

## Math 845, Spring 2009, Topics in Mathematical Physics

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Location and times: MWF from 10 am to 10:50 am, in KER 326.

Office hours: MW from 1pm to 2pm and by appointment.

Prerequisites: Math 731 or instructor permission. Recommended but not required: Math 732, Math 745.

Topics: The main object of study in this course are functional integrals, i.e., measures on spaces of functions or distributions (Sobolev spaces, Schwartz spaces) which arise in modern statistical mechanics and quantum field theory. Such measures also arise as invariant measures for some partial differential equations, i.e., infinite-dimensional dynamical systems. The goal of this course is to provide you with the essential tools needed in order to tackle research problems in the area. These are Feynman diagrams, the cluster and Mayer expansions, perturbative renormalization and the rigorous theory of the renormalization group. A typical example of measure one would like to study is that of a random function  $q : \mathbb{R}^d \rightarrow \mathbb{R}$  with a probability density distribution of the form  $\exp(-V(q))Dq$ . Here  $Dq$  should be thought of as the Lebesgue measure on the function space to which  $q$  belongs.  $V(q)$  is the integral  $dx$  over  $\mathbb{R}^d$  of a local density of the form  $a(\nabla q)^2(x) + bq(x)^2 + cq(x)^4$ . The previous description of the so called phi-four model is what can be found in any physics textbook on quantum field theory. However, providing a rigorous mathematical definition of such a measure on function space is already a challenging question. The standard approach is akin to what people do in numerical analysis, we first discretize the problem: we introduce a lattice  $(t\mathbb{Z})^d$  with mesh  $t$  in order to approximate the continuum  $\mathbb{R}^d$ , we use Riemann sums adapted to this lattice instead of integrals  $dx$ , we use finite differences instead of partial derivatives (in the  $\nabla q$  term). We also truncate the system by only considering what happens in a finite box  $B$ . Then we take the limits  $B \rightarrow \mathbb{R}^d$  (the infinite volume limit) and  $t \rightarrow 0$  (the continuum or scaling limit), and see what happens. Does one obtain in the limit a well defined measure on a suitable function or distribution space, for instance  $S'(\mathbb{R}^d)$  the space of tempered distributions? It turns out that it is usually necessary to fine tune the parameters  $a, b, c$  as  $t$  is taken to 0, this is called renormalization. In order to understand how that works, a good starting point is to look at the moments of the sought for measure as formal power series in  $c$ . These power series are best expressed using the mentioned Feynman diagrams for which we will provide a mathematically oriented introduction. This will allow us to prove the perturbative (i.e. in the sense of power series) renormalizability of the previous model in dimension  $d = 2, 3$  and 4. This is already an important result. The greatest insights in this area have been recognized by several Nobel prizes in physics: Tomonaga-Schwinger-Feynman (1965), Wilson (1982), 't Hooft-Veltman (1999), Gross-Politzer-Wilczek (2004).

The other thread we will follow in this course has to do with the infinite volume limit  $B \rightarrow \mathbb{R}^d$  on a fixed lattice. The nonperturbative methods needed for that purpose are the cluster and Mayer expansions which we will introduce. Our presentation will make use of recent technical improvements to which the instructor has contributed and which have not been taught before. We will first use these methods to provide an alternative treatment of the high-temperature expansion for the Ising model seen in the fall 08 course by Prof. Thomas. We will then proceed to the phi-four model mentioned above.

Finally, if time permits, we will present the mathematical theory which puts all these ingredients together, following the framework of D. C. Brydges and collaborators. Realistically, we will probably only cover the case of hierarchical models where one sees most clearly the main concepts at work.

Course web page: <http://people.virginia.edu/aa4cr/Math845.html>

Assessment: Attendance and class participation are required and expected. Students will also be asked to read a research article in the area and give an oral presentation, another possibility would be to write a neat and detailed set of lecture notes on some sections of this course.

References:

- - “Renormalization. An Introduction” by Manfred Salmhofer. Springer Verlag. (Required, and available at the UVa bookstore)
- - “Clustering bounds on  $n$ -point correlations for unbounded spin system” by A. Abdesselam, A. Procacci and B. Scoppola. To be posted soon on [arxiv.org](http://arxiv.org)
- - “From perturbative to constructive renormalization” by Vincent Rivasseau, Princeton Univ. Press. Available online at <http://cpth.polytechnique.fr/cpth/rivass/articles/book.ps>
- - “Lectures on the renormalisation group” by D.C. Brydges. Available online at <http://www.math.ubc.ca/db5d/Seminars/PCMIlectures/lectures.pdf>