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par

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RENORMALISATION CONSTRUCTIVE EXPLICITE

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A mes parents

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Chapitre I

Théorie constructive et renormalisation perturbative: une introduction

La théorie des champs est, si l'on veut la définir dans sa plus vaste généralité, l'étude de modèles quantiques avec une infinité de degrés de liberté. Typiquement, l'objet que l'on quantifie est un champ $(\phi(x))_{x\in\mathbb{R}^d}$. Pour chaque position x, $\phi(x)$ est une observable quantique décrite par un opérateur non borné sur l'espace de Hilbert $\mathcal{H}$ des états. Cependant, on ne peut réaliser naïvement le champ quantifié comme une fonction $x \mapsto \phi(x)$ à valeurs opératorielles car des quantités telles que $< \psi|\phi(x)^2|\psi >$ sont singulières. On doit plutôt considérer le champ comme une distribution en x à valeurs dans les opérateurs non bornés dans $\mathcal{H}$. On ne peut ainsi calculer véritablement que des quantités du type $\int \phi(x)f(x)dx$ où f est une fonction d'essai $f \in \mathcal{D}(\mathbb{R}^d)$. Ceci, d'ailleurs, est conforme à l'intuition physique de l'impossibilité de mesurer le champ en un point donné, à cause du principe d'incertitude de Heisenberg, d'après les arguments de Bohr et Rosenfeld [BR1, BR2].

Le premier cadre axiomatique rigoureux dans lequel fut formulée la théorie des champs est dû à Gårding et Wightman [GW1, GW2, W]. Grâce à leur théorème de reconstruction, il suffit pour définir l'espace $\mathcal{H}$ des états, la distribution opératorielle $\phi(x)$ et une représentation associée du groupe de Poincaré, de se donner un jeu de distributions $W_n(x_1, \ldots, x_n)$ sur $\mathbb{R}^{4n}$ vérifiant les axiomes suivants.

(W0) Propriété de distribution:

$W_0 = 1$ et pour tout $n \geq 1$, $W_n(\underline{x})$ est une distribution dans $\mathcal{S}'(\mathbb{R}^{4n})$. On utilise la notation $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^{4n}$, $x_i = (x_i^0, \vec{x}_i) \in \mathbb{R}^4$. $\mathbb{R}^4$ est l'espace-temps habituel muni de la métrique de Minkowski de signature $(1, -1, -1, -1)$. x_i^0 est la variable de temps, $\vec{x}_i$ est la variable d'espace.

(W1) Covariance relativiste:

Pour tout n, W_n est invariante par le groupe de Poincaré propre $\mathcal{P}_+^\uparrow$, c'est-à-dire pour tout $(a,\Lambda) \in \mathcal{P}_+^\uparrow$,

$$W_n(\underline{x}) = W_n(\Lambda\underline{x} + a) \tag{I.1}$$

où $\Lambda\underline{x} + a \stackrel{\text{def}}{=} (\Lambda x_1 + a, \ldots, \Lambda x_n + a)$.

(W2) Positivité:

Pour toute suite finie $f_0, f_1, \ldots, f_N$ de fonctions d'essai, $f_0 \in \mathbb{C}$, $f_n \in \mathcal{S}(\mathbb{R}^{4n})$, $n = 1, \ldots, N$, on a

$$\sum_{n,m=0}^{N} W_{n+m}(f_n^* \otimes f_m) \geq 0 \tag{I.2}$$

où $(f_n \otimes f_m)(\underline{x},\underline{y}) \stackrel{\text{def}}{=} f_n(\underline{x}).f_m(\underline{y})$ et $f_n^*(x_1, x_2, \ldots, x_n) \stackrel{\text{def}}{=} \overline{f_n(x_n, x_{n-1}, \ldots, x_1)}$.

(W3) Localité:

Pour tout n et tout $k = 1, \ldots, n-1$

$$W_n(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = W_n(x_1, \ldots, x_{k+1}, x_k, \ldots, x_n) \tag{I.3}$$

pourvu que $x_k - x_{k+1}$ soit de genre espace, c'est-à-dire $(x_k - x_{k+1})^2 < 0$.

(W4) Indépendance asymptotique des amas:

Pour tout vecteur a de genre espace, tout $k = 1, \ldots, n-1$, et tout $(\underline{x},\underline{y}) \in \mathbb{R}^{4k} \times \mathbb{R}^{4(n-k)}$

$$\lim_{s \to +\infty} W_n(\underline{x}, \underline{y} + sa) = W_k(\underline{x}).W_{n-k}(\underline{y}) \quad . \tag{I.4}$$

(W5) Condition spectrale:

L'invariance par translation de W_n, entraîne qu'il existe une distribution $w_{n-1} \in \mathcal{S}'(\mathbb{R}^{4(n-1)})$ telle que $W_n(\underline{x}) = w_{n-1}(\underline{\xi})$, où $\underline{\xi} = (\xi_1, \ldots, \xi_{n-1})$ avec $\xi_k = x_{k+1} - x_k$. Si

$$\tilde{w}_{n-1} \stackrel{\text{def}}{=} (2\pi)^{-4(n-1)} \int d^{4(n-1)}\underline{\xi} \exp\left(-\sum_{k=1}^{n-1} q_k.\xi_k\right) w_{n-1}(\underline{\xi}) \tag{I.5}$$

désigne la transformée de Fourier de w_{n-1}, le présent axiome impose la condition

$$\text{supp}(\tilde{w}_{n-1}) \subset \overline{V}_+^{n-1} \stackrel{\text{def}}{=} \{\underline{q} | q_i \in \overline{V}_+, i = 1, \ldots, n-1\} \quad . \tag{I.6}$$

$\overline{V}_+$ désigne ici le cône fermé du futur, c'est-à-dire $\overline{V}_+ \stackrel{\text{def}}{=} \{x \in \mathbb{R}^4 | x^0 \geq 0, x.x \geq 0\}$.

Les identités précédentes sur les W_n sont, bien sûr, à interpréter au sens des distributions. Les fonctions W_n correspondent aux valeurs moyennes par rapport au vide Ω de produits d'opérateurs de champs

$$W_n(x_1, \ldots, x_n) = < \Omega | \phi(x_1) \ldots \phi(x_n) | \Omega > \quad . \tag{I.7}$$

Ces axiomes fournissent les conditions nécessaires minimales pour une théorie des champs raisonnable. Encore faut-il montrer qu'un tel modèle existe. Jusqu'au début des années soixante-dix, les seuls exemples de théories vérifiant ces axiomes correspondaient à des hamiltoniens quadratiques, c'est-à-dire, des modèles sans interaction présentant peu d'intérêt pour la physique. L'objet de la théorie constructive des champs est de montrer l'existence mathématique de modèles non triviaux vérifiant les axiomes précédents. Bien qu'il soit possible, par des méthodes hamiltoniennes, de construire de tels modèles dans l'espace de Minkowski dans le cas des théories $P(\phi)$ à deux dimensions et ϕ^4 à trois dimensions [GJ1, GJ2, GJ3, GJ4], le véritable essor de la théorie constructive était dû à l'introduction des méthodes euclidiennes [Sy1, Sy2]. L'idée est de continuer analytiquement les coordonnées de temps vers l'axe imaginaire, opération appelée rotation de Wick. Il existe plusieurs formulations axiomatiques d'une théorie des champs euclidienne qui permettent de revenir au cadre minkowskien et de construire un jeu de fonctions de Wightman satisfaisant aux axiomes W0-W5. La plus ancienne est due à Nelson et fait appel à des champs aléatoires markoviens sur $\mathbb{R}^4$ euclidien [N1, N2, N3, N4]. Il existe aussi une formulation due à Frohlich qui n'est pas optimale mais qui est plus facile à vérifier sur un modèle donné [F, Si]. Enfin, les axiomes euclidiens les plus populaires, permettant de retrouver ceux de Wightman, sont dus à Osterwalder et Schrader [OS1, OS2]. Ils peuvent s'énoncer en termes de fonctions de Schwinger $S_n(x_1, \ldots, x_n)$ qui sont les continuées analytiques des fonctions de Green de la théorie considérée, c'est-à-dire des fonctions

$$G_n(x_1, \ldots, x_n) \stackrel{\text{def}}{=} < \Omega | T(\phi(x_1) \ldots \phi(x_n)) | \Omega > \tag{I.8}$$

où T désigne le T-produit habituel où l'on range les champs de droite à gauche par ordre croissant de la composante de temps x_i^0.

Ici encore, les fonctions de Schwinger S_n sont des distributions et les axiomes à vérifier sont les suivants.

(OS0) Propriété de distribution et régularité:

$S_0 = 1$, $S_n \in \mathcal{S}'(\mathbb{R}^{4n})$, il existe une semi-norme $|.|$ sur $\mathcal{S}(\mathbb{R}^4)$, et trois constantes $a, b, c \in \mathbb{R}_+^*$, telles que pour tout n et toutes fonctions d'essai $f_1, \ldots, f_n$ dans $\mathcal{S}(\mathbb{R}^4)$,

$$|S_n(f_1 \otimes \cdots \otimes f_n)| \leq a.b^n.(n!)^c. \prod_{i=1}^{n} |f_i| \quad . \tag{I.9}$$

(OS1) Invariance euclidienne:

$$S_n(Ex_1, \ldots Ex_n) = S_n(x_1, \ldots, x_n) \tag{I.10}$$

pour toute transformation E composée d'une rotation et d'une translation sur $\mathbb{R}^4$.

(OS2) Positivité par reflexion:

$$\sum_{i,j=0}^{n} (\theta f_i^* \otimes f_j) \geq 0 \tag{I.11}$$

pour toutes fonctions d'essai $f_0, \ldots, f_n$, $f_0 \in \mathbb{C}$, $f_i \in \mathcal{S}_{i,+}(\mathbb{R}^{4i})$, où $\mathcal{S}_{i,+}(\mathbb{R}^{4i})$ est l'ensemble des fonctions d'essai $f \in \mathcal{S}(\mathbb{R}^{4i})$ telles que $f(x_1, \ldots, x_i) = 0$ à moins que $0 < x_1^0 < \cdots < x_i^0$ et telles qu'elles s'annulent ainsi que leurs dérivées si $x_k = x_l$ pour $1 \leq k < l \leq i$. θf désigne la fonction $\theta f(x_1, \ldots, \theta x_i) \stackrel{\text{def}}{=} f(\theta x_1, \ldots, \theta x_i)$ où par définition $\theta(x^0, \vec{x}) \stackrel{\text{def}}{=} (-x^0, \vec{x})$ pour tout $(x^0, \vec{x}) \in \mathbb{R}^4$.

(OS3) Symétrie:

$$S_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = S_n(x_1, \ldots, x_n) \tag{I.12}$$

pour toute permutation σ de $\{1, \ldots, n\}$.

(OS4) Indépendance asymptotique des amas:

Pour tout vecteur $a \neq 0$ de $\mathbb{R}^4$, tout $k = 1, \ldots, n-1$, et tout $(\underline{x}, \underline{y}) \in \mathbb{R}^{4k} \times \mathbb{R}^{4(n-k)}$

$$\lim_{s \to +\infty} S_n(\underline{x}, \underline{y} + sa) = S_k(\underline{x}).S_{n-k}(\underline{y}) \quad . \tag{I.13}$$

Ces propriétés sont encore à prendre au sens des distributions. Les axiomes d'Osterwalder-Schrader ne sont pas strictement équivalents à ceux de Wightman. Le problème de l'équivalence n'a été résolu que très récemment par Zinoviev en introduisant une petite modification aux axiomes OS0-OS5 [Zin].

L'approche euclidienne, non seulement, facilite la construction de nouveaux modèles, mais établit un pont conceptuel avec la physique statistique. En effet, dans le cas de la théorie ϕ^4 par exemple, les fonctions de Schwinger s'écrivent, au moins formellement, comme les moments

$$S_n(x_1, \ldots, x_n) = < \phi(x_1) \ldots \phi(x_n) > \stackrel{\text{def}}{=} \int \phi(x_1) \ldots \phi(x_n) d\mu(\phi) \tag{I.14}$$

d'une mesure de probabilité $d\mu$ sur un champ aléatoire scalaire $\phi : \mathbb{R}^4 \to \mathbb{R}$, obtenue par perturbation d'une mesure gaussienne

$$d\mu(\phi) \stackrel{\text{def}}{=} \frac{1}{Z} \exp\left(-\frac{g}{4!} \int \phi^4 - \frac{m^2}{2} \int \phi^2 - \frac{a}{2} \int (\partial\phi)^2 \right) D\phi \quad . \tag{I.15}$$

$D\phi$ désigne le produit formel $\prod_{x \in \mathbb{R}^4} d\phi(x)$ des mesures de Lebesgue en chaque point de $\mathbb{R}^4$. Exception faite des théories exactement résolubles telles que les théories conformes à deux dimensions, le seul moyen que l'on connaisse, pour construire un

jeu de fonctions de Schwinger satisfaisant aux axiomes d'Osterwalder-Schrader de manière non triviale, est de partir des équations formelles (I.15) et (I.14) et d'essayer de leur donner un sens mathématiquement rigoureux. Grâce à la théorie des processus stochastique gaussiens généralisés, développée par Minlos, Gelfand, Vilenkin et Hida, on sait construire $d\mu$ lorsque $g = 0$, c'est-à-dire, la mesure gaussienne [M, GV, Hi1, Hi2, E, GJ5]. En effet, on dispose du théorème suivant:

Théorème de Bochner-Minlos

Etant donné une fonction $c : \mathcal{S}(\mathbb{R}^4) \to \mathbb{C}$ *telle que*

(i) c *est continue pour la topologie de* $\mathcal{S}(\mathbb{R}^4)$,

(ii) pour tout $n \geq 1$, *tous nombres complexes* $z_1, \ldots, z_n$ *et toutes fonctions d'essai* $f_1, \ldots, f_n$ *dans* $\mathcal{S}(\mathbb{R}^4)$

$$\sum_{i,j=1}^{n} \overline{z}_i z_j c(f_i - f_j) \in \mathbb{R}_+ \quad , \tag{I.16}$$

(iii) $c(0) = 1$,

il existe une unique mesure de probabilité $d\nu$ *sur* $\mathcal{S}'(\mathbb{R}^4)$ *muni de la tribu cylindrique telle que* c *soit la fonction caractéristique de* $d\nu$, *c'est-à-dire:*

$$\forall f \in \mathcal{S}(\mathbb{R}^4), \quad c(f) = \int e^{i<\phi,f>} d\nu(\phi) \quad . \tag{I.17}$$

La tribu cylindrique est celle engendrée par les ensembles

$$\{\phi \in \mathcal{S}'(\mathbb{R}^4) | \phi(f_1) \in B_1, \ldots, \phi(f_n) \in B_n\} \tag{I.18}$$

où $f_1, \ldots, f_n$ sont des éléments de $\mathcal{S}(\mathbb{R}^4)$ et $B_1, \ldots, B_n$ sont des boréliens de $\mathbb{R}$. Le théorème de Bochner-Minlos est valable plus généralement lorsque l'on prend à la place de $\mathcal{S}(\mathbb{R}^4)$ un espace nucléaire E, c'est-à-dire, une intersection dénombrable d'espaces de Hilbert emboîtés par des inclusions à trace. On peut, alors, construire une mesure de probabilité sur le dual topologique E^*. Dans [AR2], l'espace nucléaire utilisé est $\mathcal{S}(\mathbb{R}^4)$, tandis que dans [A], c'est l'espace des fonctions indéfiniment dérivables avec conditions aux bords périodiques dans une boîte Λ. Notons, au passage, que l'espace $\mathcal{D}(\mathbb{R}^4)$ n'est pas un espace nucléaire au sens restreint précédent, mais une limite inductive de tels espaces. On peut aussi construire des mesures gaussiennes analogues sur son dual.

Il suffit pour construire la mesure gaussienne dans (I.15), de poser

$$c(f) \stackrel{\text{def}}{=} e^{-\frac{1}{2}B(f,f)} \tag{I.19}$$

où $B(f,g)$ est la forme bilinéaire continue sur $\mathcal{S}(\mathbb{R}^4)$ définie par

$$B(f,g) \stackrel{\text{def}}{=} < f, (-\Delta + m^2)^{-1} g > \quad . \tag{I.20}$$

$< ., . >$ ici désigne le produit scalaire L^2, et Δ est le laplacien ordinaire sur $\mathbb{R}^4$ muni de la métrique euclidienne.

Cependant, la mesure $d\nu$ ainsi construite, lorsqu'on est à quatre dimensions, est supportée sur des champs ϕ qui sont des distributions singulières. Ajouter le facteur $\exp(-\frac{g}{4!}\int \phi^4)$ n'a donc pas de sens dans (I.15), car $\phi(x)^4$ n'est pas une quantité bien définie. Il est donc nécessaire de régulariser la théorie. Une façon de procéder est de se placer sur un réseau de maille $\alpha > 0$. Une autre, celle utilisée dans [AR2] et [A], est de tronquer la forme bilinéaire B en moment. En effet, B est la forme bilinéaire associée à l'opérateur de covariance C dont le noyau est donné par $C(x, y) = C(x - y)$, avec en transformée de Fourier

$$\tilde{C}(p) = \frac{1}{(2\pi)^4} \times \frac{1}{p^2 + m^2} \quad . \tag{I.21}$$

Introduire une tronquature en moment revient à remplacer $\tilde{C}$ par exemple par

$$\tilde{C}_\kappa(p) = \frac{1}{(2\pi)^4} \times \frac{e^{-\kappa(p^2+m^2)}}{p^2 + m^2} \quad . \tag{I.22}$$

Une autre régularisation utile est de confiner l'interaction ϕ^4 à un volume fini Λ. Ces précautions étant prises, (I.15) définit de façon mathématiquement rigoureuse, une mesure sur $\mathcal{S}'(\mathbb{R}^4)$. En effet, la mesure gaussienne est supportée sur des fonctions indéfiniment dérivables. Le problème est maintenant d'étudier la limite de la mesure régularisée $d\mu_{\kappa,\Lambda}$ lorsque $\Lambda \nearrow \mathbb{R}^4$ et $\kappa \to 0$. C'est une question hautement non triviale qu'il faut d'abord étudier à la lumière de la théorie des perturbations.

Lorsqu'on développe naïvement les fonctions de Schwinger en perturbation par rapport à g, les termes du développement obtenu sont sommes d'un nombre fini d'intégrales de dimension finie représentées par des diagrammes de Feynman. Lorsqu'on prend la limite $\kappa \to 0$, ces intégrales divergent. Il apparait donc que l'on ne peut construire, en général, la limite des mesures régularisées $d\mu_\kappa$ en gardant les paramètres g, m^2 et a fixés.

La théorie de la renormalisation perturbative permet de trouver la bonne façon de faire varier ces paramètres en fonction de κ pour obtenir une limite de $d\mu_\kappa$, du moins au niveau des séries formelles en g. Oublions, pour l'instant, la limite $\Lambda \nearrow \mathbb{R}^4$ et écrivons les paramètres dans (I.15) comme séries formelles dans une nouvelle constante de couplage g_r:

$$g = g_r + \sum_{n \geq 2} g_r^n c_n(\kappa) \tag{I.23}$$

$$m^2 = m_r^2 + \sum_{n \geq 1} g_r^n d_n(\kappa) \tag{I.24}$$

$$a = a_r + \sum_{n\geq 1} g_r^n e_n(\kappa) \quad . \tag{I.25}$$

Les coefficients de ces séries formelles sont obtenus en résolvant le système suivant de conditions de renormalisation:

$$\tilde{S}_4^T(0,0,0,0) = -g_r \tag{I.26}$$

$$\tilde{S}_2^T(p^2=0) = \frac{1}{m_r^2} \tag{I.27}$$

$$\frac{d}{dp^2}\tilde{S}_2^T(p^2)|_{p^2=0} = -\frac{a_r}{m_r^4}. \tag{I.28}$$

$\tilde{S}_n^T$ désigne la transformée de Fourier de la fonction de corrélation connexe ou tronquée S_n^T obtenue à partir de S_n par la définition inductive suivante:

- $S_1^T = S_1$
- pour $n \geq 2$,

$$S_n^T(x_1,\ldots,x_n) = \sum_{\pi} \prod_{I\in\pi} S_{\#(I)}^T(x_I) \quad . \tag{I.29}$$

La somme sur π est sur toutes les partitions de $\{1,\ldots,n\}$ et si $X = \{i_1,\ldots,i_p\}$, $S_{\#(I)}^T(x_I) \stackrel{\text{def}}{=} S_p^T(x_{i_1},\ldots,x_{1_p})$. Le miracle de la renormalisation est le fait qu'après le choix arbitraire fait en (I.26), (I.27) et (I.28) concernant les fonctions de Schwinger à deux et quatre points, pour toute fonction de Schwinger d'ordre quelconque exprimée comme série formelle en g_r, les coefficients de cette série *ont une limite finie lorsque* $\kappa \to 0$. Les idées fondatrices de cette méthode sont dues à Dyson, Feynman, Schwinger et Tomonaga, au lendemain de la deuxième guerre mondiale, mais la preuve rigoureuse de ce résultat a occupé les physiciens mathématiciens pendant plus d'un quart de siècle [BP, He1, He2, Zim, BL].

