each of its offsprings, of which there are $d_j - 1$ if j is different from the root 1, and d_j if $j = 1$ is the root. So in case $j \neq 1$ one has to bound

$$\begin{aligned}
\sum_{Y_j \in \mathbf{P}(\Lambda)} & \mathbb{1}\{Y_j \cap Y_k \neq \emptyset\}\ |Y_j|^{d_j - 1} \times |z(Y_j)| \\
& \le \sum_{Y_j \in \mathbf{P}(\Lambda)} \sum_{\mathbf{x} \in Y_k} \mathbb{1}\{\mathbf{x} \in Y_j\} (d_j - 1)! e^{|Y_j|} |z(Y_j)| \\
& \le |Y_k| \times ||z|| \times (d_j - 1)!
\end{aligned}$$

where k is the parent of j, and we used the inequality $\frac{x^n}{n!} \le e^x$ for x nonnegative.

In case $j = 1$ one writes

$$\begin{aligned}
\sum_{Y_1 \in \mathbf{P}(\Lambda)} & |Y_1|^{d_1} \times |z(Y_1)| \\
& \le \sum_{Y_1 \in \mathbf{P}(\Lambda)} \sum_{\mathbf{x} \in \Lambda} \mathbb{1}\{\mathbf{x} \in Y_1\}\ d_1!\ e^{|Y_1|} |z(Y_1)| \\
& \le |\Lambda| \times ||z|| \times d_1!
\end{aligned}$$

using the fact that polymers here are nonempty and contained in Λ. Hence (1) is established.

As a result

$$\begin{aligned}
\Gamma \quad \le \quad & |\Lambda|.||z|| + \sum_{p \ge 2} \frac{1}{p!} \sum_{\substack{d_1, \ldots, d_p \ge 1 \\ \Sigma d_i = 2p-2}} \sum_{\substack{\mathcal{T} \text{ tree} \\ \mathcal{T} \rightsquigarrow [p]}} \\
& \mathbb{1}\left\{ \begin{array}{c} \forall i,\ \text{degree of} \\ i \text{ in } \mathcal{T} \text{ is } d_i \end{array} \right\} \times |\Lambda| \times ||z||^p \times d_1! \times \prod_{i=2}^{p} (d_i - 1)!\ .
\end{aligned}$$

But by Cayley's second theorem (proof of this soon) which counts labelled trees with fixed vertex degrees one has:

$$\sum_{\substack{\mathcal{T} \text{ tree} \\ \mathcal{T} \rightsquigarrow [p]}} \mathbb{1}\left\{ \begin{array}{c} \forall i,\ \text{degree of} \\ i \text{ in } \mathcal{T} \text{ is } d_i \end{array} \right\} = \frac{(p-2)!}{\prod_{i=1}^{p} (d_i - 1)!}\ .$$

Thus

$$\begin{aligned}
\Gamma \quad \le \quad & |\Lambda| \times ||z|| + \sum_{p \ge 2} \frac{1}{p!} \sum_{\substack{d_1, \ldots, d_p \ge 1 \\ \Sigma d_i = 2p-2}} \frac{(p-2)!}{\prod_{i=1}^{p} (d_i - 1)!} \\
& \times |\Lambda| \times ||z||^p \times d_1! \times \prod_{i=2}^{p} (d_i - 1)!\ .
\end{aligned}$$

We now clean up the factorials and bound the left over factor of d_1 by $d_1 \le p-1$ because the root can have at most $p-1$ (everybody else) neighbors in the tree. Then we have

$$\Gamma \le |\Lambda| \times ||z|| + \sum_{p\ge 2} \frac{|\Lambda| \times ||z||^p}{p} \sum_{\substack{d_1,\ldots,d_p\ge 1\\ \Sigma d_i=2p-2}} 1$$

but, using the change of summation variables $\alpha_i = d_i - 1$, we have

$$\begin{aligned}\sum_{\substack{d_1,\ldots,d_p\ge 1\\ \Sigma d_i=2p-2}} 1 &= \sum_{\substack{\alpha_1,\ldots,\alpha_p\ge 0\\ \Sigma \alpha_i=p-2}} 1 \\ &= \binom{p-2+p-1}{p-1}\end{aligned}$$

simply by counting configurations of $p-1$ black balls in a chain of $p-2+p-1$ white or black balls. Stirling's formula shows that the binomial coefficient behaves like $\frac{1}{\sqrt{\pi p}} 2^{2p-3}$ for large p. Since we only care about the radius of convergence in z we will just use the coarse bound by 2^{2p-3}. Hence,

$$\Gamma \le |\Lambda| \times ||z|| + \sum_{p\ge 2} \frac{1}{p} \times |\Lambda| \times ||z||^p \times 2^{2p-3}$$

which proves the absolute convergence of the series, as well as the existence and analyticity of $f_\Lambda(z)$ in the ball $||z|| < \frac{1}{4}$.