Lorsqu'on calcule la fonction de corrélation connexe en transformation de Fourier $\tilde{S}_N^T(p_1,\ldots,p_N)$ comme série formelle en g_r, le coefficient de g_r^n est la somme d'amplitudes $A_G^R(p_1,\ldots,p_n)$ sur tous les graphes connexes G ayant deux types de sommets: les pattes externes numérotées de 1 à N avec coordinance 1, et n sommets internes de coordinance 4. L'amplitude renormalisée est donnée explicitement, après passage à la limite $\kappa \to 0$, par la formule de forêt de Zimmermann [Zim]:

$$A_G^R(p_1,\ldots,p_n) = \int \prod_{l\in G} \frac{d^4p_l}{(2\pi)^4} \mathcal{R}\left\{ \prod_{l\in G} \frac{1}{p_l^2+m^2} \prod_{\nu\in G} (2\pi)^4 \delta^{(4)}(\sum_l \epsilon_{\nu,l} p_l) \right\} \quad . \tag{I.30}$$

l désigne génériquement une ligne du graphe et ν un sommet. $(\epsilon_{\nu,l})$ est la matrice d'incidence définissant le graphe G avec un choix d'orientation des lignes. $\epsilon_{\nu,l} \in \{-1,0,1\}$. Pour une ligne l donnée, il existe un ν tel que $\epsilon_{\nu,l} = 1$, un autre ν'

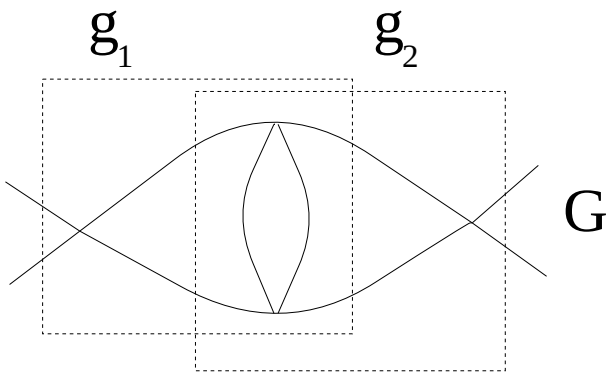

Figure 1

avec $\epsilon_{\nu',l} = -1$ et tous les autres $\epsilon_{\nu'',l}$ sont nuls. $\mathcal{R}$ est un opérateur agissant sur l'intégrand entre accolades, défini par

$$\mathcal{R} \stackrel{\text{def}}{=} \sum_{\mathcal{F}} \prod_{g \in \mathcal{F}} (-\tau_g) \tag{I.31}$$

où la somme sur $\mathcal{F}$ est sur toutes les forêts de sous-graphes superficiellement divergents de G, c'est-à-dire, les ensembles de sous-graphes à supports disjoints ou emboîtés faits de sous-graphes à deux ou quatre lignes externes. τ_g est un opérateur défini comme suit. Soit $g(x_1, \ldots, x_e)$ l'amplitude correspondant au sous-graphe g, sa transformée de Fourier s'écrit $(2\pi)^4 \delta^{(4)}(k_1 + \cdots + k_e)\hat{g}(k_1, \ldots, k_e)$. $\tau_g g$ est l'amplitude obtenue en remplaçant $\hat{g}$ par son développement de Taylor au voisinage de 0

$$\sum_{j=0}^{D(g)} \frac{1}{j!} \frac{d^j}{dt^j} \hat{g}(tk_1, \ldots, tk_e)|_{t=0} \tag{I.32}$$

où $D(g)$ est le degré de divergence superficiel de g. En dimension quelconque d

$$D(g) \stackrel{\text{def}}{=} d \times \#(\text{boucles de } G) - 2 \times \#(\text{lignes internes de } G) \tag{I.33}$$

et à quatre dimensions

$$D(g) = N(g) - 4 \tag{I.34}$$

où $N(g)$ est le nombre de lignes externes de g. L'opérateur τ_g, comme on peut le voir dans l'espace des x, correspond à mettre les variables de position des fonctions d'essai convolées avec l'amplitude de g, au même point. τ_g est ainsi appelé l'opérateur de partie locale de g. Le théorème BPHZ affirme alors que l'intégrale dans (I.30) est convergente.

Une des difficultés majeures de la preuve de ce théorème est due à la possible présence dans G de sous-diagrammes divergents s'intersectant non trivialement, par

exemple g_1 et g_2 dans la figure 1. Pour résoudre cette difficulté, il est nécessaire de découper l'intégrale sur les moments selon l'ordre relatif des $|p_l|$ (en fait, c'est plutôt celui des variables duales α_l dans la représentation paramétrique). Un tel domaine d'intégration s'appelle un secteur de Hepp. Une autre implémentation de cette idée, où l'on voit poindre celle de groupe de renormalisation, est de décomposer le propagateur comme somme de propagateurs d'échelles différentes en progression géométrique. Par exemple, on prend $\kappa = M^{-2\rho}$ avec ρ entier et on écrit:

$$C_\kappa(x,y) = \frac{1}{(2\pi)^4}\int d^4p\ e^{ip(x-y)}\frac{e^{-\kappa(p^2+m^2)}}{p^2+m^2} \tag{I.35}$$

$$= \frac{1}{(4\pi)^2}\int_\kappa^{+\infty}\frac{d\alpha}{\alpha^2}\exp\left(-\alpha m^2 - \frac{|x-y|}{4\alpha}\right) \tag{I.36}$$

$$= \sum_{i=0}^{\rho} C^i(x,y) \tag{I.37}$$

avec

$$C^0(x,y) \stackrel{\text{def}}{=} \frac{1}{(4\pi)^2}\int_1^{+\infty}\frac{d\alpha}{\alpha^2}\exp\left(-\alpha m^2 - \frac{|x-y|}{4\alpha}\right) \tag{I.38}$$

et, pour $i \geq 1$

$$C^i(x,y) \stackrel{\text{def}}{=} \frac{1}{(4\pi)^2}\int_{M^{-2i}}^{M^{-2(i-1)}}\frac{d\alpha}{\alpha^2}\exp\left(-\alpha m^2 - \frac{|x-y|}{4\alpha}\right) \quad . \tag{I.39}$$

Les moments p contribuent à la transformée de Fourier du propagateur C^i sont typiquement d'ordre de grandeur $|p| \approx M^i$. Par ailleurs, il existe des constantes K et δ telles qu'uniformément en i, x et y

$$|C^i(x,y)| \leq KM^{2i}\exp(-\delta M^i|x-y|) \quad . \tag{I.40}$$

En utilisant cette décomposition, on peut montrer des résultats plus forts que le théorème BPHZ [dCR, FMRS1, FMRS2, R1]:

Théorème de Weinberg uniforme:

L'amplitude d'un graphe sans sous-graphes divergents est bornée par $(\mathrm{Cst})^n$*, où* n *est le nombre de sommets.*

Théorème BPHZ uniforme:

L'amplitude renormalisée d'un graphe de Feynman est bornée par $(\mathrm{Cst})^n.n!$.

Pour aller plus loin, c'est-à-dire, construire les distributions S_n satisfaisant les axiomes d'Osterwalder-Schrader, on doit trouver un moyen de resommer la série des perturbations. Un simple coup d'œil à (I.15) suffit à se convaincre qu'il est sans espoir de chercher un rayon d'analyticité en g autour de zéro, pour la série des perturbations, même en présence des régularisations, pour des théories bosoniques.

Au vu du théorème de Weinberg uniforme, la série des perturbations semble diverger non pas à cause de la taille des amplitudes des graphes mais plutôt à cause de la prolifération combinatoire de tels graphes.

En effet, pour le terme en g^n d'une fonction à N points, la somme sur les graphes est due à l'application du théorème de Wick pour intégrer gaussiennement un monôme dans les champs. On a, alors, à sommer sur les appariements deux à deux des $N+4n$ champs présents. On a donc un facteur $(N+4n-1)!! \stackrel{\text{def}}{=} (N+4n-1)(N+4n-3)\ldots 5.3.1$. La situation a l'air encore plus grave pour la théorie renormalisée après retrait des régularisations, car la borne sur un graphe seul est d'ordre $(\text{Cst})^n.n!$. En fait, cette dernière factorielle est un artefact de la procédure de renormalisation adoptée précédemment. On soustrait à un sous-graphe divergent sa partie locale quel que soit l'ordre de grandeur des moments de ses lignes internes par rapport à ceux des lignes externes. En fait, il suffit pour guérir la divergence des diagrammes de faire cette soustraction uniquement lorsque les lignes internes sont de haut moment par rapport aux lignes externes. Les soustractions supplémentaires inutiles de (I.30) sont en fait responsables de la mauvaise borne en $n!$ dans le théorème de BPHZ uniforme. Il existe une méthode de renormalisation perturbative qui pallie à cet inconvénient, c'est le développement avec des constantes de couplage effectives. C'est une version non perturbative de ce type de procédure de renormalisation qui fut utilisée avec succès dans [FMRS3] et que j'ai améliorée dans mon dernier article [A].

Chapitre II

Développements en amas à une tranche

Si la théorie des perturbations diverge c'est, d'une part, parce qu'elle ne considère pas les bons degrés de liberté et, d'autre part, développe les couplages entre eux de façon redondante. En effet, si $d\mu^i$ est la mesure gaussienne correspondant au propagateur C^i de (I.38) et (I.39) et si $\mathcal{D}_i$ est un découpage de $\mathbb{R}^4$ en boîtes Δ de taille M^{-i}, on a la borne suivante (redémontrée dans [AR2])

La borne gaussienne:

$$\left| \int d\mu^i(\phi^i)\phi^i(x_1)\dots\phi^i(x_n) \right| \leq \left(\prod_{k=1}^{n} K.M^i \right) \times \prod_{\Delta\in\mathcal{D}_i} \sqrt{n(\Delta)!} \qquad \text{(II.1)}$$

où K est une constante, et pour tout $\Delta \in \mathcal{D}_i$, $n(\Delta)$ compte le nombre de points parmi $x_1, \dots, x_n$ qui appartiennent à la boîte Δ.

On voit donc que si factorielle il y a, du fait de la somme combinatoire sur les graphes, ce n'est pas une factorielle globale $\sqrt{n!}$ comme on pourrait s'y attendre, mais un produit de factorielles locales par boîte du réseau. Le danger c'est de développer trop de champs au même endroit. On voit ainsi que les bons degrés de liberté à considérer sont non pas les valeurs du champ total $\phi(x)$ vues indépendamment, mais pour chaque i et $\Delta \in \mathcal{D}_i$ la restriction de ϕ^i à Δ. En effet, le champ ϕ est remplacé par $\sum_{i=0}^{\rho} \phi^i$, somme de champs gaussiens indépendants avec propagateurs C^i, $0 \leq i \leq \rho$. Ce découpage en degrés de liberté élémentaires est en fait plus conforme aux canons de la physique quantique que l'omniprésente théorie des perturbations. L'ensemble des valeurs de $\phi^i(x)$ pour $x \in \Delta$, avec Δ une boîte donnée de $\mathcal{D}_i$, est une quantité qu'il ne faut surtout pas sonder en détail à cause du principe d'incertitude. En effet, ϕ^i fluctue à des échelles de moment d'ordre M^i tandis que Δ a une extension spatiale M^{-i}. Cette analyse fine du contenu du champ brut ϕ s'apparente à l'analyse microlocale, c'est-à-dire, à la fois en position et en fréquence. Une façon

possible de faire la même chose est de décomposer ϕ sur une base d'ondelettes [BaF].

Une fois les bons degrés de liberté identifiés, on voit qu'aussi bien la limite $\Lambda \nearrow \mathbb{R}^4$ que celle $\kappa \to 0$ se conçoivent comme une exhaustion par paquets finis de l'ensemble dénombrable de ces degrés de liberté.

C'est une opération de limite thermodynamique, d'où l'apparition dans un contexte de théorie des champs de méthodes de développement en amas et de Mayer provenant de la mécanique statistique. Prenons le cas d'un modèle à une tranche, c'est-à-dire, avec troncature ultraviolette donnée par κ et infrarouge donnée par la masse m, sur le propagateur C_κ de la mesure gaussienne. Le développement en amas permet, par exemple, d'exprimer la fonction de partition

$$Z(\Lambda) \stackrel{\text{def}}{=} \int d\mu_{C_\kappa}(\phi) \exp\left(-\frac{g}{4!}\int_\Lambda \phi^4\right) \tag{II.2}$$

sous forme commode pour étudier la limite thermodynamique $\Lambda \nearrow \mathbb{R}^4$:

$$Z(\Lambda) = \sum_\pi \prod_{Y\in\pi} A(Y) \quad . \tag{II.3}$$

Ici π est sommée sur les partitions de l'ensemble des boîtes Δ dans Λ d'un réseau fixé $\mathcal{D}$ de boîtes de côté unité. Un élément Y de π est un ensemble fini de boîtes appelé polymère. $A(Y)$ est un nombre associé à Y appelé l'activité ou l'amplitude du polymère. L'équation (II.3) s'obtient par un développement de Taylor avec reste intégral des couplages, ici dus à la covariance $C_\kappa(x,y)$ entre les différentes boîtes du réseau. On cherche à exprimer les amplitudes $A(Y)$ par des sommes sur des arbres reliant les boîtes de Y, avec pour chaque lien $\{\Delta,\Delta'\}$ d'un tel arbre, un propagateur explicite entre les deux boîtes Δ et Δ'. La décroissance exponentielle de $C_\kappa(x,y)$ fait que $A(Y)$ possède une décroissance en arbre, et devient très petit lorsque Y s'étale. D'autre part, en prenant g suffisamment petit on recupère un facteur $u(g)^{\#(Y)}$ (où $u(g)\to 0$ si $g\to 0$), qui pénalise les grands polymères.

Si A_0 est l'amplitude d'un polymère fait d'un seul cube, on a à partir de (II.3)

$$Z(\Lambda) = A_0^{\#(\Lambda)} \sum_{n\geq 0} \frac{1}{n!} \sum_{\substack{Y_1,\dots,Y_n\subset\Lambda \\ Y_i \text{ disjoints} \\ \#(Y_i)\geq 2}} \tilde{A}(Y_1)\dots\tilde{A}(Y_n) \tag{II.4}$$

avec $\tilde{A}(Y) \stackrel{\text{def}}{=} A_0^{-\#(Y)} A(Y)$. Le développement de Mayer permet alors de calculer le logarithme de $Z(\Lambda)$ sous la forme

$$\log Z(\Lambda) = \#(\Lambda).\log A_0 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{Y_1,\dots,Y_n\subset\Lambda \\ \#(Y_i)\geq 2}} \tilde{A}(Y_1)\dots\tilde{A}(Y_n)\psi^T(Y_1,\dots,Y_n) \tag{II.5}$$

où $\psi^T(Y_1,\dots,Y_n)$ est un coefficient combinatoire. Si $\psi(Y_1,\dots,Y_n)$ est la fonction valant 1 si les Y_i sont disjoints et 0 sinon, ψ^T est la fonction connexe obtenue à partir de ψ de la même façon que nous avons obtenu S_n^T à partir de S_n dans (I.29).

Une limite intéressante, dans le cas du modèle à une tranche, est une quantité intensive : la pression

$$p \stackrel{\text{def}}{=} \lim_{\Lambda \nearrow \mathbb{R}^4} -\frac{\log Z(\Lambda)}{\#(\Lambda)} \quad . \tag{II.6}$$

On peut, maintenant, l'écrire

$$p = -\log A_0 - \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{Y_1,\dots,Y_n \subset \Lambda \\ \#(Y_i)\geq 2 \\ \Delta_0 \in \cup Y_i}} \tilde{A}(Y_1)\dots\tilde{A}(Y_n)\psi^T(Y_1,\dots,Y_n) \tag{II.7}$$

où Δ_0 est une boîte fixée de $\mathcal{D}$ et les polymères Y_i ne sont plus restreints à un volume Λ.

On peut démontrer que la série donnant p converge normalement lorsque $\Lambda \nearrow \mathbb{R}^4$ et donc montre rigoureusement l'existence de la limite (II.6). Les ingrédients essentiels sont la propriété de décroissance en arbre de $A(Y)$, ainsi que le petit facteur par cube de Y, et enfin que $|\psi^T(Y_1,\dots,Y_n)|$ est borné par le nombre d'arbres sur $\{1,\dots,n\}$ dont les liens $\{i,j\}$ sont tels que $Y_i \cap Y_j \neq \emptyset$. Le dernier résultat s'appelle le théorème de Rota [Ro].

Les développements en amas donnant (II.3) sont d'un formalisme assez lourd d'où l'intérêt de les affiner et de les simplifier. Les premiers développements dus à Glimm, Jaffe et Spencer [GJS] utilisaient une représentation des propagateurs à l'aide de chemins de Wiener et produisaient des polymères Y sans trous, c'est-à-dire, faits de cubes collés les uns aux autres. Une amélioration décisive a été apportée par Brydges, Battle et Fedrebush [BaF, Bat, B1, BrF], avec des formules plus explicites pour les $A(Y)$, et des polymères admettant la possibilité de trous, c'est-à-dire, pouvant sauter d'un cube à un autre qui ne lui est pas adjacent. Cependant, la formule la plus agréable à manier, pour des théories bosoniques, est due à Brydges et Kennedy [BK,B2]. C'est la formule II.1 de [AR1] (annexe de ce chapitre). Dans [AR1] nous apportons une preuve entièrement algébrique de cette formule démontrée par Brydges et Kennedy comme sous-produit d'une équation de Hamilton-Jacobi quelque peu miraculeuse. Par ailleurs, nous en apportons une généralisation appelée formule de jungle ([AR1], formule IV.1) dans le cas où plusieurs développements en amas sont superposés avec differents types de liens et règle de priorité. Ces formules de jungle sont utiles lorsqu'il faut faire dans un modèle à une tranche un développement préalable de grand/petit champ comme dans [KMR]. Les formules de jungle permettent aussi d'écrire les deux étapes (II.3) et (II.5) en une seule avec une somme sur un seul arbre fait de deux types de liens: des liens de propagateurs et des liens de Mayer rendant compte d'une intersection $Y_i \cap Y_j \neq \emptyset$ entre polymères ([AR1], formule V.C.19).

Néanmoins le but initial des formules de jungle qui était de les utiliser pour écrire de façon explicite des développements multi-tranches (pour étudier la limite $\kappa \to 0$) a été abandonné.

Par ailleurs, nous introduisons dans [AR1] une formule, totalement nouvelle, de forêt avec racine afin d'exprimer comme somme sur des arbres avec racine, aussi bien les amplitudes $A(Y)$ que les coefficients $\psi^T(Y_1, \ldots, Y_n)$, redémontrant au passage le théorème de Rota. Néanmoins, cette formule ne préserve pas la positivité de la covariance C_κ interpolée. La formule ne s'applique donc pas aux théories bosoniques. Par contre, dans le cas de théories fermioniques, c'est la formule la plus optimale pour montrer des bornes sur la série des perturbations comme par exemple l'existence d'un rayon d'analyticité non nul en présence de régularisations. Rappelons que l'intégrale fermionique d'un monôme $\psi(x_1) \ldots \psi(x_n)\overline{\psi}(y_1) \ldots \overline{\psi}(y_n)$ s'exprime à l'aide d'un déterminant $det(C(x_i, y_j)_{\substack{1 \le i \le n \\ 1 \le j \le n}})$ où C est le propagateur fermionique. L'argument essentiel [IM1] est le principe de Pauli, qui se manifeste par l'annulation du déterminant si $x_i = x_j$ car il y a deux lignes identiques par exemple. Le type d'interpolation utilisé dans ([AR1], formule II.3) et modifiant le déterminant précédent conserve la propriété précitée. Ce n'est pas le cas de la formule de Brydges et Kennedy ou de l'approche de [FKLT] conduisant à faire de fastidieuses intégrations par parties de certains champs.

Chapitre III

Développements dans l'espace des phases

Pour étudier en plus de la limite $\Lambda \nearrow \mathbb{R}^4$, la limite $\kappa \to 0$, le développement en amas du Chapitre II n'est pas suffisant. Il faut découper le champ ϕ sous la forme $\sum_{i=0}^{\rho} \phi^i$ comme indiqué précédemment et écrire $Z(\Lambda)$, par exemple, sous la forme (II.3), les polymères Y étant cette fois dans l'espace des phases [GJ4, GK, FMRS3, MRS, Bal]. Celà signifie que Y est un ensemble de boîtes Δ pouvant appartenir à des réseaux différents $\mathcal{D}_i$, $i = 0, \ldots, \rho$ mais, néanmoins, contenues dans le volume Λ. Les boîtes correspondent aux bons degrés de liberté introduits au début du Chapitre II. Il existe, alors, deux types de couplages entre boîtes de $\cup \mathcal{D}_i$: des couplages horizontaux dus aux propagateurs C^i entre deux boîtes de la même échelle i, et des couplages verticaux entre quatre boîtes d'échelles différentes mais localisées à la même position en espace, c'est-à-dire, d'intersection non vide. Les couplages verticaux sont dus à l'interaction locale ϕ^4. Il s'agit encore d'obtenir une décroissance en arbre pour les $A(Y)$, ainsi qu'un facteur petit par cube. La borne (I.40) montre que pour un terme $\int_\Delta \phi^{i_1} \ldots \phi^{i_4}$ développé, avec $i_1 > \cdots > i_4$, Δ d'échelle i_1, on récupère un facteur M^{-4i_1} de volume pour Δ et des facteurs d'échelle $M^{i_1}, \ldots, M^{i_4}$ pour les champs, correspondant à leur taille caractéristique imposée par la mesure gaussienne. On a donc un facteur de décroissance verticale entre les fréquences $i_1, \ldots, i_4$, qui est $M^{-|i_1 - i_2|}.M^{-|i_1 - i_3|}.M^{|i_1 - i_4|}$. Néanmoins, pour arriver à borner la somme $\sum_{Y | \Delta_0 \in Y} |A(Y)|$, il faut, lorsque l'on somme la position d'un petit cube d'échelle i dans un grand cube d'échelle $j < i$, posséder un facteur $M^{-(4+\epsilon)|i-j|}$ ce qui n'est pas garanti pour tous les graphes du développement.

Ceux qui posent problème sont les graphes, possédant des sous-graphes formés de cubes d'échelle plus haute que i donné, dont les lignes externes sont d'échelle plus basse que i et sont au nombre de deux ou quatre. On retrouve ainsi les sous-graphes divergents de l'analyse perturbative du Chapitre I.

Dans [AR2] (annexe de ce chapitre) nous avons introduit une nouvelle méthode de développement dans l'espace des phases qui s'applique au modèle déjà difficile du problème infrarouge de ϕ^4 à quatre dimensions.

Néanmoins, c'est une approche qui devrait fonctionner pour tous les modèles renormalisables asymptotiquement libres et même pour des modèles non renormalisables avec un paramètre petit supplémentaire comme ϕ^4 infrarouge à trois dimensions avec un grand nombre de composantes.

Dans [AR2] nous montrons une borne du type $\sum_{Y|\Delta_0 \in Y} |A(Y)| K^{\#(Y)} < +\infty$ pour tous les polymères faits de graphes exempts de sous-graphes divergents tels que ceux mentionnés plus haut. C'est l'analogue constructif du théorème de Weinberg.

Parmi les nouveautés de l'approche de [AR2], l'interpolation des couplages horizontaux et verticaux est traitée de la même façon dans les deux cas grâce à une formule *linéaire*. L'interpolation quartique des couplages verticaux dans [FMRS3, R1] rend difficile l'obtention de formules explicites pour les $A(Y)$. Notre interpolation linéaire préserve aussi la positivité de la covariance et de l'interaction.

Nous avons, aussi, introduit une décomposition de grand/petit champ, écrite ainsi pour la première fois en détail dans un contexte multi-tranches dans [AR2].

Ces deux différences avec [FMRS3, R1] font que, bien qu'inspiré des idées de ces travaux antérieurs, [AR2] n'en est pas une simple version rédigée de façon détaillée. La mise au point des bornes pour le nouveau développement était longue et très délicate. Il a fallu introduire plusieurs améliorations techniques pour la démonstration des bornes.

D'abord, la Proposition 1 d'[AR2] fournit un critère quasi minimal pour sommer sur la forme d'un polymère dans l'espace des phases, lorqu'il est restreint à contenir une boîte Δ_0 fixée. En effet, on peut très bien sommer des polymères Y avec des trous dans la direction verticale, c'est-à-dire, couplant directement des cubes de fréquences éloignées, contrairement à [FMRS3]. Ces polymères à trous sont effectivement engendrés par le nouveau développement. Cette disparité avec l'approche classique du groupe de renormalisation est plus marquée encore dans [A] (annexe du Chapitre IV). En effet, notre méthode est plus proche de l'ancienne renormalisation perturbative dans l'esprit de (I.30). Une amélioration par rapport à [FMRS3, R1, BDH2] est que l'on ne demande pas que le pas de groupe de renormalisation M soit grand, ce qui est un prérequis surprenant et même contraire à l'idée physique de morceler le passage du microscopique au macroscopique en une succession d'étapes où l'on gagne un facteur d'échelle borné à chaque fois.

Nous introduisons aussi un lemme permettant de borner aisément les factorielles locales du type (II.1). C'est le Lemme 17 de [AR2] de déplacement des factorielles locales.

Le Lemme 18 de [AR2] sert à payer à peu de frais (un facteur $M^{-\epsilon|i-j|}$ au lieu de $M^{-(4+\epsilon)|i-j|}$) des sommes de petits cubes Δ dans un grand $\overline{\Delta}$, à condition que Δ soit restreint à un polymère Y déjà connu pour lequel on dispose d'un facteur petit

par cube.

Enfin, dans le Lemme 19 d'[AR2] (idée de J. Magnen) on améliore [FMRS3] en ce sens que la domination, c'est-à-dire, une borne de Hölder quartique, des $\partial\phi$ est faite uniquement avec le ϕ^4 de l'interaction et un terme de fluctuation en $\partial\partial\phi$ qui se comporte bien vis-à-vis des factorielles locales.

Chapitre IV

Renormalisation constructive explicite

Le dernier article de ma thèse [A] (annexe du Chapitre IV) est une suite à [AR2] où l'on traite le modèle ϕ^4_4 infrarouge complet, c'est-à-dire, y compris les graphes présentant des sous-graphes divergents.

Le type d'interpolation est le même que dans [AR2]; néanmoins, la nouveauté est surtout dans le développement de Mayer. Dans [FMRS3] aussi bien que [GK] ou [BDH2], on procède inductivement échelle par échelle et à chaque étape on fait un pas de développement en amas, ce qui fait apparaître des fonctions à deux et quatre points qu'il faut renormaliser. Cependant, ces fonctions baignent dans une mer de polymères du vide (sans pattes externes) et de fonctions convergentes (avec au moins six pattes externes de basse fréquence). De plus, toutes ces fonctions ont entre elles des contraintes de non intersection ou de cœur dur. En d'autres termes, elles ne sont pas factorisées. Pour résoudre ce problème et définir les contre-termes, il faut intercaler un développement de Mayer à chaque étape. Cette procédure rend l'écriture d'une formule explicite pour la pression ou les fonctions de Schwinger, analogue de (I.30), extrêmement difficile. Cet aspect a, sans doute, contribué à la réputation d'inaccessibilité de la théorie constructive, culminant dans l'étude rigoureuse des théories de jauge [Bal, MRS]. Je propose dans [A] une approche qui simplifie cet aspect (mais ne le trivialise pas pour autant) basée sur une seule application d'un développement en amas et d'un développement de Mayer. On obtient, ainsi, une formulation convergente de la pression considérablement plus explicite que celle de [FMRS3]. On résoud aussi l'itération d'un grand nombre de pas de renormalisation contrairement à l'approche de [GK] et [BDH2] où l'accent est mis sur l'étude d'un pas uniquement. Dans [BDH2] par exemple, l'opération de groupe de renormalisation agit sur des fonctions $A(Y, \phi)$, des polymères et du champ de basse fréquence ϕ, que l'on ne connaît pas explicitement. Il faut donc obtenir des bornes inductives pour toutes les fonctions de ce type dans une boule d'un espace de Banach bien

choisi, et pas uniquement pour celles qui résultent effectivement du découpage de la théorie. Ceci n'est pas le cas de [A] où l'on développe entièrement la théorie avant de borner globalement le développement obtenu.

Terminons par une ébauche, sur le cas plus simple de la formule (II.5), de cette nouvelle approche du développement de Mayer. Soit Z donné par une formule du type

$$Z = \sum_{n \geq 0} \frac{1}{n!} \sum_{\substack{Y_1, \dots, Y_n \\ \text{disjoints}}} A(Y_1) \dots A(Y_n) \quad . \tag{IV.1}$$

On appelle configuration de Mayer toute suite $\mathcal{M} = (Y_1, \dots, Y_n)$ de polymères. On pose $\sigma(\mathcal{M}) = n!$, $\mathcal{A}(\mathcal{M}) = A(Y_1) \dots A(Y_n)$. Pour calculer $\log Z$ on résoud dans les variables $\mathcal{C}(\mathcal{M})$ l'équation

$$1 = e^{-\sum_{\mathcal{M}} \mathcal{C}(\mathcal{M})} . Z \tag{IV.2}$$

c'est-à-dire:

$$1 = \sum_{n \geq 0} \frac{1}{n!} \sum_{r \geq 0} \frac{1}{r!} \sum_{Y_1, \dots, Y_n} \sum_{\mathcal{M}_1, \dots, \mathcal{M}_r} (-\mathcal{C}(\mathcal{M}_1)) \dots (-\mathcal{C}(\mathcal{M}_r)) A(Y_1) \dots A(Y_n) \tag{IV.3}$$

Les $\mathcal{C}(\mathcal{M})$ correspondent à des *contre-termes* du vide pour compenser les termes $A(Y_i)$.

On recherche les solutions sous la forme

$$\mathcal{C}(\mathcal{M}) = \frac{\psi(\mathcal{M})}{\sigma(\mathcal{M})} \mathcal{A}(\mathcal{M}) \quad . \tag{IV.4}$$

Dans (IV.3) on peut, à partir de $Y_1, \dots, Y_n$, $\mathcal{M}_1, \dots, \mathcal{M}_r$, construire par *concaténation* la configuration de Mayer

$$\mathcal{M}_{\text{big}} \stackrel{\text{def}}{=} (Y_1, \dots, Y_n, Y_1^1, \dots, Y_{m_1}^1, \dots, Y_1^r, \dots, Y_{m_r}^r) \tag{IV.5}$$

où $\mathcal{M}_s = (Y_1^s, \dots, Y_{m_s}^s)$ pour $s = 1, \dots, r$. On regroupe (IV.3) par valeurs de $\mathcal{M}_{\text{big}}$ et on symétrise par rapport à l'ordre des polymères dans $\mathcal{M}_{\text{big}}$, puis on procède par identification. On obtient alors pour ψ la formule simple suivante

$$\psi(\mathcal{M}) = \sum_{W} \prod_{V \in W} (-\psi(\mathcal{M}(\mathcal{M}, V))) \quad . \tag{IV.6}$$

Ici, $\mathcal{M} = (Y_1, \dots, Y_n)$, W est sommé sur les ensembles de parties propres et disjointes de $\{1, \dots, n\}$, tels que si $i \neq j$ sont dans le complémentaire de $\cup_{V \in W} V$, alors $Y_i \cap Y_j = \emptyset$. Si $V = \{i_1, \dots, i_k\}$, $1 \leq i_1 < \dots < i_k \leq n$, $\mathcal{M}(\mathcal{M}, V) \stackrel{\text{def}}{=} (Y_{i_1}, \dots, Y_{i_n})$.

On peut résoudre cette équation inductive immédiatement par

$$\psi(\mathcal{M}) = \sum_{\mathcal{F}} (-1)^{\#(\mathcal{F})} \tag{IV.7}$$

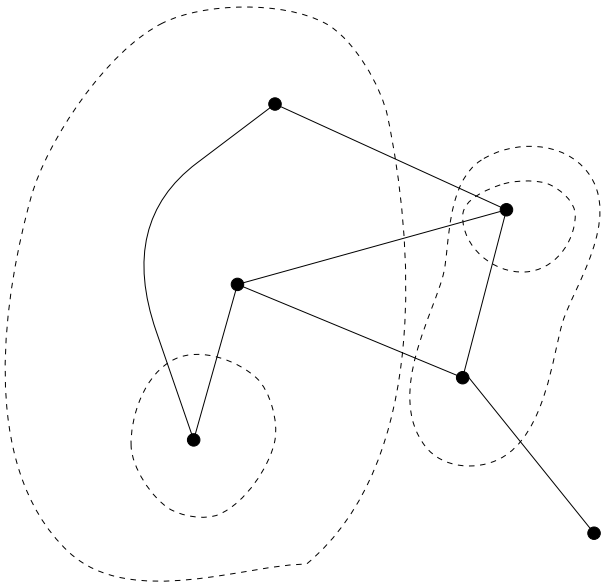

Figure 2

où $\mathcal{F}$ est sommée sur les forêts de parties propres de $\{1,\ldots,n\}$ telles que si $i \neq j$ et $Y_i \cap Y_j \neq \emptyset$, alors il existe $V \in \mathcal{F}$ tel que $i \in V$ et $j \notin V$ ou inversement.

Si l'on trace sur $\{1,\ldots,n\}$ le graphe G des relations d'intersection $Y_i \cap Y_j \neq \emptyset$ entre indices i, j, alors on voit qu'une forêt $\mathcal{F}$ est permise si tous les liens de G sont coupés par $\mathcal{F}$ (figure 2).

On peut montrer facilement que $\psi(\mathcal{M})$ est nul si G ne connecte pas $\{1,\ldots,n\}$, et donc que l'on peut se limiter dans (IV.7) à des forêts faites de paquets connexes. Il faut, cependant, montrer que les multiples compensations de signe dans (IV.7) permettent de réduire cette somme à une somme sur des arbres. C'est possible; en effet, on peut sommer sur $\mathcal{F}$ en séparant deux cas.

1er cas: Il existe un V maximal dans $\mathcal{F}$ contenant 1.

Dans ce cas, V est une partie propre de $\{1,\ldots,n\}$. On peut donc considérer les composantes connexes du complémentaire $X_1,\ldots,X_\rho$ à $\mathcal{F}$. Comme on peut rajouter ou enlever les ensembles $X_1,\ldots,X_\rho$ à $\mathcal{F}$ sans contrainte, la somme du premier cas est nulle. En effet, X_ν est une partie propre de $\{1,\ldots,n\}$, X_ν ne dérange pas la structure de forêt de $\mathcal{F}$ qui est faite de paquets connexes, et les liens sortant de X_ν vont vers V et sont déjà coupés par V lui-même (figure 3).

2ème cas: 1 n'est dans aucun $V \in \mathcal{F}$.

Dans ce cas, si $i_1,\ldots,i_\alpha$ sont les voisins de 1 dans G, comme les liens $\{1, i_\beta\} \in G$ doivent être coupés, chaque i_β est nécessairement dans un V maximal. Le même raisonnement que dans le premier cas montre que dans (IV.7) il ne reste que les forêts où le complémentaire de l'union de $\{1\}$ et des V maximaux est vide. On a donc typiquement la situation de la figure 4 pour les parties maximales de $\mathcal{F}$.

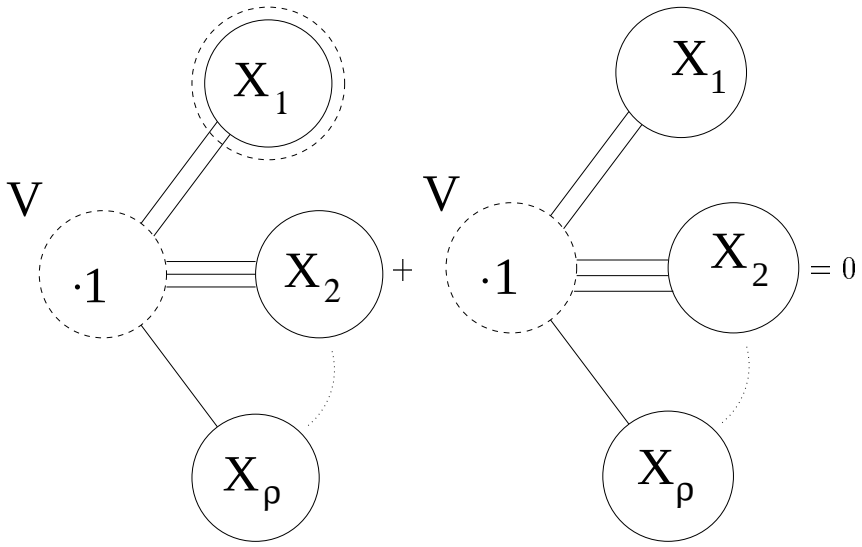

Figure 3

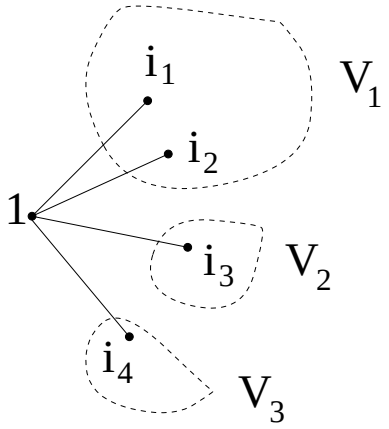

Figure 4

On voit ainsi pousser un arbre à partir de 1. La somme sur les forêts survivantes dans (IV.7) se ramène à une somme sur des arbres avec racine 1. Plusieurs choix d'arbres sont possibles pour une forêt survivante. C'est dû au choix, par exemple pour rattacher V_1 à 1 dans la figure 4, entre le lien $\{1, i_1\}$ et celui $\{1, i_2\}$. Par contre, et c'est le point essentiel, à un arbre donné correspond au plus une forêt survivante. Elle s'obtient comme l'ensemble des branches de l'arbre lorsqu'on coupe tous les liens en commençant par le racine et en progressant vers les feuilles.

Dans [A], c'est une variation élaborée sur ce thème qui est utilisée. On a besoin, pour définir une configuration de Mayer $\mathcal{M}$, de coder, non seulement, une suite de polymères, mais aussi une suite de liens ϕ^4, ϕ^2 ou $(\partial\phi)^2$ entre ces polymères, ainsi que l'ensemble (appelé l'histoire de $\mathcal{M}$) de toutes les opérations de renormalisation τ_g internes à la configuration. Dans [A], une configuration de Mayer $\mathcal{M}$ est l'analogue, dans la formule de Zimmermann (I.30), d'un graphe de Feynman muni d'une forêt de sous-graphes divergents $\mathcal{F}$. Des coefficients $\mathcal{C}(\mathcal{M})$ sont introduits comme précédemment pour compenser les termes du vide (calcul de logarithme de la fonction de partition), mais aussi pour faire disparaître les parties locales de sous-graphes divergents (renormalisation).

Explicit constructive renormalization

(Prépublication, Ecole Polytechnique)

IV.1 Introduction

Whereas the perturbative renormalization has matured into a now well understood theoretic framework, one cannot make such claim for constructive renormalization. In particular there is no analog of Zimmermann's forest formula which expresses in a simple and clear way the amplitude of a renormalized Feynman graph for all orders of perturbation theory, as a pathway to proving rigorously its finiteness when cut-offs are removed. In one constructive approach, one defines a single renormalization group step on lattice potentials or polymer functions [GK, Bal, B1, B2, B3, BY, BDH1, BDH2, Por], and proves iterated bounds on these functions. However it does not seem easy in this way to solve the recursion consisting in applying a great number of times the renormalization group operation.

Our method treats in a single step all these iterations, it bears thus more resemblance to the spirit of the older perturbation theory. We expand our theory into a series of pieces alternating in sign, and we match divergent graphs with their counter-terms, i.e. we gather the pieces into packets of objects of the same size but opposite signs. The initial expansion is not the perturbative expansion, of course, but a truncation of it, i.e. a cluster expansion. This involves a preliminary large versus small field expansion in the spirit of [AR2], then the couplings between scales due to vertices of the theory, and between spatial locations at the same scale due to the covariance propagator, are treated on the same footing. They are linearly interpolated, in contrast with [FMRS3] where the vertical decoupling is quartic. This feature allows to write explicitly the amplitudes of the polymers appearing in our expansion.

To solve the renormalization recursion, we need to introduce the notion of Mayer configuration. This is a combinatoric object that is stable by the renormalization operation. It consists roughly of a sequence of polymers that have a summable decay, linked by vertices on which several renormalization operations, like the operators τ and $1-\tau$ in Zimmermann's formula taking the local or the renormalized part of a divergent subgraph respectively.

We can define an amplitude for such Mayer configurations, and the result for the pressure, the Schwinger functions or the variation of effective constants is expressed as an expansion over these configurations. In this expansion each configuration $\mathcal{M}$ comes with an amplitude factor $\mathcal{A}(\mathcal{M})$ times a purely combinatorial factor $\psi(\mathcal{M})$. This factor is a generalization of the well known factor appearing in a standard Mayer

expansion for making the thermodynamic limit of a model of statistical physics. Recall that the latter had, in order to show the convergence of the expansion, to be bounded by a tree expansion. This is called Rota's theorem. This is indeed what we will do in this more complicated situation.

Section VI.3 presents the algebraic definition of our expansion scheme, whereas in Section VI.4 we sketch the convergence proof. The methods used for the bounds were explained in great detail in [AR2], we therefore concentrated our attention to carefully presenting the proofs of two main novelties. These are the above mentioned bound by a tree expansion, and the sum over the locations of the polymers making a Mayer configuration.