In order to show that $\mathcal{Z}_\Lambda = \exp\ f_\Lambda$, first note that for a sequence of polymers $Y_1, \ldots, Y_p$ one has

$$\begin{aligned}1\!\!1\left\{\begin{array}{c}\text{the } Y_i \text{ are}\\ \text{disjoint}\end{array}\right\} &= \prod_{\{i,j\}\in[p]^{(2)}} \left(1 - 1\!\!1_{\{Y_i\cap Y_j\neq\emptyset\}}\right) \\ &= \sum_{\mathrm{g}\subset[p]^{(2)}} \prod_{\{i,j\}\in\mathrm{g}} \left(-1\!\!1_{\{Y_i\cap Y_j\neq\emptyset\}}\right) \\ &= \sum_{\pi \text{ partition of } [p]} \prod_{J\in\pi} \left[\sum_{\substack{\mathrm{g}_J\subset J^{(2)}\\ \mathrm{g}_J\rightsquigarrow J}} \prod_{\{i,j\}\in\mathrm{g}_J} \left(-1\!\!1_{\{Y_i\cap Y_j\neq\emptyset\}}\right)\right] \\ &= \sum_{\pi \text{ partition of } [p]} \prod_{J\in\pi} \phi^{\mathrm{T}}(Y_J) \qquad (2)\end{aligned}$$

where we simply wrote $\phi^{\mathrm{T}}(Y_J) = \phi^{\mathrm{T}}(Y_{j_1}, \ldots, Y_{j_k})$ for any subset $J = \{j_1, \ldots, j_k\}$ of $[p]$. Now consider the series of nonnegative terms

$$\mathcal{U} = \sum_{p\ge 0} \sum_{Y_1,\ldots,Y_p\in\mathbf{P}(\Lambda)} \sum_{k\ge 0} \sum_{J_1,\ldots,J_k\subset[p]} 1\!\!1\left\{\begin{array}{c}\text{the } J_i \text{ form a}\\ \text{partition of } [p]\end{array}\right\} \frac{1}{p!k!} \left(\prod_{i=1}^{k} |\phi^{\mathrm{T}}(Y_{J_i})|\right) \left(\prod_{j=1}^{p} |z(Y_j)|\right)$$

which a priori belongs to the completed half line $[0,+\infty]$. Without having to worry about convergence, one can rearrange it using the discrete version of Tonelli's Theorem as

$$\mathcal{U}=\sum_{p\ge 0}\frac{1}{p!}\sum_{k\ge 0}\frac{1}{k!}\sum_{J_1,\ldots,J_k\subset[p]}$$
$$\mathbb{1}\left\{\begin{array}{c}\text{the } J_i \text{ form a}\\ \text{partition of } [p]\end{array}\right\}\prod_{i=1}^{k}\left(\sum_{(Y_j)_{j\in J_i}}|\phi^{\mathrm{T}}(Y_{J_i})|\prod_{j\in J_i}|z(Y_j)|\right)$$
$$=\sum_{p\ge 0}\frac{1}{p!}\sum_{k\ge 0}\frac{1}{k!}\sum_{p_1,\ldots,p_k\ge 1}\sum_{J_1,\ldots,J_k\subset[p]}\mathbb{1}\{\forall i,|J_i|=p_i\}$$
$$\mathbb{1}\left\{\begin{array}{c}\text{the } J_i \text{ form a}\\ \text{partition of } [p]\end{array}\right\}\prod_{i=1}^{k}\left(\sum_{Y_1,\ldots,Y_{p_i}}|\phi^{\mathrm{T}}(Y_1,\ldots,Y_{p_i})|\prod_{j=1}^{p_i}|z(Y_j)|\right)$$

since the sums over the polymers only depend on the cardinalities p_i of the sets J_i. Thus

$$\begin{aligned}\mathcal{U} &= \sum_{p\ge 0}\frac{1}{p!}\sum_{k\ge 0}\frac{1}{k!}\sum_{\substack{p_1,\ldots,p_k\ge 1\\ \Sigma p_i=p}}\frac{p!}{p_1!\ldots p_k!}\prod_{i=1}^{k}\left(\sum_{Y_1,\ldots,Y_{p_i}}|\phi^{\mathrm{T}}(Y_1,\ldots,Y_{p_i})|\prod_{j=1}^{p_i}|z(Y_j)|\right)\\ &= \sum_{k\ge 0}\frac{1}{k!}\sum_{p_1,\ldots,p_k\ge 1}\frac{1}{p_1!\ldots,p_k!}\prod_{i=1}^{k}\left(\sum_{Y_1,\ldots,Y_{p_i}}|\phi^{\mathrm{T}}(Y_1,\ldots,Y_{p_i})|\prod_{j=1}^{p_i}|z(Y_j)|\right)\\ &= \exp(\Gamma)<+\infty\ .\end{aligned}$$