IV.2 The model

Let $M \geq 2$ be a fixed integer. For each integer $i \geq 0$ we introduce a discretization of $\mathbb{R}^4$ by boxes of size M^i. We pose

$$\begin{aligned} \mathcal{D}_i &\stackrel{\text{def}}{=} \{\Delta \subset \mathbb{R}^4 | \exists (n_1, \ldots, n_4) \in \mathbb{Z}^4, \\ &\Delta = [n_1 M^i, (n_1+1)M^i[\times \cdots \times [n_4 M^i, (n_4+1)M^i[\} \end{aligned} \tag{IV.8}$$

and $\mathcal{D} \stackrel{\text{def}}{=} \cup_{i \geq 0} \mathcal{D}_i$. If $\Delta \in \mathcal{D}$, the unique $i \geq 0$ such that $\Delta \in \mathcal{D}_i$ is denoted by $i(\Delta)$. To regularize the theory we introduce an integer $N \geq 0$ to define an infrared cut-off on momenta of order M^{-N}, and also a finite volume box $\Lambda \subset \mathbb{R}^4$ of the form $[x_1, x_1 + L[\times \cdots \times [x_4, x_4 + L[$. We require that L be a multiple of M^N and that Λ be in fact a union of boxes in $\mathcal{D}_N$. The set of boxes $\Delta \in \mathcal{D}$ with $i(\Delta) = i$ and $\Delta \subset \Lambda$ is denoted by $\mathcal{D}_i(\Lambda)$, and we set $\Delta(\Lambda, N) \stackrel{\text{def}}{=} \cup_{0 \leq i \leq N} \mathcal{D}_i(\Lambda)$.

In accordance with our vertical picture of phase space with high frequencies above lower ones, we introduce the following terminology. If $\Delta_1, \Delta_2 \in \mathcal{D}(\Lambda, N)$ and $i(\Delta_1) < i(\Delta_2)$ and $\Delta_1 \subset \Delta_2$, we say that Δ_1 is *above* Δ_2. If furthermore $i(\Delta_1) = i(\Delta_2) - 1$ we say that Δ_1 is *just above* Δ_2 and also that Δ_1 and Δ_2 are *vertically neighboring.*

We let C_{free} be a free boundary condition covariance on $\mathbb{R}^4$ that is the inverse of an ultraviolet regularized Laplacian, namely:

$$C_{\text{free}}(x,y) \stackrel{\text{def}}{=} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2} \left(\eta(p^2) - \eta(M^{2(N+1)}p^2)\right) \quad , \tag{IV.9}$$

where η is a smooth function $\mathbb{R} \mapsto [0,1]$ such that $\eta(x) = 1$ if $|x| \leq 1$ and $\eta(x) = 0$ if $|x| \geq 2$. We define also the corresponding periodic boundary condition covariance on Λ:

$$C(x,y) \stackrel{\text{def}}{=} \sum_{z \in (L\mathbb{Z})^4} C_{\text{free}}(x, y+z) \quad . \tag{IV.10}$$

This sum converges since C_{free} has exponential decay at length scale M^N. This allows to construct on the dual of the nuclear space of smooth periodic functions on Λ, a Gaussian measure $d\mu_C$ (see [E, GJ5]). This measure is in fact supported on smooth functions since C_{free} and C are smooth as a consequence of the ultraviolet regularization due to the $\eta(p^2)$ term in (IV.9) (see [E, GJ5]). The quantity we consider is the well defined partition function

$$Z(\Lambda, N) \stackrel{\text{def}}{=} \int d\mu_C(\phi) \exp\left(-\mu \int_\Lambda \phi(x)^2 dx - \lambda \int_\Lambda \phi(x)^4 dx\right) \quad . \tag{IV.11}$$

Our purpose is to define a converging expansion for the free energy per volume of the critical model:

$$f \stackrel{\text{def}}{=} \lim_{N \to +\infty} \lim_{\Lambda \to \mathbb{R}^4} -\frac{\log Z(\Lambda, N)}{|\Lambda|} \quad , \tag{IV.12}$$

where $|\Lambda|$ denotes the volume of Λ

The first step is to extract a $(\partial\phi)^2$ term from the Gaussian measure to put it in the exponential of the interaction. In this particular model such a precaution is not necessary for two reasons. First, the $(\partial\phi)^2$ does not diverge and therefore the wave function renormalization can be avoided. Second, the effective wave function renormalization constant is positive and there is no need to add a positive contribution from the Gaussian part to preserve the global positivity of the effective potentials involved in the interaction part. However we go at it to advertise the full power of our formalism that can be easily tailored to suit other situations where such treatment may be necessary, for instance infrared ϕ^4 in $4 - \epsilon$ dimensions (see [BDH2]).

We therefore introduce an ϵ verifying $0 < \epsilon < \frac{1}{2}$ to be fixed later and write

$$d\mu_C(\phi) = d\mu_{\tilde{C}}(\phi) \mathcal{N}_{\Lambda,N} \exp\left(-\epsilon \int_\Lambda \partial\phi(x)^2 dx\right) \tag{IV.13}$$

where $\mathcal{N}_{\Lambda,N} \stackrel{\text{def}}{=} \det(\tilde{C}C^{-1})^{\frac{1}{2}}$ is the change in normalization and $\tilde{C}$ is the covariance of the remaining Gaussian part. It is easy to check that

$$\mathcal{N}_{\Lambda,N} = \exp\left(-\frac{1}{2} \sum_{p \in (\frac{2\pi}{L}\mathbb{Z})^4} \log\left(1 - 2\epsilon\left(\eta(p^2) - \eta(M^{2(N+1)}p^2)\right)\right)\right) \tag{IV.14}$$

and

$$\tilde{C}(x, y) = \sum_{z \in (l\mathbb{Z})^4} \tilde{C}_{\text{free}}(x, y + z) \tag{IV.15}$$

where

$$\tilde{C}_{\text{free}}(x, y) \stackrel{\text{def}}{=} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2} \frac{\left(\eta(p^2) - \eta(M^{2(N+1)}p^2)\right)}{1 - 2\epsilon\left(\eta(p^2) - \eta(M^{2(N+1)}p^2)\right)} \quad . \tag{IV.16}$$

Remark that $\tilde{C}_{\rm free}$ and $\tilde{C}$ have the same smoothness properties as $C_{\rm free}$ and C. Now $Z(\Lambda,N)=\mathcal{N}_{\Lambda,N}.\tilde{Z}(\Lambda,N)$, with

$$\tilde{Z}(\Lambda,N)\stackrel{\rm def}{=}\int d\mu_{\tilde{C}}(\phi)\exp\left(-\int_\Lambda\left\{\epsilon\partial\phi(x)^2+\mu\phi(x)^2+\lambda\phi(x)^4\right\}dx\right)\quad . \tag{IV.17}$$

Since, taking the limit of a Riemann sum and applying the theorem of dominated convergence, we have

$$\lim_{N\to+\infty}\lim_{\Lambda\to\mathbb{R}^4}-\frac{\log\mathcal{N}_{\Lambda,N}}{|\Lambda|}=\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\log\left(1-2\epsilon\eta(p^2)\right)\quad , \tag{IV.18}$$

it remains to study the limit

$$\tilde{f}\stackrel{\rm def}{=}\lim_{N\to+\infty}\lim_{\Lambda\to\mathbb{R}^4}-\frac{\log\tilde{Z}(\Lambda,N)}{|\Lambda|}\quad . \tag{IV.19}$$

Following the renormalization group strategy we replace the fluctuating ϕ with covariance $\tilde{C}$ by a sum $\sum_{0\le i\le N}\phi_i$ of independent Gaussian random fields ϕ_i with covariances

$$\tilde{C}_i(x,y)\stackrel{\rm def}{=}\sum_{z\in(l\mathbb{Z})^4}\tilde{C}_{i,\rm free}(x,y+z) \tag{IV.20}$$

where

$$\tilde{C}_{i,\rm free}(x,y)\stackrel{\rm def}{=}\int\frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2}\frac{\left(\eta(M^{2i}p^2)-\eta(M^{2(i+1)}p^2)\right)}{1-2\epsilon\left(\eta(p^2)-\eta(M^{2(i+1)}p^2)\right)} \tag{IV.21}$$

Since $\tilde{C}(x,y)=\sum_{0\le i\le N}\tilde{C}_i(x,y)$, the law of $\sum_{0\le i\le N}\phi_i$ is the same as that of ϕ, which legitimates the operation.

IV.3 The algebra of the expansion

IV.3.1 The Mayer configurations

A *Mayer configuration* is a compound $\mathcal{M}=(e,\mathcal{G},\mathcal{Y},\mathcal{L},\mathcal{H})$ containing the following data.

- e is an integer $e\ge 0$. A Mayer configuration can be viewed as a constructive analog of a Feynman diagram, e corresponds then to the number of its external legs. If $e=0$ we say that $\mathcal{M}$ is a *vacuum Mayer configuration.*

- $\mathcal{G}$ is a sequence $(\gamma_1,\ldots,\gamma_e)$ of derivation multi-indices, i.e. each γ_i is in $\mathbb{N}^4$. We will say that $\mathcal{M}$ is of type $(e,(\gamma_1,\ldots,\gamma_e))$. If $\alpha=(\alpha^1,\ldots,\alpha^4)$ is a derivation multi-index we denote by $|\alpha|$ the sum $\alpha_1+\cdots+\alpha_4$.

- $\mathcal{Y}$ is a sequence $(Y_1, \ldots, Y_k)$ of *polymers*, i.e. finite subsets of $\mathcal{D}$ with at least two elements.

- $\mathcal{L}$ is sequence $(l_1, \ldots, l_p)$ of *links*. Such a link l is by definition a sequence of pairs $((P_1, \omega_1), \ldots, (P_r, \omega_r))$. For each i, $1 \leq i \leq r$, P_i is a *location* i.e. either a pair (j, Δ) with j an integer such that $1 \leq j \leq k$ and Δ is an element of Y_j, or an integer c, $1 \leq c \leq e$ labeling an *external leg* of $\mathcal{M}$.

An ω_i, for $1 \leq i \leq r$, is a derivation multi-index. An ω_i is also called the initial derivation multi-index of the i-th leg of l. Indeed later on some renormalization operations may act on this leg by producing new derivations. In addition we impose on l the restriction that $\sum_{1 \leq i \leq r}(1 + |\omega_i|) \leq 4$, and that the boxes Δ appearing in the locations listed in l have a nonempty intersection i.e. that are included one in another.

- $\mathcal{H}$ is the *history* of $\mathcal{M}$. Namely, it is a set of quadruplets

$$T = (\mathrm{status}(T), \underline{\Delta}(T), \mathrm{dir}(T), \mathfrak{L}(T)) \tag{IV.22}$$

we call renormalization operations.

status(T) must be an element of the pair $\{\mathrm{loc}, \mathrm{ren}\}$, it indicates whether we are defining the local part or the renormalized part of the "subgraph" we are considering, like the splitting τ_g versus $1 - \tau_g$ of the old perturbative renormalization (see [R1]). $\underline{\Delta}(T)$ is an element of $\mathcal{D}$. dir(T) is a vector in $\{\overrightarrow{0}, \overrightarrow{e_1}, \ldots, \overrightarrow{e_4}, -\overrightarrow{e_1}, \ldots, -\overrightarrow{e_4}\}$ where $(\overrightarrow{e_1}, \ldots, \overrightarrow{e_4})$ is the canonical base of $\mathbb{R}^4$.

$\mathfrak{L}(T)$ is a sequence of quadruplets $((i_1, j_1, \alpha_1, \beta_1), \ldots, (i_u, j_u, \alpha_u, \beta_u))$ where for each v, $1 \leq v \leq u$, i_v is an index $1 \leq i_v \leq p$ labeling a link l_{i_v} in the previously defined sequence $\mathcal{L}$; j_v is a label $1 \leq j_v \leq r_v$ where r_v is the length of the sequence $((P_1, \omega_1), \ldots, (P_{r_u}, \omega_{r_u}))$ defining the link l_{i_v}. j_v is said to label a particular *leg* in the link l_{i_v}. Finally α_v is a derivation multi-index with nonnegative entries, while β_v is a derivation multi-index with at most one nonzero entry that has to be 1 or -1.

We impose that $\mathrm{dir}(T) = \overrightarrow{0}$ if status$(T) =$ ren. We impose also that if $\mathrm{dir}(T) = \overrightarrow{0}$ then either all the β_v's are 0, or only two of them are nonzero , one with a 1 and the other with a -1 at the same entry. If $\mathrm{dir}(T) = \pm\overrightarrow{e}_\nu$ then there must be only one non zero β_v with a -1 at the ν-th entry.. The α_v's represent the spatial derivations operating on the corresponding fields due to the Taylor expansion comparing their original positions with a local part smeared in $\underline{\Delta}(T)$. The β_v's represent the effect of an additional integration by part creating a face term namely the face of $\underline{\Delta}(T)$ in the direction dir(T). This subtlety is only relevant for the $\phi\partial\partial\phi$ counterterms that have to be converted into $(\partial\phi)^2$ in order to preserve the positivity of the effective potentials.

The previous definition of a Mayer configuration is too broad for our purposes, therefore we now define *admissible Mayer configurations* or simply AMC's.

For any $T \in \mathcal{H}$ in some Mayer configuration $\mathcal{M}$, we define $\mathrm{cut}(T)$ the set of links that are *cut* by the operation T. Namely if

$$\mathfrak{L}(T) = ((i_1^T, j_1^T, \alpha_1^T, \beta_1^T), \dots, (i_{u(T)}^T, j_{u(T)}^T, \alpha_{u(T)}^T, \beta_{u(T)}^T)) \tag{IV.23}$$

then $\mathrm{cut}(T) \stackrel{\mathrm{def}}{=} \{i_1^T, \dots, i_{u(T)}^T\} \subset \{1, \dots, p\}$. We introduce also $\mathrm{up}(T)$ the set of indices s, $1 \leq s \leq k$, labeling polymers of $\mathcal{M}$, such that there exists a link l_i in $\mathcal{L}$ with $i \in \mathrm{cut}(T)$ containing a pair (P_a, ω_a) with $P_a = (s, \Delta)$ and $i(\Delta) \leq i(\underline{\Delta}(T))$. If s_1 and s_2 in $\{1, \dots, k\}$ are two distinct indices labeling polymers, we say that s_1 and s_2 are *connected* if there exist a link l_i in $\mathcal{L}$ featuring two elements of the form $((s_1, \Delta_1), \omega_1)$ and $((s_2, \Delta_2), \omega_2)$, or if there exist a $T \in \mathcal{H}$ such that $\mathfrak{L}(T)$ contains two entries $(i_{v_1}, j_{v_1}, \alpha_{v_1}, \beta_{v_1})$ and $(i_{v_2}, j_{v_2}, \alpha_{v_2}, \beta_{v_2})$ such that the j_{v_1}-th entry of $l_{i_{v_1}}$ is of the form $((s_1, \Delta_1), \omega_1)$ and the j_{v_2}-th entry of $l_{i_{v_2}}$ is of the form $((s_2, \Delta_2), \omega_2)$. We define $U(T)$ to be the set of indices s, $1 \leq s \leq k$, for which there exists a sequence $s_1, \dots, s_\nu$ in $\{1, \dots, k\}$, $\nu \geq 1$, with $s_1 = s$ and $s_\nu \in \mathrm{up}(T)$ and such that s_μ and $s_{\mu+1}$ are connected for any μ, $1 \leq \mu < \nu$.

In the last statement we *do not* take into account a connection between s_μ and $s_{\mu+1}$ if it is due to a link l_i with $i \in \mathrm{cut}(T)$. We introduce a relation $\preceq$ between the elements of $\mathcal{H}$. By definition $T_1 \preceq T_2$ if and ony if $U(T_1) \subset U(T_2)$.

- Our first requirement on $\mathcal{M}$ to be an AMC is that $\preceq$ be a partial ordering i.e. that $T_1 \preceq T_2$and $T_2 \preceq T_1$ implies $T_1 = T_2$. We use the notation $T_1 \prec T_2$ if $T_1 \preceq T_2$ and $T_1 \neq T_2$.

We list now the other requirements on $\mathcal{M}$ to be an AMC.

- For any $T \in \mathcal{H}$ and any box $\Delta \in Y_s$ for some $s \in U(T)$, we have $i(\Delta) \leq i(\underline{\Delta}(T))$.

- For any $T \in \mathcal{H}$, the pairs $(i_1, j_1), \dots, (i_{u(T)}, j_{u(T)})$, labeling legs of links, featuring in $\mathfrak{L}(T)$ are distinct.

- For any $T \in \mathcal{H}$, any pair (i_v, j_v) featuring in $\mathfrak{L}(T)$ is such that the j_v-th leg of the link l_{i_v} is either an external leg index c, $1 \leq c \leq e$, or a pair (s, Δ) with $i(\Delta) > i(\underline{\Delta}(T))$.

- For any $T \in \mathcal{H}$, any link label i, $1 \leq i \leq p$, and any j labeling a leg of l_i, if l_i contains a leg of the form $((s, \Delta), \omega)$ with $s \in U(T)$, and if also the j-th leg of l_i is of the form $((s', \Delta'), \omega')$ with $i(\Delta') > i(\underline{\Delta}(T))$, or (c, ω'') with $1 \leq c \leq e$, then the pair (i, j) has to be listed in $\mathfrak{L}(T)$.

To explain the remaining requirements some more definitions are needed. We introduce for any $T \in \mathcal{H}$ and any v, $1 \leq v \leq u(T)$ the *past derivation multi-index*

$$\gamma_v^{T,\mathrm{past}} \stackrel{\mathrm{def}}{=} \omega(i_v^T, j_v^T) + \sum_{T' \prec T} \sum_{1 \leq w \leq u(T')} \mathbb{1}_{\{(i_w^{T'}, j_W^{T'}) = (i_v^T, j_v^T)\}} \cdot (\alpha_w^{T'} + \beta_w^{T'}) \tag{IV.24}$$

where $\omega(i_v^T, j_v^T)$ is the derivation multi-index ω appearing in the j_v^T-th leg of the i_v^T-th link, and $\mathbb{1}_{\{\dots\}}$ denotes the characteristic function of the condition between

brackets. Note that the right hand side sum on w contains at most one nonzero term. We define the *superficial degree of divergence* of T

$$\mathrm{div}(T) \stackrel{\mathrm{def}}{=} 4 - \sum_{1 \le v \le u(T)} (1 + |\gamma_v^{T,\mathrm{past}}|) \quad . \tag{IV.25}$$

We now require that

- For any $T \in \mathcal{H}$, $\mathrm{div}(T) \ge 0$.
- If $\mathrm{status}(T) = \mathrm{loc}$ then $\sum_{1 \le v \le u(T)} |\alpha_v^T| \le \mathrm{div}(T)$.
- If $\mathrm{status}(T) = \mathrm{ren}$ then $\sum_{1 \le v \le u(T)} |\alpha_v^T| = \mathrm{div}(T) + 1$.
- For any $T \in \mathcal{H}$, if there is in $\mathfrak{L}(T)$ a $\beta_{v_1}^T$ containing a -1 entry, then $u(T) = 2$, and if $\{v_2\} = \{1,2\} \backslash \{v_1\}$, we demand the following : $|\alpha_{v_2}^T + \gamma_{v_2}^{T,\mathrm{past}}| = 0$, $|\alpha_{v_1}^T + \gamma_{v_1}^{T,\mathrm{past}}| = 2$ and $\alpha_{v_1}^T + \gamma_{v_1}^{T,\mathrm{past}} + \beta_{v_1}^T \ge 0$.

If i is a link index and j a leg index in l_i then we denote by $P(i,j)$ the location of this leg. Another requirement is

- If $e \ge 1$, there exists a T in the history of $\mathcal{M}$ such that $\mathrm{status}(T) = \mathrm{loc}$, $u(T) = e$, and $P(i_1^T, j_1^T), \ldots, P(i_{u(T)}^T, j_{u(T)}^T)$ are equal respectively to $1, \ldots, e$. Besides we ask that for any v, $1 \le v \le u(T)$,

$$\gamma_{P(i_v^T, j_v^T)} = \gamma_v^{T,\mathrm{past}} + \alpha_v^T + \beta_v^T \quad . \tag{IV.26}$$

Such a T is necessarily unique in $\mathcal{H}$. We denote it by $T_{\mathrm{out}}(\mathcal{M})$. We require also that any polymer of $\mathcal{M}$ is made by boxes Δ with $i(\Delta) \le i(\underline{\Delta}(T_{\mathrm{out}}(\mathcal{M})))$.