As a result, if one does the same calculation again but without the $|\cdot|$'s, the series are absolutely convergent and the manipulations are justified. One therefore gets

$$\exp f_\Lambda(z)=\sum_{p\ge 0}\sum_{Y_1,\ldots,Y_p\in\mathbf{P}(\Lambda)}\sum_{k\ge 0}\sum_{J_1,\ldots,J_k\subset[p]}$$
$$\mathbb{1}\left\{\begin{array}{c}\text{the } J_i \text{ form a}\\ \text{partition of } [p]\end{array}\right\}\frac{1}{p!k!}\left(\prod_{i=1}^{k}\phi^{\mathrm{T}}(Y_{J_i})\right)\left(\prod_{j=1}^{p}z(Y_j)\right)\ ,$$

namely,

$$\begin{aligned}\exp f_\Lambda(z) &= \sum_{p\ge 0}\sum_{Y_1,\ldots,Y_p\in\mathbf{P}(\Lambda)}\sum_{\pi\ \text{partition of}\ [p]}\frac{1}{p!}\left(\prod_{J\in\pi}\phi^{\mathrm{T}}(Y_J)\right)\left(\prod_{j=1}^{p}z(Y_j)\right)\\ &= \mathcal{Z}_\Lambda(z)\end{aligned}$$

because of (2). ■

Remark 2 *We essentially followed the presentation due to Cammarota [1] which perhaps is the simplest and most direct. The $||z||$ small condition is often referred to as the Kotecký-Preiss criterion [4]. However, it is not optimal. The best available bounds can be found in [2].*

We can isolate from the proof of Theorem 1 the following lemma.

Lemma 2 *For polymer activities $z(Y)$, $Y \in \mathbf{P}(\Lambda)$, such that $||z|| < \frac{1}{4}$ we have the bound*

$$\sup_{\mathbf{z}\in\Lambda} \sum_{p\ge 1} \frac{1}{p!} \sum_{Y_1,\ldots,Y_p\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z} \in \cup_{q=1}^{p} Y_q\}$$
$$\times|\phi^{\mathrm{T}}(Y_1,\ldots,Y_p)| \times |z(Y_1)|\ldots|z(Y_p)| \le \frac{4||z||}{1-4||z||} \ . \qquad (3)$$

Proof: First fix some $\mathbf{z}$ and write

$$\mathbb{1}\{\mathbf{z} \in \cup_{q=1}^{p} Y_q\} \le \sum_{q_0=1}^{p} \mathbb{1}\{\mathbf{z} \in Y_{q_0}\}$$

so that the left-hand side Γ_0 of (3), without the sup over $\mathbf{z}$, will satisfy

$$\begin{aligned}\Gamma_0 \quad \le \quad & \sum_{Y\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z} \in Y\}|z(Y)| \\ & + \sum_{p\ge 2} \frac{1}{p!} \sum_{q_0=1}^{p} \sum_{\substack{d_1,\ldots,d_p\ge 1 \\ \Sigma d_i = 2p-2}} \sum_{\substack{\mathcal{T} \text{ tree} \\ \mathcal{T}\rightsquigarrow[p]}} \mathbb{1}\left\{ \begin{array}{c} \forall i, \text{ degree of} \\ i \text{ in } \mathcal{T} \text{ is } d_i \end{array} \right\} M_{q_0}(\mathcal{T})\end{aligned}$$

with

$$M_{q_0}(\mathcal{T}) = \sum_{Y_1,\ldots,Y_p\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z} \in Y_{q_0}\} \left(\prod_{\{i,j\}\in\mathcal{T}} \mathbb{1}\{Y_i \cap Y_j \ne \emptyset\} \right) \left(\prod_{i=1}^{p} |z(Y_i)| \right) \ .$$

By the same argument as for inequality (1) one has

$$M_{q_0}(\mathcal{T}) \le ||z||^p \times d_{q_0}! \times \prod_{\substack{i=1 \\ i\ne q_0}}^{p} (d_i - 1)! \ .$$

The only difference is that one *has* to use q_0 as a root for the recursive leaf summation procedure, and one does not have to pay a large volume factor $|\Lambda|$ at the last stage of summing over Y_{q_0}. This last sum is done by

$$\begin{aligned}& \sum_{Y_{q_0}\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z} \in Y_{q_0}\} \ |Y_{q_0}|^{d_{q_0}} \times |z(Y_{q_0})| \\ & \quad \le \sum_{Y_{q_0}\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z} \in Y_{q_0}\} \ d_{q_0}! \ e^{|Y_{q_0}|} |z(Y_{q_0})| \\ & \quad \le ||z|| \times d_{q_0}!\end{aligned}$$

where there is *no* factor of $|\Lambda|$.

Then we proceed with the same bounds as before, this time making use of the left over $\frac{1}{p}$ to beat the sum over q_0 from 1 to p. In sum, we get

$$\begin{aligned}\Gamma_0 &\leq ||z|| + \sum_{p\geq 2} ||z||^p \times 2^{2p-3} \\ &\leq \sum_{p\geq 1} (4||z||)^p\end{aligned}$$

and the result follows. ■

Remark 3 *The important feature of this lemma is that the bound it provides is uniform in Λ, so it still holds if we work on the infinite lattice $\mathbb{Z}^d$ instead of a finite volume Λ.*

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