We introduce on each $\mathcal{D}_{\le i} \stackrel{\mathrm{def}}{=} \mathcal{D}_0 \cup \ldots \cup \mathcal{D}_i$ the projection on $\mathcal{D}_i$, pr_i, that associates to each box $\Delta \in \mathcal{D}_{\le i}$ the unique box $\mathrm{pr}_i(\Delta)$ in $\mathcal{D}_i$ that contains Δ. For any $T \in \mathcal{H}$ we define $\mathrm{supp}(T)$ the set of boxes $\Delta \in \mathcal{D}$ such that $i(\Delta) \le i(\underline{\Delta}(T))$ and there exists a link l_i with $i \in \mathrm{cut}(T)$ in which Δ appears in the location of some of its legs. We say that a set $X \subset \mathcal{D}_i$ for some $i \ge 0$, is a *corner set* if for any Δ, $\Delta' \in X$, $d_2^\Lambda(\Delta, \Delta') = 0$. Here

$$d_2^\Lambda(\Delta, \Delta') \stackrel{\mathrm{def}}{=} \inf\{d_2^\Lambda(x,y) | x \in \Delta, y \in \Delta'\} \tag{IV.27}$$

with

$$d_2^\Lambda(x, y \stackrel{\mathrm{def}}{=} \inf\{d_2(x, y+z) | z \in (L\mathbb{Z})^4\} \tag{IV.28}$$

where d_2 denotes the Euclidean distance on $\mathbb{R}^4$. d_2^Λ is simply the induced distance on the torus $\frac{\mathbb{R}^4}{(L\mathbb{Z})^4}$. Now we state our final requirement on $\mathcal{M}$ to be an AMC

- For any $T \in \mathcal{H}$, $i(\underline{\Delta}(T))$ is the smallest i such that $\mathrm{supp}(T) \subset \mathcal{D}_{\le i}$ and $\mathrm{pr}_i(\mathrm{supp}(T))$ is a corner set, and furthermore $\underline{\Delta}(T)$ belongs to it.

IV.3.2 The expansion

IV.3.2.1 The effective coupling constants

We suppose that for each AMC $\mathcal{M}$ in Λ, i.e. made of polymers whose boxes are in Λ, is associated a number $\mathcal{C}(\mathcal{M})$. We will explain later the definition and computational rule for a $\mathcal{C}(\mathcal{M})$. We assume also that these numbers are *translation invariant* in $\frac{\mathbb{R}^4}{(L\mathbb{Z})^4}$. More precisely, for any $i, 0 \leq i \leq N$, and any vector $\overrightarrow{v} \in (M^i\mathbb{Z})^4$, we define the translation $\tau_{\overrightarrow{v}}$ on $\frac{\mathbb{R}^4}{(L\mathbb{Z})^4}$ identified with Λ. $\tau_{\overrightarrow{v}}$ is a bijection on the set of boxes $\mathcal{D}_i(\Lambda)$. Assume $\overrightarrow{v} \in (M^i\mathbb{Z})^4$ and $\mathcal{M}$ is a Mayer configuration in Λ such that every Δ in a polymer of $\mathcal{M}$ satisfies $i(\Delta) \leq i$, and every $\underline{\Delta}(T)$ for T in the history of $\mathcal{M}$ satisfies $i(\underline{\Delta}(T)) \leq i$. We can define the translate $\tau_{\overrightarrow{v}}(\mathcal{M})$ as the Mayer configuration obtained from $\mathcal{M}$ by repacing any of its polymers Y_s by $\tau_{\overrightarrow{v}}(Y_s)$, and any Δ featuring in a leg of its links by $\tau_{\overrightarrow{v}}(\Delta)$ and any $\underline{\Delta}(T)$ by $\tau_{\overrightarrow{v}}(\underline{\Delta}(T))$. The translation invariance now means that whenever $\tau_{\overrightarrow{v}}(\mathcal{M})$ is defined we have $\mathcal{C}(\tau_{\overrightarrow{v}}(\mathcal{M})) = \mathcal{C}(\mathcal{M})$.

We require also a *face symmetry* for the *face Mayer configurations* i.e. the configurations $\mathcal{M}$ such that $\mathrm{dir}(T_{\mathrm{out}}(\mathcal{M})) \neq \overrightarrow{0}$. A face admissible Mayer configuration is said FAMC, the remaining ones with $e \geq 1$ are called CAMC for *core admissible Mayer configuration.*

If $\mathcal{M}$ is a FAMC, we define $\mathrm{opp}(\mathcal{M})$ the *opposite* of $\mathcal{M}$ as the face Mayer configuration obtained from $\mathcal{M}$ by changing $\mathrm{dir}(T_{\mathrm{out}}(\mathcal{M}))$ into $-\mathrm{dir}(T_{\mathrm{out}}(\mathcal{M}))$. Now face symmetry means that for any FAMC $\mathcal{M}$ we have $\mathcal{C}(\mathrm{opp}(\mathcal{M})) = -\mathcal{C}(\mathcal{M})$. This allows for any $e \geq 1$ and sequence of multi-indices $(\gamma_1, \ldots, \gamma_e)$ such that $r + |\gamma_1| + \cdots + |\gamma_e| \leq 4$, and any i, $0 \leq i \leq N$, to define the *effective coupling constants* $\lambda^i_{e,(\gamma_1,\ldots,\gamma_e)}$.

For $i = 0$ we let
- $\lambda^0_{4,(0,0,0,0)} \stackrel{\mathrm{def}}{=} \lambda$
- $\lambda^0_{2,(0,0)} \stackrel{\mathrm{def}}{=} \mu$
- $\lambda^0_{2,(\gamma,\gamma)} \stackrel{\mathrm{def}}{=} \epsilon$ for any γ with $|\gamma| = 1$.

We put all other $\lambda^0_{e,(\gamma_1,\ldots,\gamma_e)}$ equal to zero. Now let $i \geq 1$ and suppose we have defined all the $\lambda^{i-1}_{e,(\gamma_1,\ldots,\gamma_e)}$. Let us choose a $\Delta \in \mathcal{D}_{i-1}$ then we pose for any e and $(\gamma_1, \ldots, \gamma_e)$:

$$\lambda^i_{e,(\gamma_1,\ldots,\gamma_e)} \stackrel{\mathrm{def}}{=} \lambda^{i-1}_{e,(\gamma_1,\ldots,\gamma_e)} - \sum_{\mathcal{M}} \mathcal{C}(\mathcal{M}) \quad , \tag{IV.29}$$

where the sum is over all AMC $\mathcal{M}$ of type $(e, (\gamma_1, \ldots, \gamma_e))$ that is not a FAMC and such that $i(\underline{\Delta}(T_{\mathrm{out}}(\mathcal{M}))) = \Delta$. It is easy to check that by the translation invariance condition this quantity is independent of the choice of Δ. The sum over $\mathcal{M}$ is a priori

infinite but for the moment we work formally. The convergence will be a consequence of our subsequent definition of the $\mathcal{C}(\mathcal{M})$'s and the arguments of section IV.4.

IV.3.2.2 The large versus small field expansion

We put in the functional integral defining $\tilde{Z}(\Lambda, N)$ a decomposition of unity into characteristic functions testing whether the fields are large or not in some region of $\mathcal{D}$ inside Λ. First we introduce a C^∞ step function χ

$$\chi(u) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \ 0 \leq t \leq \frac{1}{2} \\ e^2 \left(1 - e^{-\frac{1}{t-\frac{1}{2}}}\right) e^{-\frac{1}{t-1}} & \text{if } \ \frac{1}{2} \leq t \leq 1 \\ 0 & \text{if } \ t \geq 1 \end{cases} \tag{IV.30}$$

that interpolates smoothly between 1 and 0 on the interval $[0, 1]$. We choose also a constant $\epsilon_1 > 0$.

Now if Γ is some subset of $\mathcal{D}(\Lambda, N)$, and $(\phi_i)_{0 \leq i \leq N}$ is a configuration of the fields, we pose

$$\chi_\Gamma((\phi_i)_{0\leq i\leq N}) = \prod_{\Delta\in\Gamma}(1-\chi)\left((\lambda^{i(\Delta)}_{4,(0,0,0,0)})^{(1+\epsilon_1)}\int_\Delta\left(\sum_{i\in I_\Gamma(\Delta)}\phi_i(x)\right)^4 dx\right)$$

$$\times \prod_{\Delta\in\mathcal{D}(\Lambda,N)\setminus\Gamma}\chi\left((\lambda^{i(\Delta)}_{4,(0,0,0,0)})^{(1+\epsilon_1)}\int_\Delta\left(\sum_{i\in I_\Gamma(\Delta)}\phi_i(x)\right)^4 dx\right) \quad , \tag{IV.31}$$

where $I_\Gamma(\Delta)$ is the largest interval of the form $\{i(\Delta), i(\Delta)+1, \ldots, i\}$ such that for any of its elements j, $j > i(\Delta)$ we have $\mathrm{pr}_j(\Delta) \in \Gamma$. The cubes of Γ are *large field cubes*, those of $\mathcal{D}(\Lambda, N)\setminus\Gamma$ are *small field cubes*. It is an elementary algebraic computation to check that

$$1 = \sum_{\Gamma\subset\mathcal{D}(\Lambda,N)} \chi_\Gamma((\phi_i)_{0\leq i\leq N}) \tag{IV.32}$$

(a proof can be found in [AR2], lemma 4).

If $\Delta \in \mathcal{D}_i(\Lambda) \cap \Gamma$ and $\Delta' \in \mathcal{D}_{i-1}(\Lambda)$ with $1 \leq i \leq N$, we say that Δ' and Δ are *glued*. This notion of connectedness defines a partition of $\mathcal{D}(\Lambda, N)$ into connected components we denote by $\Pi_\Gamma(\emptyset)$. An element of $\Pi_\Gamma(\emptyset)$ is call a *large field block*.

IV.3.2.3 The cluster expansion

It is not difficult to see that, thanks to our definition of the effective coupling constants, we can rewrite

$$\tilde{Z}(\Lambda, N) = \sum_{\Gamma\subset\mathcal{D}(\Lambda,N)} H_\Gamma \tag{IV.33}$$

where

$$H_\Gamma \stackrel{\text{def}}{=} \int d\mu_{\tilde{C}_0}(\phi_0) \otimes \cdots \otimes d\mu_{\tilde{C}_N}(\phi_N)\ \chi_\Gamma((\phi_i)_{0\le i\le N}) \exp\left(-I((\phi_i)_{0\le i\le N})\right) \quad \text{(IV.34)}$$

with

$$I((\phi_i)_{0\le i\le N}) \stackrel{\text{def}}{=} \int_\Lambda \Bigg\{ \epsilon \sum_{i_1,i_2=0}^{N} \partial^\nu \phi_{i_1}(x) \partial_\nu \phi_{i_2}(x) + \mu \sum_{i_1,i_2=0}^{N} \phi_{i_1}(x)\phi_{i_2}(x)$$

$$+\lambda \sum_{i_1,i_2,i_3,i_4=0}^{N} \phi_{i_1}(x)\phi_{i_2}(x)\phi_{i_3}(x)\phi_{i_4}(x) \Bigg\} \quad \text{(IV.35)}$$

which can be reformulated as

$$\begin{aligned} I((\phi_i)_{0\le i\le N}) = &\sum_{e=1}^{4} \sum_{(\gamma_1,\dots,\gamma_e)} \sum_{\Delta_1,\dots,\Delta_e \in \mathcal{D}(\Lambda,N)} \\ &\lambda^{\min(i(\Delta_1),\dots,i(\Delta_e))}_{e,(\gamma_1,\dots,\gamma_e)} \int_{\Delta_1\cap\dots\cap\Delta_e} dx\ \partial^{\gamma_1}\phi_{i(\Delta_1)}(x)\dots\partial^{\gamma_e}\phi_{i(\Delta_e)}(x) \\ +\sum_{e=1}^{4} \sum_{(\gamma_1,\dots,\gamma_e)} &\sum_{\Delta_1,\dots,\Delta_e \in \mathcal{D}(\Lambda,N)} \sum_{\substack{\mathcal{M}\ \text{CAMC} \\ \text{type}\ (e,(\gamma_1,\dots,\gamma_e)) \\ i(\underline{\Delta}(T_{\text{out}}(\mathcal{M}))) < \min(i(\Delta_1),\dots,i(\Delta_e))}} \\ &\mathcal{C}(\mathcal{M}) \int_{\underline{\Delta}(T_{\text{out}}(\mathcal{M}))} dx\ \partial^{\gamma_1}\phi_{i(\Delta_1)}(x)\dots\partial^{\gamma_e}\phi_{i(\Delta_e)}(x) \\ +\sum_{(\gamma_1,\gamma_2)} &\sum_{\Delta_1,\Delta_2 \in \mathcal{D}(\Lambda,N)} \sum_{\substack{\mathcal{M}\ \text{FAMC} \\ \text{type}\ (2,(\gamma_1,\gamma_2)) \\ i(\underline{\Delta}(T_{\text{out}}(\mathcal{M}))) < \min(i(\Delta_1),i(\Delta_2))}} \\ &\mathcal{C}(\mathcal{M}) \int_{\partial\underline{\Delta}(T_{\text{out}}(\mathcal{M}))} d\Sigma(x)\ \partial^{\gamma_1}\phi_{i(\Delta_1)}(x)\partial^{\gamma_2}\phi_{i(\Delta_2)}(x) \end{aligned} \quad \text{(IV.36)}$$

Here $\partial\underline{\Delta}(T)$, for some T in the history of $\mathcal{M}$ such that $\mathrm{dir}(T) \neq \overrightarrow{0}$, denotes the face of $\underline{\Delta}(T)$ in the direction $\mathrm{dir}(T)$. Naturally $d\Sigma(x)$ denotes the ordinary (unoriented) surface measure on $\partial\underline{\Delta}(T)$.

Now let us choose ξ to be a smooth function on $\mathbb{R}$ such that for any $x \in \mathbb{R}$, $0 \le \xi(x) \le 1$, $\xi(-x) = \xi(x)$, $\xi = 1$ on $[-\frac{1}{4}, \frac{1}{4}]$, and $\xi = 0$ on $]-\infty, -\frac{3}{4}] \cup [\frac{3}{4}, +\infty[$ and satisfying

$$\sum_{n\in\mathbb{Z}} \xi(x-n) = 1 \quad \text{(IV.37)}$$

for any $x \in \mathbb{R}$.

Now if $\Delta = [n_1 M^i, (n_1+1)M^i[\times \cdots \times [n_4 M^i, (n_4+1)M^i[$ is some box in $\mathcal{D}_i$ we define the function χ_Δ on $\mathbb{R}^4$ by

$$\chi_\Delta(x_1,x_2,x_3,x_4) \stackrel{\text{def}}{=} \xi\left(M^{-i}x_1 - (n_1+\frac{1}{2})\right)\ldots\xi\left(M^{-i}x_4 - (n_4+\frac{1}{2})\right) \qquad \text{(IV.38)}$$

Cearly $\sum_{\Delta\in\mathcal{D}_i(\Lambda)}\chi_\Delta$ forms a smooth partition of the unity on the torus $\frac{\mathbb{R}^4}{(L\mathbb{Z})^4}$ identified with Λ, for any i, $0\le i\le N$. We can now write for any x_1, x_2 in this torus and any i, $0\le i\le N$,

$$\tilde{C}_i(x_1,x_2) = \sum_{\Delta_1,\Delta_2\in\mathcal{D}_i(\Lambda)} \chi_{\Delta_1}(x_1)\tilde{C}_i(x_1,x_2)\chi_{\Delta_2}(x_2) \quad . \qquad \text{(IV.39)}$$

The cluster expansion we use relies on the introduction of interpolation parameters multiplying each term in the sums contained in (IV.36) and (IV.39). We use the formulation of [AR2] (section III.2.1), we recall briefly here.

Let us suppose we have a finite set of indices L, and $H : (t_l)_{l\in L} \mapsto H((t_l)_{l\in L})$ is a smooth function in some domain of $\mathbb{R}^L$. We denote by $\mathbf{1}$ the vector where all the t_l are set equal to 1. We suppose we have a choice map $\mathcal{C}$ which associates to any sequence $(l_1,\ldots,l_k)$ in L, a subset $\mathcal{C}((l_1,\ldots,l_k))$ of L, such that if $k_1\le k_2$, $\mathcal{C}((l_1,\ldots,l_{k_2})) \subset \mathcal{C}((l_1,\ldots,l_{k_1}))$. A sequence $\mathfrak{g} = (l_1,\ldots,l_k)$ is said allowed if for any a,$1\le a\le k$, $l_a \in \mathcal{C}((l_1,\ldots,l_{a-1}))$. Note that the empty sequence $\emptyset$ is always allowed. We now have the following interpolation formula whose proof is given in [AR2] (lemma 1).

$$H(\mathbf{1}) = \sum_{\substack{\mathfrak{g}=(l_1,\ldots,l_k)\\ \text{allowed}}} \int_{1>h_1>\cdots>h_k>0} dh_1\ldots dh_k \frac{\partial^k H}{\partial t_{l_1}\ldots\partial t_{l_k}}(\mathcal{T}_\mathfrak{g}(\mathbf{h})) \qquad \text{(IV.40)}$$

where $\mathbf{h}$ denotes the vector $(h_1,\ldots,h_k)$, and $\mathcal{T}_\mathfrak{g}(\mathbf{h})$ is the $(t_l)_{l\in L}$ vector defined in the following way:

- if $l\notin\mathcal{C}(\emptyset)$ then $t_l = 1$,
- if $l\in\mathcal{C}(\emptyset)\backslash\mathcal{C}((l_1))$ then $t_l = h_1$,
- if $l\in\mathcal{C}((l_1))\backslash\mathcal{C}((l_1,l_2))$ then $t_l = h_2$,

$\vdots$

- if $l\in\mathcal{C}((l_1,\ldots,l_{k-1}))\backslash\mathcal{C}((l_1,\ldots,l_k))$ then $t_l = h_k$,
- if $l\in\mathcal{C}((l_1,\ldots,l_k))$ then $t_l = 0$.

Summation on k includes all possible values.

Now we define precisely in the present context the objects L, H and $\mathcal{C}$.

The elements of L we call *cluster links* fall into four categories

- We have the *horizontal links* making the set $L_{\rm hor}$. Such a link l is characterized by a pair of boxes $\Delta_1(l)$ and $\Delta_2(L)$ in $\mathcal{D}(\Lambda,N)$ such that $i(\Delta_1(l))=i(\Delta_2(l))$. In that case we pose $i(l)=i(\Delta_1(l))$.

- We have the *effective links* $l\in L_{\rm eff}$. To define such a link l one needs to specify a type $(e(l),(\gamma_1(l),\dots,\gamma_{e(l)}(l)))$ and a sequence of boxes $\Delta_1(l),\dots,\Delta_{e(l)}(l)$ in $\mathcal{D}(\Lambda,N)$ such that $\Delta_1(l)\cap\cdots\cap\Delta_{e(l)}(l)\neq\emptyset$.

- We have *core counter-term links* $l\in L_{\rm core}$. They are characterized by a type $(e(l),(\gamma_1(l),\dots,\gamma_{e(l)}(l)))$, a sequence of boxes $\Delta_1(l),\dots,\Delta_{e(l)}(l)$ in $\mathcal{D}(\Lambda,N)$ and a CAMC $\mathcal{M}(l)$ such that $\mathcal{M}(l)$ is of type $(e(l),(\gamma_1(l),\dots,\gamma_{e(l)}(l)))$, and $\underline{\Delta}(T_{\rm out}(\mathcal{M}(l)))$ is strictly contained in $\Delta_1(l)\cap\cdots\cap\Delta_{e(l)}(l)$.

- Finally we have the *face counter-term links* $l\in L_{\rm face}$. They are characterized by a type $(e(l),(\gamma_1(l),\dots,\gamma_{e(l)}(l)))$ a sequence of boxes $\Delta_1(l),\dots,\Delta_{e(l)}(l)$ in $\mathcal{D}(\Lambda,N)$ and a FAMC $\mathcal{M}(l)$ such that $\mathcal{M}(l)$ is of type $(e(l),(\gamma_1(l),\dots,\gamma_{e(l)}(l)))$, and $\partial\underline{\Delta}(T_{\rm out}(\mathcal{M}(l)))$ is contained in the closure of $\Delta_1(l)\cap\cdots\cap\Delta_{e(l)}(l)$. Besides, we demand that $i(\underline{\Delta}(T_{\rm out}(\mathcal{M}(l))))<\min(i(\Delta_1(l)),\dots,i(\Delta_{e(l)}(l)))$.

We now define H or in fact $H_\Gamma((t_l)_{l\in L})$ separately for each large field region Γ.

$$H_\Gamma((t_l)_{l\in L})\stackrel{\rm def}{=}\int d\mu_{\tilde C_0((t_l)_{l\in L})}(\phi_0)\otimes\cdots\otimes d\mu_{\tilde C_N((t_l)_{l\in L})}(\phi_N)$$

$$\exp\Big\{-\sum_{l\in L_{\rm eff}}t_l.\lambda^{\min(i(\Delta_1(l)),\dots,i(\Delta_{e(l)}(l)))}_{e(l),(\gamma_1(l),\dots,\gamma_{e(l)}(l))}\int_{\Delta_1(l)\cap\cdots\cap\Delta_{e(l)}(l)}dx$$

$$\partial^{\gamma_1(l)}\phi_{i(\Delta_1(l))}(x)\dots\partial^{\gamma_{e(l)}(l)}\phi_{i(\Delta_{e(l)}(l))}(x)$$

$$-\sum_{l\in L_{\rm core}}t_l.\mathcal{C}(\mathcal{M}(l))\int_{\underline{\Delta}(T_{\rm out}(\mathcal{M}(l)))}dx\;\partial^{\gamma_1(l)}\phi_{i(\Delta_1(l))}(x)\dots\partial^{\gamma_{e(l)}(l)}\phi_{i(\Delta_{e(l)}(l))}(x)$$

$$-\sum_{l\in L_{\rm face}}t_l.\mathcal{C}(\mathcal{M}(l))\int_{\partial\underline{\Delta}(T_{\rm out}(\mathcal{M}(l)))}d\Sigma(x)\;\partial^{\gamma_1(l)}\phi_{i(\Delta_1(l))}(x)\partial^{\gamma_2(l)}\phi_{i(\Delta_2(l))}(x)\Big\}\quad\text{(IV.41)}$$

where for each i, $0\le i\le N$, $\tilde C_i((t_l)_{l\in L})$ is the covariance defined by

$$\tilde C_i((t_l)_{l\in L})(x_1,x_2)\stackrel{\rm def}{=}\sum_{\substack{l\in L_{\rm hor}\\ i(l)=i}}t_l\chi_{\Delta_1(l)}(x_1)\tilde C_i(x_1,x_2)\chi_{\Delta_2(l)}(x_2)\quad.\qquad\text{(IV.42)}$$

It is clear from our definition that for any Γ, $H_\Gamma(\mathbf{1})=H_\Gamma$ introduced in (IV.33).

We now define the choice map $\mathcal{C}_\Gamma$ depending on the large field region. First for any sequence of cluster links we call also a *graph* $\mathfrak{g}=(l_1,\dots,l_k)$ we introduce $\Pi_\Gamma(\mathfrak{g})$ the partition of $\mathcal{D}(\Lambda,N)$ into *strongly connected sets*. We first define for any $l\in L$ the *support* $\mathrm{supp}(l)$ of l.

- If $l\in L_{\rm hor}$, supp is the set made by $\Delta_1(l)$, $\Delta_2(l)$ and every $\Delta\in\mathcal{D}_{i(l)}(\Lambda)$ such that $d_2^\Lambda(\Delta,\Delta_1(l))=0$ or $d_2^\Lambda(\Delta,\Delta_2(l))=0$.

- If $l \in L_{\rm eff}$, $\mathrm{supp}(l) \stackrel{\rm def}{=} \{\Delta_1(l), \dots, \Delta_{e(l)}(l)\}$.

- If $l \in L_{\rm core}$, $\mathrm{supp}(l) \stackrel{\rm def}{=} \{\Delta_1(l), \dots, \Delta_{e(l)}(l), \underline{\Delta}(T_{\rm out}(\mathcal{M}(l)))\}$.

- If $l \in L_{\rm face}$, $\mathrm{supp}(l) \stackrel{\rm def}{=} \{\Delta_1(l), \dots, \Delta_{e(l)}(l), \underline{\Delta}_1(l), \underline{\Delta}_2(l)\}$. where $\underline{\Delta}_1(l)$ and $\underline{\Delta}_2(l)$ are the two boxes in $\mathcal{D}_{i(\underline{\Delta}(T_{\rm out}(\mathcal{M}(l))))}(\Lambda)$ sharing $\partial\underline{\Delta}(T_{\rm out}(\mathcal{M}(l)))$ as a common face. Obviously one of them is $\underline{\Delta}(T_{\rm out}(\mathcal{M}(l)))$.

Now given $V \subset \mathcal{D}(\Lambda, N)$, we define $\mathfrak{g}_V$ as the subsequence of $\mathfrak{g}$ obtained by keeping only the cluster links whose support is a subset of V. Suppose we are given a partition π_0 of $\mathcal{D}(\Lambda, N)$. We say that V is strongly connected for $\mathfrak{g}$ with respect to π_0 if it satisfies the following conditions.

- V is *connected*, i.e. given Δ and Δ' in V there exists a sequence $\Delta_1, \dots, \Delta_b$ in V such that $\Delta_1 = \Delta$, $\Delta_2 = \Delta'$ and for any a, $1 \le a < b$, Δ_a, Δ_{a+1} fall both in the support of a cluster link in $\mathfrak{g}_V$ or in some block of π_0.

- for any $i \le \max\{i(\Delta) | \Delta \in V\}$ we consider the set $V_i \stackrel{\rm def}{=} \{\mathrm{pr}_i(\Delta) | i(\Delta) \le i, \Delta \in V\}$. Two elements Δ_1, Δ_2 of V_i are said *joined* if there exist Δ_1', $\Delta_2' \in V$ such that $i(\Delta_1') \le i$, $i(\Delta_2') \le i$, $\mathrm{pr}_i(\Delta_1') = \Delta_1$, $\mathrm{pr}_i(\Delta_2') = \Delta_2$ and Δ_1', Δ_2' fall in the support of some link in $\mathfrak{g}_V$ or in a block of π_0. Now if $X \subset V_i$ is some connected component of V_i for this notion, we define the *order* of X, $\mathrm{ord}(X)$, by the following. If there is a $\Delta \in V$, such $i(\Delta) \le i$ and $\mathrm{pr}_i(\Delta) \in X$, as well as a Δ', $i(\Delta') > i$ such that Δ and Δ' belong to the same block of π_0 then $\mathrm{ord}(X) = +\infty$. If not we consider the links $l \in \mathfrak{g}_V$ such that $\mathrm{supp}(l)$ contains a box $\Delta_{\rm high}$ with $i(\Delta_{\rm high}) \le i$ and $\mathrm{pr}_i(\Delta_{\rm high}) \in X$, and a box $\Delta_{\rm low}$ with $i(\Delta_{\rm low}) > i$. Those can only be effective or counterterm links, and the possible boxes $\Delta_{\rm low}$ are among $\Delta_1(l), \dots, \Delta_{e(l)}(l)$. We sum over these boxes the corresponding numbers $1 + |\gamma_c(l)|$, for $1 \le c \le e(l)$, and we sum the results over the above mentioned links $l \in \mathfrak{g}_V$ to obtain finally $\mathrm{ord}(X)$.

Now in order for V to be strongly connected we need that for every i and X, $\mathrm{ord}(X) \ge 5$. In other words the high frequency function that projects on X has a strictly negative superficial degree of divergence.

Now if $\Pi_\Gamma(\mathfrak{g})$ is the set of maximal strongly connected sets of $\mathfrak{g}$ with respect to $\pi_0 = \Pi_\Gamma(\emptyset)$. It is not difficult to see that $\Pi_\Gamma(\mathfrak{g})$ is indeed a partition of $\mathcal{D}(\Lambda, N)$ and that $\Pi_\Gamma(\emptyset)$ is a coherent notation. Besides $\Pi_\Gamma(\emptyset)$ is finer than $\Pi_\Gamma(\mathfrak{g})$.

Remark that our notion of strong connectedness is weaker than the 4-vertex irreducibility used in [AR2].

If $V \subset \mathcal{D}(\Lambda, N)$ we introduce the *vertical closure* of V as

$$\overline{V} \stackrel{\rm def}{=} \{\Delta \in \mathcal{D}(\Lambda, N) | \exists (\Delta_1, \Delta_2) \in V^2, \Delta_1 \subset \Delta \subset \Delta_2\} \quad . \tag{IV.43}$$

We then have to define the choice map $\mathcal{C}_\Gamma$. Given a sequence of cluster links $\mathfrak{g}$, a link $l \in L$ is said *forbidden* by $\mathfrak{g}$ if it falls into one of the following categories:

- $l \in (L_{\rm hor} \cup L_{\rm eff})$ and there is a block $B \in \Pi_\Gamma(\mathfrak{g})$ such that $\mathrm{supp}(l) \subset B$,

- $l \in (L_{\rm core} \cup L_{\rm face})$ and there is is a block $B \in \Pi_\Gamma(\mathfrak{g})$ such that $\Delta_1(l), \dots, \Delta_{e(l)}(l)$ are in B and $\mathrm{supp}(l) \subset \overline{B}$.

In both cases such a B is unique and we say that l is *forbidden by* B. Now $\mathcal{C}_\Gamma(\mathfrak{g})$ is by definition the set of the links in L that are not forbidden.

Now we can write the output of the identity IV.40 when used with the previous rules to compute H_Γ and thus $\tilde{Z}(\Lambda, N)$. We have:

$$\begin{aligned}
\tilde{Z}(\Lambda,N) = & \sum_{\Gamma \subset \mathcal{D}(\Lambda,N)} \sum_{\substack{\mathfrak{g}=(l_1,\dots,l_k)\\ \text{allowed}}} \\
& \int_{1>h_1>\dots>h_k>0} dh_1 \dots dh_k \int d\mu_{\tilde{C}_0(\mathbf{h})}(\phi_0) \otimes \dots \otimes d\mu_{\tilde{C}_N(\mathbf{h})}(\phi_N) \\
& \prod_{\substack{1\le m\le k\\ l_m \in L_{\rm hor}}} \left(\frac{1}{2} \int_\Lambda dx_{m,1} \int_\Lambda dx_{m,2}\ \chi_{\Delta_1(l_m)}(x_{m,1}) \chi_{\Delta_2(l_m)}(x_{m,2}) \right. \\
& \qquad \left. \tilde{C}_{i(l_m)}(x_{m,1},x_{m,2}) \frac{\delta}{\delta\phi_{i(l_m)}(x_{m,1})} \frac{\delta}{\delta\phi_{i(l_m)}(x_{m,2})} \right) \times \chi_\Gamma\left((\phi_i)_{0\le i\le N}\right) \\
& \times \prod_{\substack{1\le m\le k\\ l_m \in L_{\rm eff}}} \left(-\lambda^{\min(i(\Delta_1(l_m)),\dots,i(\Delta_{e(l_m)}(l_m)))}_{e(l_m),(\gamma_1(l_m),\dots,\gamma_{e(l_m)}(l_m))} \int_{\Delta_1(l_m)\cap\dots\cap\Delta_{e(l_m)}(l_m)} dx_m \right. \\
& \qquad \left. \partial^{\gamma_1(l_m)} \phi_{i(\Delta_1(l_m))}(x_m) \dots \partial^{\gamma_{e(l_m)}(l_m)} \phi_{i(\Delta_{e(l_m)}(l_m))}(x_m) \right) \\
& \times \prod_{\substack{1\le m\le k\\ l_m \in L_{\rm core}}} \left(-\mathcal{C}(\mathcal{M}(l_m)) \int_{\underline{\Delta}(T_{\rm out}(\mathcal{M}(l_m)))} dx_m \right. \\
& \qquad \left. \partial^{\gamma_1(l_m)} \phi_{i(\Delta_1(l_m))}(x_m) \dots \partial^{\gamma_{e(l_m)}(l_m)} \phi_{i(\Delta_{e(l_m)}(l_m))}(x_m) \right) \\
& \times \prod_{\substack{1\le m\le k\\ l_m \in L_{\rm face}}} \left(-\mathcal{C}(\mathcal{M}(l_m)) \int_{\partial\underline{\Delta}(T_{\rm out}(\mathcal{M}(l_m)))} d\Sigma(x_m) \right. \\
& \qquad \left. \partial^{\gamma_1(l_m)} \phi_{i(\Delta_1(l_m))}(x_m) \partial^{\gamma_2(l_m)} \phi_{i(\Delta_2(l_m))}(x_m) \right) \\
& \times \exp\left(-\sum_{j=0}^{k} (h_j - h_{j+1}) \sum_{B\in\Pi_\Gamma((l_1,\dots,l_j))} I_B\left((\phi_i)_{0\le i\le N}\right) \right) \quad . \qquad \text{(IV.44)}
\end{aligned}$$

In this formula, $\mathbf{h}$ denotes the vector $(h_1, \dots, h_k)$, the functional derivations $\frac{\delta}{\delta\phi}$ may act upon any an factor on the right, each χ_Δ is periodized on $\Lambda \simeq \frac{\mathbb{R}^4}{(L\mathbb{Z})^4}$ and

we have posed $h_0 \stackrel{\text{def}}{=} 1$ and $h_{k+1} \stackrel{\text{def}}{=} 0$. Furthermore $\tilde{C}_i(\mathbf{h})$ is defined by

$$\tilde{C}_i(\mathbf{h})(x_1,x_2) = \sum_{j=0}^{k}(h_j - h_{j+1}) \sum_{B\in\Pi_\Gamma((l_1,\ldots,l_j))} \left(\sum_{\substack{\Delta\in\breve{B}\\ i(\Delta)=i}} \chi_\Delta(x_1)\right) \tilde{C}_i(x_1,x_2) \left(\sum_{\substack{\Delta\in\breve{B}\\ i(\Delta)=i}} \chi_\Delta(x_2)\right) \qquad \text{(IV.45)}$$

where $\breve{B}$ denotes the *interior* of B:

$$\breve{B} \stackrel{\text{def}}{=} \{\Delta \in B | \forall \Delta' \in \mathcal{D}_{i(\Delta)}(\Lambda), (d_2^\Lambda(\Delta,\Delta') = 0 \Rightarrow \Delta' \in B)\} \qquad \text{(IV.46)}$$

Finally $I_B((\phi_i)_{0\le i\le N})$ is the sum of the contributions, as appearing in (IV.42), of the links in $L_{\text{eff}} \cup L_{\text{core}} \cup L_{\text{face}}$ that are forbidden by B.

It is not difficult to check that thanks to our definition of the effective coupling constants,

$$I_B\left((\phi_i)_{0\le i\le N}\right) = \int_\Lambda dx \left\{ \sum_{e,(\gamma_1,\ldots\gamma_e)} \lambda^{i_B(x)}_{e,(\gamma_1,\ldots\gamma_e)} \left(\sum_{i_1\in I_B(x)} \partial^{\gamma_1}\phi_{i_1}(x)\right)\ldots\left(\sum_{i_e\in I_B(x)} \partial^{\gamma_e}\phi_{i_e}(x)\right)\right\} \qquad \text{(IV.47)}$$

where $I_B(x)$ denotes the *section* of B at x defined by $I_B(x) \stackrel{\text{def}}{=} \{i, 0\le i\le N | \exists \Delta \in B, i(\Delta) = i, x\in\Delta\}$ and $i_B(x) \stackrel{\text{def}}{=} \min(I_B(x))$. The fact that the faced counter-term links do not appear in the interpolated exponential is due to their banning rule and the face symmetry. Indeed for such a link l, to be forbidden by B, both the cubes $\underline{\Delta}_1(l)$ and $\underline{\Delta}_2(l)$ touching the face $\partial\underline{\Delta}(T_{\text{out}}(\mathcal{M}(l)))$ have to be in $\overline{B}$. But this condition is also shared by the link corresponding to the "flipped" Mayer configuration i.e. the one obtained by translating $\text{opp}(\mathcal{M}(l))$ by the vector

$$M^{i(\underline{\Delta}(T_{\text{out}}(\mathcal{M}(l))))}.\text{dir}((T_{\text{out}}(\mathcal{M}(l)))) \quad .$$

Since this flipped configuration has the opposite coefficient $\mathcal{C}(\mathcal{M})$, we see that the contributions of the forbidden face counter-term links cancel each other.

Note that $\tilde{C}_i(\mathbf{h})$ is clearly from (IV.45) a positive bilinear form on the nuclear space of smooth periodic functions on Λ. Therefore the Gaussian measures $d\mu_{\tilde{C}_i(\mathbf{h})}$ are well defined.

The stability of the interpolated interaction in the exponential factor of (IV.44) will follow if we prove that $I_B((\phi_i)_{0\le i\le N})$ is bounded from below. For that we must

know the renormalization group flow of the effective coupling constants. In particular we must show that $\lambda^i_{e,(\gamma_1,\dots,\gamma_e)}$ is zero except for $\lambda^i_{4,(0,0,0,0)}$, $\lambda^i_{2,(0,0)}$ and $\lambda^i_{2,(\gamma,\gamma)}$ with $|\gamma| = 1$. These correspond to the usual coupling constant, mass and wave function renormalizations, respectively. We will have to show that the running coupling constants of the ϕ^4 and the $(\partial\phi)^2$ terms are positive. The $\lambda^i_{2,(0,0)}$ is allowed to be negative and will in fact be if we try to approach the critical point of the infrared ϕ^4_4 model. This study is postponed to section IV.4.

We now have to write (IV.44) as a polymer expansion like we did in [AR2] (theorem 1). Therefore we have to group together every features like Γ, $\mathfrak{g}$, the integration with respect to $\mathbf{h}$ and the fields $(\phi_i)_{0\le i\le N}$ themselves, according to the block of the final partition $\Pi_\Gamma(\mathfrak{g})$ to which they belong.

For any $Y \in \Pi_\Gamma(\mathfrak{g})$, we define $\Gamma_Y \stackrel{\text{def}}{=} \Gamma \cap Y$. Note that Γ_Y has to verify the following property. If $\Delta \in \Gamma_Y$ and $i(\Delta) > 1$ then any Δ' just above Δ has to belong to Y. We then say that Γ_Y is *internal* to Y. This property does not depend on what happen in the other blocks of $\Pi_\Gamma(\mathfrak{g})$, therefore we can make the sums $\sum_{\Gamma_Y}$ for $Y \in \Pi_\Gamma(\mathfrak{g})$ that compose $\sum_\Gamma$, independently. Likewise if $\mathfrak{g}_{\text{ext}}$ is the sequence of links obtained from $\mathfrak{g}$ by removing all the disjoint subsequences $\mathfrak{g}_Y$, for $Y \in \Pi_\Gamma(\mathfrak{g})$, we have to show that we can sum independently on $\mathfrak{g}_{\text{ext}}$ and the $\mathfrak{g}_Y$'s.

Note that any link in $\mathfrak{g}_{\text{ext}}$ is in $\mathcal{C}_\Gamma(\mathfrak{g})$. Else there would be a $Y \in \Pi_\Gamma(\mathfrak{g})$ that forbids l. It is not difficult to see that $Y \cup \text{supp}(l)$ would then be strongly connected by $\mathfrak{g}$ and the maximality of Y implies $\text{supp}(l) \subset Y$ which is a contradiction by definition of $\mathfrak{g}_{\text{ext}}$. Note also that the condition of allowedness for $\mathfrak{g}$ with respect to Γ is equivalent to that of all the $\mathfrak{g}_Y$'s, $Y \in \Pi_\Gamma(\mathfrak{g})$, relatively to Γ_Y respectively. The proof is similar to that of lemmas 6 and 7 in [AR2]. Given a partition π of $\mathcal{D}(\Lambda, N)$ we say that a graph $\mathfrak{g}_{\text{ext}}$ is π-*trivial* if $\mathfrak{g}$ does not connect the blocks of π into bigger strongly connected sets. This means that a strongly connected set for $\mathfrak{g}_{\text{ext}}$ with respect to the partition π must be contained in some element of π.

Now summing over the allowed $\mathfrak{g}$ with $\Pi_\Gamma(\mathfrak{g})$ fixed equal to π, is the same as summing independently over each $\mathfrak{g}_Y$, $Y \in \pi$, allowed with respect to Γ_Y and made by links whose support is in Y, and over $\mathfrak{g}_{\text{ext}}$ made by links whose support is not in an element of π and which is π-trivial. Finally one has to sum over the intertwinings of the sequences $\mathfrak{g}_{\text{ext}}$ and $\mathfrak{g}_Y$, for $Y \in \pi$, to recompose the full sequence $\mathfrak{g}$. As is explained in [AR2] the effect of this sum over intertwinings is to release, in the integration on $\mathbf{h}$, the constraints $1 > h_1 > \cdots > h_k > 0$ that are between any h_i and h_j if l_i and l_j are not in the same $\mathfrak{g}_Y$ of $\mathfrak{g}_{\text{ext}}$.

Finally we want to factorize the functional integrations between each block of $\pi = \Pi_\Gamma(\mathfrak{g})$. For that we decompose the interpolated covariance $\tilde{C}_i(\mathbf{h})$ according to

$\tilde{C}_i(\mathbf{h}) = \sum_{Y\in\pi} \tilde{C}_{i,Y}(\mathbf{h})$, where

$$\tilde{C}_{i,Y}(\mathbf{h})(x_1,x_2) \stackrel{\text{def}}{=} \sum_{j=0}^{k}(h_j - h_{j+1}) \sum_{\substack{B\in\Pi_\Gamma((l_1,\dots,l_j))\\ B\subset Y}} \left(\sum_{\substack{\Delta\in\breve{B}\\ i(\Delta)=i}} \chi_\Delta(x_1)\right) \tilde{C}_i(x_1,x_2) \left(\sum_{\substack{\Delta\in\breve{B}\\ i(\Delta)=i}} \chi_\Delta(x_2)\right) . \qquad \text{(IV.48)}$$

Likewise we replace each fluctuating Gaussian field ϕ_i by a sum $\sum_{Y\in\pi}\phi_{i,Y}$ of independent Gaussian fields $\phi_{i,Y}$ with covariances $\tilde{C}_{i,Y}(\mathbf{h})$ respectively. Now each field $\partial^\gamma\phi_i(x)$ appearing in the integrand of (IV.44) becomes $\sum_{Y\in\pi}\partial^\gamma\phi_{i,Y}(x)$. Now if Δ the box containing x at scale i, is in some Y then almost surely $\partial^\gamma\phi_{i,Y'}(x) = 0$ for any $Y'\in\pi$, $Y'\neq Y$. Indeed the $\chi_{\Delta'}$ with $\Delta'\in\breve{Y}'$, $i(\Delta') = i(\Delta)$, vanish on a neighborhood of Δ, thanks to our choice of ξ defining our smooth partition of unity. Therefore $\tilde{C}_{i,Y}(\mathbf{h})(y,y) = 0$ for y in a neighborhood of Δ. Since we deal with smooth fields $\phi_{i,Y}$, the "almost surely" locally for fixed x can be upgraded (by considering the x's with rational coordinates) to a global "almost surely" for all x in Δ.

What we get when we collect the pieces of the expansion in some $Y\in\pi$ is a *polymer amplitude*. Given a family $\mathcal{J} = ((i_q,y_q,\alpha_q))_{1\le q\le r}$ where i_q is a scale index in $\{0,\dots,N\}$, y_q is a point in Λ, and α_q a derivation multi-index, we introduce the polymer amplitude

$$\begin{aligned}\mathcal{A}(Y;\mathcal{J}) \stackrel{\text{def}}{=} &\sum_{\Gamma_Y}\sum_{\mathfrak{g}_Y=(l_1,\dots,l_k)} \int_{1>h_1>\dots>h_k>0} dh_1\dots dh_k \int d\mu_{\tilde{C}_{0,Y}(\mathbf{h})}(\phi_{0,Y})\otimes\dots\otimes d\mu_{\tilde{C}_{N,Y}(\mathbf{h})}(\phi_{N,Y})\\ &\prod_{\substack{1\le m\le k\\ l_m\in L_{\text{hor}}}}\left(\frac12\int_\Lambda dx_{m,1}\int_\Lambda dx_{m,2}\ \chi_{\Delta_1(l_m)}(x_{m,1})\chi_{\Delta_2(l_m)}(x_{m,2})\right.\\ &\left.\tilde{C}_{i(l_m)}(x_{m,1},x_{m,2})\frac{\delta}{\delta\phi_{i(l_m),Y}(x_{m,1})}\frac{\delta}{\delta\phi_{i(l_m),Y}(x_{m,2})}\right)\\ &\times\partial^{\alpha_1}\phi_{i_1,Y}(y_1)\dots\partial^{\alpha_r}\phi_{i_r,Y}(y_r)\times\chi_{\Gamma_Y,Y}\left((\phi_{i,Y})_{0\le i\le N}\right)\\ \times&\prod_{\substack{1\le m\le k\\ l_m\in L_{\text{eff}}}}\left(-\lambda^{\min(i(\Delta_1(l_m)),\dots,i(\Delta_{e(l_m)}(l_m)))}_{e(l_m),(\gamma_1(l_m),\dots,\gamma_{e(l_m)}(l_m))}\int_{\Delta_1(l_m)\cap\dots\cap\Delta_{e(l_m)}(l_m)}dx_m\right.\\ &\left.\partial^{\gamma_1(l_m)}\phi_{i(\Delta_1(l_m)),Y}(x_m)\dots\partial^{\gamma_{e(l_m)}(l_m)}\phi_{i(\Delta_{e(l_m)}(l_m)),Y}(x_m)\right)\end{aligned}$$

$$\times \prod_{\substack{1\le m\le k\\ l_m\in L_{\rm core}}} \left(-\mathcal{C}(\mathcal{M}(l_m)) \int_{\underline{\Delta}(T_{\rm out}(\mathcal{M}(l_m)))} dx_m \right.$$
$$\left. \partial^{\gamma_1(l_m)}\phi_{i(\Delta_1(l_m)),Y}(x_m)\dots\partial^{\gamma_{e(l_m)}(l_m)}\phi_{i(\Delta_{e(l_m)}(l_m)),Y}(x_m) \right)$$
$$\times \prod_{\substack{1\le m\le k\\ l_m\in L_{\rm face}}} \left(-\mathcal{C}(\mathcal{M}(l_m)) \int_{\partial\underline{\Delta}(T_{\rm out}(\mathcal{M}(l_m)))} d\Sigma(x_m) \right.$$
$$\left. \partial^{\gamma_1(l_m)}\phi_{i(\Delta_1(l_m)),Y}(x_m)\partial^{\gamma_2(l_m)}\phi_{i(\Delta_2(l_m)),Y}(x_m) \right)$$
$$\times \exp\left(-\sum_{j=0}^{k}(h_j-h_{j+1}) \sum_{\substack{B\in\Pi_{\Gamma_Y}((l_1,\dots,l_j))\\ B\subset Y}} I_B\left((\phi_{i,Y})_{0\le i\le N}\right) \right) \quad . \tag{IV.49}$$

Here Γ_Y is summed over large field regions that are internal to Y. $\chi_{\Gamma_Y,Y}$ denotes the product of factors in χ_{Γ_Y}, as defined in (IV.31), corresponding to boxes Δ in Y. $\mathfrak{g}_Y$ is summed over graphs that are made by links with support in Y, are allowed with respect to Γ_Y and make Y strongly connected.

Finally we can now rewrite (IV.44) as a polymer expansion:

$$\tilde{Z}(\Lambda,N) = \sum_{\pi} \sum_{\mathfrak{g}_{\rm ext}=(l_1,\dots,l_{p_{\rm ext}})} \frac{1}{p_{\rm ext}!} \prod_{\substack{1\le m\le p_{\rm ext}\\ l_m\in L_{\rm eff}}} \left(-\lambda_{e(l_m),(\gamma_1(l_m),\dots,\gamma_{e(l_m)}(l_m))}^{\min(i(\Delta_1(l_m)),\dots,i(\Delta_{e(l_m)}(l_m)))} \right.$$
$$\left. \int_{\Delta_1(l_m)\cap\dots\cap\Delta_{e(l_m)}(l_m)} dx_m \right) \prod_{\substack{1\le m\le p_{\rm ext}\\ l_m\in L_{\rm core}}} \left(-\mathcal{C}(\mathcal{M}(l_m)) \int_{\underline{\Delta}(T_{\rm out}(\mathcal{M}(l_m)))} dx_m \right)$$
$$\prod_{\substack{1\le m\le p_{\rm ext}\\ l_m\in L_{\rm face}}} \left(-\mathcal{C}(\mathcal{M}(l_m)) \int_{\partial\underline{\Delta}(T_{\rm out}(\mathcal{M}(l_m)))} d\Sigma(x_m) \right) \prod_{Y\in\pi} \mathcal{A}(Y;\mathcal{J}(\mathfrak{g}_{\rm ext},Y)) \quad . \tag{IV.50}$$

Here π is summed over the partitions of $\mathcal{D}(\Lambda,N)$, and $\mathfrak{g}_{\rm ext}$ over the graphs that are made by links whose support is not contained in an element of π, and such that $\mathfrak{g}_{\rm ext}$ is π-trivial. $\frac{1}{p_{\rm ext}!}$ is the volume resulting from the integration of the h parameters attached to the links of $\mathfrak{g}_{\rm ext}$. Finally $\mathcal{J}(\mathfrak{g}_{\rm ext},Y)$ is the family obtained by putting in lexicographic order in (m,c), $1\le m\le k$, $1\le c\le e(l_m)$, the triplets $(i(\Delta_c(l_m)),x_m,\gamma_c(l_m))$ such that $\Delta_c(l_m)\in Y$.

Note that this rewriting was possible because when considering the piece of covariance or interaction in some Y, in formula (IV.44), they were expressed by sums of the type $\sum_{j=0}^{k}(h_j-h_{j-1}).u_j$. The latter, in fact reduces to $\sum_{\nu=0}^{\rho}(h_{j_\nu}-h_{j_{\nu+1}})u_{j_\nu}$

where $j_0 = 0$, $j_{\rho+1} = k+1$ and $(l_{j_1}, \ldots, l_{j_\rho})$ is the subsequence $\mathfrak{g}_Y$ of $\mathfrak{g}$. Indeed u_j which is either

$$\sum_{\substack{B \in \Pi_\Gamma((l_1,\ldots,l_j)) \\ B \subset Y}} \left(\sum_{\substack{\Delta \in \check{B} \\ i(\Delta)=i}} \chi_\Delta(x_1) \right) \tilde{C}_i(x_1, x_2) \left(\sum_{\substack{\Delta \in \check{B} \\ i(\Delta)=i}} \chi_\Delta(x_2) \right) \tag{IV.51}$$

for the covariance or

$$\sum_{\substack{B \in \Pi_\Gamma((l_1,\ldots,l_j)) \\ B \subset Y}} I_B\left((\phi_{i,Y})_{0 \le i \le N}\right) \tag{IV.52}$$

for the interaction potential, varies only when j is some j_ν. Actually, only the links with support in Y are relevant to the computation of the partition induced on Y by $\Pi_\Gamma((l_1, \ldots, l_j))$.

Note that if Y is reduced to a single cube Δ then $\breve{Y} = \emptyset$ and thus $\tilde{C}_{i,Y}(\mathbf{h})$ is identically zero and so are the fields $\phi_{i,Y}$, $0 \le i \le N$, almost surely. As a result, the only Γ_Y giving a non zero contribution is $\emptyset$, for which we have

$$\chi_{\Gamma_Y,Y}\left((\phi_{i,Y})_{0 \le i \le N}\right) = \chi\left((\lambda^{i(\Delta)}_{4,(0,0,0,0)})^{(1+\epsilon_1)} \int_\Delta \phi_{i,Y}(x)^4 dx \right) \quad . \tag{IV.53}$$

Therefore $\mathcal{A}(Y; \mathcal{J}) = 1$ if $\mathcal{J}$ is empty, and $\mathcal{A}(Y, \mathcal{J}) = 0$ otherwise. This is a nice feature in the cluster expansion business since we do not have in order to compute $\log \tilde{Z}(\Lambda, N)$, to divide by the normalization of isolated cubes called also *monomers*.

IV.3.2.4 The Mayer expansion

We present here a new point of view for performing the well known Mayer expansion, which is more suitable to the present situation. The goal is to compute $\log \tilde{Z}(\Lambda, N)$. The way we proceed is by solving in the coefficients $\mathcal{C}(\mathcal{M})$ where $\mathcal{M}$ is a VAMC, the equation

$$1 = \exp\left(- \sum_{\mathcal{M} \text{ VAMC}} \mathcal{C}(\mathcal{M}) \right) \cdot \tilde{Z}(\Lambda, N) \tag{IV.54}$$

i.e.

$$1 = \sum_{r \ge 0} \frac{1}{r!} \sum_{\substack{\mathcal{M}_1, \ldots, \mathcal{M}_r \\ \text{VAMC}}} (-\mathcal{C}(\mathcal{M}_1)) \ldots (-\mathcal{C}(\mathcal{M}_r)) \cdot \tilde{Z}(\Lambda, N) \quad . \tag{IV.55}$$

In the last line we decompose $\tilde{Z}(\Lambda, N)$ according to the polymer expansion (IV.50). A term in the obtained sum is indexed by r, $\mathcal{M}_1, \ldots, \mathcal{M}_r$, π and $\mathfrak{g}_{\text{ext}}$.

If $Y_1, \dots, Y_s$ are the elements of π that are not isolated cubes, since they are disjoint, summing over π is the same as summing over sequences $Y_1, \dots, Y_s$ with a $\frac{1}{s!}$ symmetrization factor. Therefore the equation to solve in the $\mathcal{C}(\mathcal{M})$'s is

$$1 = \sum_{r \geq 0} \sum_{\substack{\mathcal{M}_1, \dots, \mathcal{M}_r \\ \text{VAMC}}} \sum_{s \geq 0} \sum_{\substack{(Y_1, \dots, Y_s) \\ \text{disjoint} \\ \#(Y_i) \geq 2}} \sum_{\mathfrak{g}_{\text{ext}}} \frac{1}{r! s! p_{\text{ext}}!} (-\mathcal{C}(\mathcal{M}_1)) \dots (-\mathcal{C}(\mathcal{M}_r))$$

$$\prod_{\substack{1 \leq m \leq p_{\text{ext}} \\ l_m \in L_{\text{eff}}}} \left(-\lambda^{\min(i(\Delta_1(l_m)), \dots, i(\Delta_{e(l_m)}(l_m)))}_{e(l_m), (\gamma_1(l_m), \dots, \gamma_{e(l_m)}(l_m))} \int_{\Delta_1(l_m) \cap \dots \cap \Delta_{e(l_m)}(l_m)} dx_m \right)$$

$$\prod_{\substack{1 \leq m \leq p_{\text{ext}} \\ l_m \in L_{\text{core}}}} \left(-\mathcal{C}(\mathcal{M}(l_m)) \int_{\underline{\Delta}(T_{\text{out}}(\mathcal{M}(l_m)))} dx_m \right)$$

$$\prod_{\substack{1 \leq m \leq p_{\text{ext}} \\ l_m \in L_{\text{face}}}} \left(-\mathcal{C}(\mathcal{M}(l_m)) \int_{\partial \underline{\Delta}(T_{\text{out}}(\mathcal{M}(l_m)))} d\Sigma(x_m) \right) \prod_{Y \in \pi} \mathcal{A}(Y; \mathcal{J}(\mathfrak{g}_{\text{ext}}, Y)) \quad . \qquad \text{(IV.56)}$$

The goal is to write the right hand side as a sum over VAMC's, and checking that the coefficients for any non empty VAMC is zero. For that we have to show how to gather the information contained in $\mathcal{M}_1, \dots, \mathcal{M}_r$, $Y_1, \dots, Y_s$ and $\mathfrak{g}_{\text{ext}}$ to construct by a *concatenation* procedure a new VAMC $\mathcal{M}_{\text{big}}$.

First the polymer sequence in $\mathcal{M}_{\text{big}}$ is obtained by listing in the following order $Y_1, \dots, Y_s$, the polymer sequence of $\mathcal{M}_1$, ..., the polymer sequence of $\mathcal{M}_r$, then the polymer sequences of $\mathcal{M}(l_m)$, $1 \leq m \leq p_{\text{ext}}$, whenever $l_m \in L_{\text{core}} \cup L_{\text{face}}$.

Now the link sequence of $\mathcal{M}_{\text{big}}$ is likewise obtained by listing with the order of $\mathfrak{g}_{\text{ext}}$ the links of the latter that are in L_{eff}, then we list the links of $\mathcal{M}_1, \dots, \mathcal{M}_r$, and that of $\mathcal{M}(l_m)$, $1 \leq m \leq p_{\text{ext}}$ such that $l_m \in L_{\text{core}} \cup L_{\text{face}}$. There is a slight lack of precision in the last definition since these links are not a priori links with legs $((i, \Delta), \omega)$ where i is a polymer index of $\mathcal{M}_{\text{big}}$. Therefore a little treatment is necessary for each of the links we used in order to be admissible as links of $\mathcal{M}_{\text{big}}$.

If l_m is a link of $\mathfrak{g}_{\text{ext}}$ in L_{eff}, we replace it by the sequence of legs

$$(((j_1, \Delta_1(l_m)), \gamma_1(l_m)), \dots, ((j_{e(l_m)}, \Delta_{e(l_m)}(l_m)), \gamma_{e(l_m)}(l_m))) \qquad \text{(IV.57)}$$

where j_ν is the unique j, $1 \leq j \leq s$, such that $\Delta_\nu(l_m) \in Y_j$. Indeed $Y_1, \dots, Y_s$ are disjoint.

If l is some link of $\mathcal{M}_u$, $1 \leq u \leq r$ and $((i, \Delta), \omega)$ is a leg of l, we need only to change the polymer index i in $\mathcal{M}_u$ into the new index in $\mathcal{M}_{\text{big}}$ where the corresponding polymer was placed when constructing the polymer sequence of $\mathcal{M}_{\text{big}}$.

If l is some link of $\mathcal{M}(l_m)$, $1 \leq m \leq p_{\text{ext}}$, $l_m \in (L_{\text{core}} \cup L_{\text{face}})$ we need to do the same operation. Besides if $(c, \gamma_c(l_m))$, $1 \leq c \leq e(l_m)$, is an external leg of l, we

put instead $((j, \Delta_c(l_m)), \gamma_c(l_m))$ where j is the unique index $1 \leq j \leq s$ such that $\Delta_c(l_m) \in Y_j$.

Now the history of $\mathcal{M}_{\rm big}$ is simply the union of the histories of $\mathcal{M}_1, \ldots, \mathcal{M}_r$ and $\mathcal{M}(l_m)$ such that $1 \leq m \leq p_{\rm ext}$ and $l_m \in (L_{\rm core} \cup L_{\rm face})$. Again the only modification is to change the index of a polymer or a link into its new location in the polymer or link sequence of $\mathcal{M}_{\rm big}$.

We have now to split the contribution of $\mathcal{M}_{\rm big}$ in (IV.56) by introducing a decomposition of the "dangerous subgraphs" into renormalized and local part. This is close in spirit to the introduction of the substraction operators τ_g and $1 - \tau_g$ in Zimmermann's treatment of perturbative renormalization (see [R1]).

The idea is to collect the polymers by connectedness with respect to the links of $\mathcal{M}_{\rm big}$ and with respect to the notion of overlap, above a certain scale. If a connected component communicates with the lower frequencies by links in $\mathfrak{g}_{\rm ext}$ whose lower legs are $\partial^{\gamma_1}\phi_{i_1}(x_1), \ldots, \partial^{\gamma_e}\phi_{i_e}(x_e)$ with $|\gamma_1| + \cdots + |\gamma_e| + e \leq 4$, i.e. that component is "dangerous", we perform the following operations. We suppose we have chosen a cube $\underline{\Delta}$ of reference and write

$$\partial^{\gamma_1}\phi_{i_1}(x_1) \ldots \partial^{\gamma_e}\phi_{i_e}(x_e) = \frac{1}{|\underline{\Delta}|} \int_{\underline{\Delta}} dz \partial^{\gamma_1}\phi_{i_1}(z) \ldots \partial^{\gamma_e}\phi_{i_e}(z)$$

$$+ \sum_{c=1}^{e} \sum_{\nu=1}^{4} \frac{1}{|\underline{\Delta}|} \int_{\underline{\Delta}} dz \int_0^1 dt \ (x_c - z)_\nu$$

$$\partial^{\gamma_1}\phi_{i_1}(z + t(x_1 - z)) \ldots \partial^{\gamma_c + e_\nu}\phi_{i_c}(z + t(x_c - z)) \ldots \partial^{\gamma_e}\phi_{i_e}(z + t(x_e - z)) \quad \text{(IV.58)}$$

where e_ν is the multi-index with 1 at entry ν, and 0 at the other entries. If $|\gamma_1| + \cdots + |\gamma_e| + e = 4$, we call the first term in (IV.58) the *local part* and the second expression the *renormalized part*. If $|\gamma_1| + \cdots + |\gamma_e| + e < 4$ then the product of fields in the second expression is still "dangerous" since $|\gamma_1| + \cdots + |\gamma_c + e_\nu| + \cdots + |\gamma_e| + e \leq 4$. Therefore we apply to it the same expansion as (IV.58) and we repeat the operation as long as there is a product of fields located in the interpolated positions with multi-indices of derivation $\alpha_1, \ldots, \alpha_e$ such that $|\alpha_1| + \cdots + |\alpha_e| + e \leq 4$. In the final expression the terms where the fields are located at the same point z integrated in $\underline{\Delta}$ go into the local part and the remaining trems make the renormalized part. This inductive procedure stops after at most four steps in the case of a ϕ^4_4 model. We give here the decomposition associated with the really important field monomials i.e. the ϕ^2, $\partial\phi\partial\phi$, $\phi\partial\partial\phi$ and the ϕ^4 terms.

- The local part of $\phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3)\phi_{i_4}(x_4)$ is

$$\mathrm{loc}(\phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3)\phi_{i_4}(x_4)) \stackrel{\rm def}{=} \frac{1}{|\underline{\Delta}|} \int_{\underline{\Delta}} dz \phi_{i_1}(z)\phi_{i_2}(z)\phi_{i_3}(z)\phi_{i_4}(z) \quad . \quad \text{(IV.59)}$$

-The renormalized part is

$$\mathrm{ren}(\phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3)\phi_{i_4}(x_4)) \stackrel{\mathrm{def}}{=} \frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz \int_0^1 dt\ (x_1-z)^\nu$$

$$\partial_\nu\phi_{i_1}(x_1+t(x_1-z))\phi_{i_2}(x_2+t(x_2-z))\phi_{i_3}(x_3+t(x_3-z))\phi_{i_4}(x_4+t(x_4-z))$$

$$+\frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz \int_0^1 dt\ (x_2-z)^\nu\partial_\nu\phi_{i_2}(x_2+t(x_2-z))$$

$$\phi_{i_1}(x_1+t(x_1-z))\phi_{i_3}(x_3+t(x_3-z))\phi_{i_4}(x_4+t(x_4-z))$$

$$+\frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz \int_0^1 dt\ (x_3-z)^\nu\partial_\nu\phi_{i_3}(x_3+t(x_3-z))$$

$$\phi_{i_1}(x_1+t(x_1-z))\phi_{i_2}(x_2+t(x_2-z))\phi_{i_4}(x_4+t(x_4-z))$$

$$+\frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz \int_0^1 dt\ (x_4-z)^\nu\partial_\nu\phi_{i_4}(x_4+t(x_4-z))$$

$$\phi_{i_1}(x_1+t(x_1-z))\phi_{i_2}(x_2+t(x_2-z))\phi_{i_3}(x_3+t(x_3-z)) \quad . \tag{IV.60}$$

- Likewise

$$\mathrm{loc}(\phi_{i_1}(x_1)\partial_\mu\partial_\nu\phi_{i_2}(x_2)) \stackrel{\mathrm{def}}{=} \frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz\phi_{i_1}(z)\partial_\mu\partial_\nu\phi_{i_2}(z) \tag{IV.61}$$

- and

$$\mathrm{ren}(\phi_{i_1}(x_1)\partial_\mu\partial_\nu\phi_{i_2}(x_2)) \stackrel{\mathrm{def}}{=} \frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz \int_0^1 dt$$

$$(x_1-z)^\rho\partial_\rho\phi_{i_1}(x_1+t(x_1-z))\partial_\mu\partial_\nu\phi_{i_2}(x_2+t(x_2-z))$$

$$+\frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz \int_0^1 dt(x_2-z)^\rho\phi_{i_1}(x_1+t(x_1-z))\partial_\rho\partial_\mu\partial_\nu\phi_{i_2}(x_2+t(x_2-z)) \quad . \tag{IV.62}$$

- Similarly

$$\mathrm{loc}(\partial_\mu\phi_{i_1}(x_1)\partial_\nu\phi_{i_2}(x_2)) \stackrel{\mathrm{def}}{=} \frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz\partial_\mu\phi_{i_1}(z)\partial_\nu\phi_{i_2}(z) \tag{IV.63}$$

- and

$$\mathrm{ren}(\partial_\mu\phi_{i_1}(x_1)\partial_\nu\phi_{i_2}(x_2)) \stackrel{\mathrm{def}}{=} \frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz \int_0^1 dt\ (x_1-z)^\rho$$

$$\partial_\rho\partial_\mu\phi_{i_1}(x_1+t(x_1-z))\partial_\nu\phi_{i_2}(x_2+t(x_2-z))$$

$$+\frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz \int_0^1 dt (x_2-z)^\rho \partial_\mu \phi_{i_1}(x_1+t(x_1-z))\partial_\rho\partial_\nu\phi_{i_2}(x_2+t(x_2-z)) \ . \quad \text{(IV.64)}$$

- Now more complicated

$$\begin{aligned}
\mathrm{loc}(\phi_{i_1}(x_1)\phi_{i_2}(x_2)) &\stackrel{\mathrm{def}}{=} \frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}} dz_1 \phi_{i_1}(z_1)\phi_{i_2}(z_1) \\
&+\frac{1}{|\underline{\Delta}|^2}\int_{\underline{\Delta}} dz_1 \int_{\underline{\Delta}} dz_2 (x_1-z_1)^\nu \partial_\nu \phi_{i_1}(z_2)\phi_{i_2}(z_2) \\
&+\frac{1}{|\underline{\Delta}|^2}\int_{\underline{\Delta}} dz_1 \int_{\underline{\Delta}} dz_2 (x_2-z_1)^\nu \phi_{i_1}(z_2)\partial_n u \phi_{i_2}(z_2) \\
+\frac{1}{|\underline{\Delta}|^3}\int_{\underline{\Delta}} dz_1 \int_{\underline{\Delta}} dz_2 \int_{\underline{\Delta}} dz_3 \int_0^1 dt_1 &(x_1-z_1)^\nu (z_1+t_1(x_1-z_1)-z_2)^\mu \partial_\mu\partial_\nu\phi_{i_1}(z_3)\phi_{i_2}(z_3) \\
+\frac{1}{|\underline{\Delta}|^3}\int_{\underline{\Delta}} dz_1 \int_{\underline{\Delta}} dz_2 \int_{\underline{\Delta}} dz_3 \int_0^1 dt_1 &(x_1-z_1)^\nu (z_1+t_1(x_2-z_1)-z_2)^\mu \partial_\nu\phi_{i_1}(z_3)\partial_\mu\phi_{i_2}(z_3) \\
+\frac{1}{|\underline{\Delta}|^3}\int_{\underline{\Delta}} dz_1 \int_{\underline{\Delta}} dz_2 \int_{\underline{\Delta}} dz_3 \int_0^1 dt_1 &(x_2-z_1)^\nu (z_1+t_1(x_1-z_1)-z_2)^\mu \partial_\mu\phi_{i_1}(z_3)\partial_\nu\phi_{i_2}(z_3) \\
+\frac{1}{|\underline{\Delta}|^3}\int_{\underline{\Delta}} dz_1 \int_{\underline{\Delta}} dz_2 \int_{\underline{\Delta}} dz_3 \int_0^1 dt_1 &(x_2-z_1)^\nu (z_1+t_1(x_2-z_1)-z_2)^\mu \phi_{i_1}(z_3)\partial_\mu\partial_\nu\phi_{i_2}(z_3)
\end{aligned} \quad \text{(IV.65)}$$

- and

$$\begin{aligned}
\mathrm{ren}(\phi_{i_1}(x_1)\phi_{i_2}(x_2)) = \frac{1}{|\underline{\Delta}|^3}\int_{\underline{\Delta}} dz_1 \int_{\underline{\Delta}} dz_2 \int_{\underline{\Delta}} dz_3 \int_0^1 dt_1 \int_0^1 dt_2 \int_0^1 dt_3 \\
(x_1-z_1)^\nu (t_1x_1+(1-t_1)z_1-z_2)^\mu (t_2t_1x_1+t_2(1-t_1)z_1+(1-t_2)z_2-z_3)^\rho \\
\partial_\rho\partial_\mu\partial_\nu\phi_{i_1}(t_3t_2t_1x_1+t_3t_2(1-t_1)z_1+t_3(1-t_2)z_2+(1-t_3)z_3) \\
\phi_{i_2}(t_3t_2t_1x_2+t_3t_2(1-t_1)z_1+t_3(1-t_2)z_2+(1-t_3)z_3) \\
+ \text{ 7 other terms as ugly as this one .}
\end{aligned} \quad \text{(IV.66)}$$

Now the previous decompositions for the contribution of $\mathcal{M}_{\mathrm{big}}$ have to be performed beginning with high frequency scales and progressing towards the low frequencies. Let us suppose we have treated all dangerous parts strictly above some fixed scale i, $0 \le i \le N$. We define now the dangerous parts at scale i.

If $Y_1, \ldots, Y_{s_{\mathrm{big}}}$ denotes the polymer list of $\mathcal{M}_{\mathrm{big}}$ we consider $S \stackrel{\mathrm{def}}{=} \{s | 1 \le s \le s_{\mathrm{big}}, Y_s \subset \mathcal{D}_{\le i}\}$. We say that s_1 and s_2 in S are *joined* if $Y_{s_1} \cap Y_{s_2} \neq \emptyset$ i.e. Y_{s_1} and Y_{s_2} overlap, or if there is a link of $\mathcal{M}_{\mathrm{big}}$ having a leg with position (s_1, Δ_1) and another

(s_2, Δ_2). We consider $R \subset S$ some connected component according to this notion of junction. Suppose there are links in $\mathcal{M}_{\rm big}$ coming out or R and that their legs that are not located in R have positions (s, Δ) with $s \notin R$ and $i(\Delta) > i$. Suppose also there is no polymer Y_s, $s \notin S$ overlapping with some $Y_{s'}, s' \in R$. We suppose that a polymer index set corresponding to one of the Mayer configurations $\mathcal{M}_1, \dots, \mathcal{M}_r$ and $\mathcal{M}(l_m)$ for $1 \leq m \leq p$, $l_m \in (L_{\rm core} \cup L_{\rm face})$, is either disjoint from R or strictly contained in it.

We consider the links that come out of R, and more particularly their legs that are in R. If (s, Δ) is the position of such a leg, we project Δ on scale i i.e. look at ${\rm pr}_i(\Delta)$. Let X_R be the set off all these projections and suppose that i is the smallest scale index such that X_R is a corner set. We suppose also that $e + |\gamma_1| + \cdots + |\gamma_e| \leq 4$, where e is the total number of legs of the links coming out or R, and $\gamma_1, \dots, \gamma_e$ are the derivation multi-indices acting on them *after taking into account the decompositions of the previously treated dangerous parts, either due to decompositions of* $\mathcal{M}_{\rm big}$ *at higher frequency or due to renormalization operations* T *coming from the histories of the Mayer configurations used to build* $\mathcal{M}_{\rm big}$.

If all these assumptions are realized we say that R is a *dangerous part at scale* i. Now two situations can occur either R has been treated already by the history of a Mayer configuration building $\mathcal{M}_{\rm big}$, i.e. there is already a T in the history of $\mathcal{M}_{\rm big}$ with the same external legs as those of R, or not. In the second case one performs the decomposition into renormalized and local parts, as explained before, of the corresponding monomial $\partial^{\gamma_1}\phi_{i_1}(x_1) \dots \partial^{\gamma_e}\phi_{i_e}(x_e)$ appearing in the product of polymer amplitudes in (IV.56). For that we sum over the choices of $\underline{\Delta}$ in the corner set X_R and divide by $\frac{1}{\#(X_R)}$. We chose also a numbering from 1 to e of these external legs and divide by the number of choices. Each time we perform a new decomposition we add to the history of $\mathcal{M}_{\rm big}$ the corresponding T that encodes it precisely. Knowing T means that we know exactly what term was selected in the local or the renormalized part in the equations (IV.59), (IV.60), (IV.61), (IV.62), (IV.63), (IV.64), (IV.65) or (IV.66).

Note that an additional decomposition is introduced for a local part term of the form

$$\text{Cst} \times \int_{\underline{\Delta}} dz \ \phi_{i_1}(z) \partial_\mu \partial_\nu \phi_{i_2}(z) \quad . \tag{IV.67}$$

We integrate it by parts to obtain

$$\text{Cst} \times \int_{\underline{\Delta}} dz \ \partial_\mu \phi_{i_1}(z) \partial_\nu \phi_{i_2}(z)$$

$$+\text{Cst} \times \int_{\partial_{\underline{\Delta}_+}} d\Sigma(z) \ \phi_{i_1}(z) \partial_\nu \phi_{i_2}(z) - \text{Cst} \times \int_{\partial_{\underline{\Delta}_-}} d\Sigma(z) \ \phi_{i_1}(z) \partial_\nu \phi_{i_2}(z) \tag{IV.68}$$

where $\partial\underline{\Delta}_+$ is the face of $\underline{\Delta}$ in the direction $\overrightarrow{e}_\mu$, and $\partial\underline{\Delta}_-$ is the face of $\underline{\Delta}$ in the direction $-\overrightarrow{e}_\mu$. We obtain therefore a new renormalization operation by choosing

one of these three terms. If it is the first one then $\mathrm{dir}(T) \stackrel{\mathrm{def}}{=} \overrightarrow{0}$ and $\beta_1 \stackrel{\mathrm{def}}{=} \overrightarrow{e}_\mu$ and $\beta_2 \stackrel{\mathrm{def}}{=} -\overrightarrow{e}_\mu$. If it is the second then $\mathrm{dir}(T) \stackrel{\mathrm{def}}{=} \overrightarrow{e}_\mu$ and $\beta_1 \stackrel{\mathrm{def}}{=} \overrightarrow{0}$ and $\beta_2 \stackrel{\mathrm{def}}{=} -\overrightarrow{e}_\mu$. If it is the third then $\mathrm{dir}(T) \stackrel{\mathrm{def}}{=} -\overrightarrow{e}_\mu$ and $\beta_1 \stackrel{\mathrm{def}}{=} \overrightarrow{0}$ and $\beta_2 \stackrel{\mathrm{def}}{=} -\overrightarrow{e}_\mu$.

We now define the coefficients $\mathcal{C}(\mathcal{M})$ to be of the form

$$\mathcal{C}(\mathcal{M}) = \frac{\psi(\mathcal{M})\mathcal{A}(\mathcal{M})}{\sigma(\mathcal{M})} \tag{IV.69}$$

where $\mathcal{A}(\mathcal{M})$ is the *amplitude* of the Mayer configuration $\mathcal{M}$, $\sigma(\mathcal{M})$ is its *symmetry factor*, and $\psi(\mathcal{M})$ is its *Mayer coefficient* that has to be tuned in order for equation (IV.56) to hold.

First we define $\mathcal{A}(\mathcal{M})$. If $\mathcal{M}$ is a VAMC, we start with

$$\mathcal{A}_0(\mathcal{M}) \stackrel{\mathrm{def}}{=} \prod_{1\le m\le p} \left(-\lambda_{e_m,(\omega_{m,1},\ldots,\omega_{m,e_m})}^{\min(i(\Delta_{m,1}),\ldots,i(\Delta_{m,e_m}))} \int_{\Delta_{m,1}\cap\cdots\cap\Delta_{m,e_m}} dx_m \right) \prod_{1\le s\le k} \mathcal{A}(Y_s, \mathcal{J}(\mathcal{L}, Y_s)) \tag{IV.70}$$

where $l_1, \ldots, l_p$ are the links of $\mathcal{M}$, $Y_1, \ldots, Y_k$ are its polymers, $\Delta_{m,1}, \ldots, \Delta_{m,e_m}$ are the boxes featuring in the position of the legs of l_m and $\omega_{m,1}, \ldots \omega_{m,e_m}$ are the initial derivation multi-indices of these legs. $\mathcal{A}(Y_s, \mathcal{J}(\mathcal{L}, Y_s))$ is the polymer amplitude of Y_s with respect to the family $\mathcal{J}$ defined in the same manner as in (IV.50) but with the collection $\mathcal{L}$ of links instead of $\mathfrak{g}_{\mathrm{ext}}$.

Then $\mathcal{A}(\mathcal{M})$ is obtained from $\mathcal{A}_0(\mathcal{M})$ by performing all the operations encoded by the T's in the history of $\mathcal{M}$. The case of a CAMC or a FAMC is similar, the external legs playing no role in defining the amplitude of $\mathcal{M}$.

The symmetry factor $\sigma(\mathcal{M})$ is the cardinal of the symmetry group $\mathfrak{G}(\mathcal{M}_{\mathrm{big}})$ of $\mathcal{M}$ i.e. the group of relabeling operations that preserve the amplitude of $\mathcal{M}$.

- First there are the permutations of $\{1, \ldots, k\}$, the label set of the polymer sequence of $\mathcal{M}$. It is understood that such a permutation modifies also the labels s in a position (s, Δ) of a leg in a link of $\mathcal{M}$.

- Then there are also the permutations of $\{1, \ldots, p\}$ the label set of the link sequence $\mathcal{L}$ of $\mathcal{M}$. Such a permutation modifies the labels i appearing in a quadruplet (i, j, α, β) featuring in $\mathfrak{L}(T)$ where T is in the history of $\mathcal{M}$.

Note that the history $\mathcal{H}$ being an unordered set there is no permutation group acting on it. Therefore

$$\sigma(\mathcal{M}) \stackrel{\mathrm{def}}{=} k! \times p! \quad . \tag{IV.71}$$

We suppose that $\psi(\mathcal{M})$ is invariant by these transformations. We will see later how this can be achieved.

The right hand side of (IV.56) can now be rewritten as

$$\sum_{\mathcal{M}_{\rm big}\ \rm VAMC} \mathcal{A}(\mathcal{M}_{\rm big})\Omega(\mathcal{M}_{\rm big}) \tag{IV.72}$$

with

$$\Omega(\mathcal{M}_{\rm big}) \stackrel{\rm def}{=} \sum_{\substack{\rm cluster \\ \rm producing\ \mathcal{M}_{\rm big}}} \frac{1}{r!s!p_{\rm ext}!} \prod_{1\le u\le r} \left(-\frac{\psi(\mathcal{M}_u)}{\sigma(\mathcal{M}_u)}\right) \times \prod_{\substack{1\le m\le p_{\rm ext} \\ l_m\in(L_{\rm core}\cup L_{\rm face})}} \left(-\frac{\psi(\mathcal{M}(l_m))}{\sigma(\mathcal{M}(l_m))}\right) \tag{IV.73}$$

where the sum is over cluster decompositions i.e all ways to build $\mathcal{M}_{\rm big}$ by concatenation and renormalization splittings from a sequence $(Y_1,\dots,Y_s)$ of disjoint polymers, a sequence $\mathcal{M}_1,\dots,\mathcal{M}_r$ of VAMC's and a sequence of external links $\mathfrak{g}_{\rm ext}$ as in equation (IV.56). Now we can symmetrize the expression by application of the transformations in $\mathfrak{G}(\mathcal{M}_{\rm big})$ which do not change its amplitude:

$$\sum_{\mathcal{M}_{\rm big}\ \rm VAMC} \mathcal{A}(\mathcal{M}_{\rm big})\Omega(\mathcal{M}_{\rm big}) = \sum_{\mathcal{M}_{\rm big}\ \rm VAMC} \frac{\mathcal{A}(\mathcal{M}_{\rm big})}{\sigma(\mathcal{M}_{\rm big})} \times \sum_{\tau\in\mathfrak{G}(\mathcal{M}_{\rm big})} \Omega(\tau(\mathcal{M}_{\rm big}))\,. \tag{IV.74}$$

Now

$$\sum_{\tau\in\mathfrak{G}(\mathcal{M}_{\rm big})} \Omega(\tau(\mathcal{M}_{\rm big})) = \sum_{W} \sum_{\tau\in\mathfrak{G}(\mathcal{M}_{\rm big})} \sum_{\substack{\rm cluster \\ \rm producing\ \tau(\mathcal{M}_{\rm big})}} \frac{1}{r!s!p_{\rm ext}!}$$

$$\prod_{1\le u\le r} \left(-\frac{\psi(\mathcal{M}_u)}{\sigma(\mathcal{M}_u)}\right) \times \prod_{\substack{1\le m\le p_{\rm ext} \\ l_m\in(L_{\rm core}\cup L_{\rm face})}} \left(-\frac{\psi(\mathcal{M}(l_m))}{\sigma(\mathcal{M}(l_m))}\right) \tag{IV.75}$$

where W is the set of subsets of the index set of the polymer sequence of $\mathcal{M}_{\rm big}$, that are transformed by τ into the index sets of the polymers of $\mathcal{M}_1,\dots,\mathcal{M}_r$ or $\mathcal{M}(l_m)$, $1\le m\le p_{\rm ext}$, $l_m\in(L_{\rm core}\cup L_{\rm face})$, viewed in their concatenated Mayer configuration, here $\tau(\mathcal{M}_{\rm big})$. The sum over τ is the same, but the sum over the cluster decompositions is now *conditioned by the constraint of knowing* W.

It is not difficult to see that if we know W and τ we know exactly r, s, $p_{\rm ext}$, $Y_1,\dots,Y_s$ and $\mathcal{M}_1,\dots,\mathcal{M}_r$ together with the sequence $(\mathcal{M}(l_m))_m$ for $1\le m\le p_{\rm ext}$, $l_m\in(L_{\rm core}\cup L_{\rm face})$ we call the elementary Mayer configurations, and also the renormalization splittings. The only thing that remains to be summed over is $\mathfrak{g}_{\rm ext}$. We know the sequence of effective links in it as well as the sequence of links in $L_{\rm core}\cup L_{\rm face}$, but we have to sum over all intertwinings of these sequences. This accounts for a factor $\frac{p_{\rm ext}!}{p_{\rm eff}!p_{\rm cont}!}$ where $p_{\rm eff}$ is the number of effective links and $p_{\rm cont}$ the number of links in $L_{\rm core}\cup L_{\rm face}$.

As a result

$$\sum_{\tau\in\mathfrak{G}(\mathcal{M}_{\rm big})}\Omega(\tau(\mathcal{M}_{\rm big}))=\sum_{W}\sum_{\tau\in\mathfrak{G}(\mathcal{M}_{\rm big})}\frac{1}{r!s!p_{\rm eff}!p_{\rm cont}!}\prod_{\mathcal{M}\ \text{elementary}}\left(-\frac{\psi(\mathcal{M})}{\sigma(\mathcal{M})}\right)\ . \tag{IV.76}$$

Now if W is specified, by symmetry of the coefficients $\psi(\mathcal{M})$, the terms in the sum over τ are independent of τ. We just have to count such transformations. The action of τ on the polymer indices must leave W fixed therefore there are $s!r!\times\prod_{\mathcal{M}\ \text{elementary}}k_{\mathcal{M}}!$ possibilities, where $k_{\mathcal{M}}$ is number of polymers in an elementary configuration $\mathcal{M}$. The action of τ on the links of $\mathcal{M}_{\rm big}$ can shuffle together the effective links, $p_{\rm eff}!$ possibilities, it can also shuffle the groups of links corresponding respectively to the elementary Mayer configurations that are not VAMC's, $p_{\rm cont}!$ possibilities. Finally it can permute the links in each group therefore accounting for $\prod_{\mathcal{M}\ \text{elementary}}p_{\mathcal{M}}!$ possibilities, where $p_{\mathcal{M}}$ is number of links in an elementary configuration $\mathcal{M}$. As a result

$$\sum_{\tau\in\mathfrak{G}(\mathcal{M}_{\rm big})}\Omega(\tau(\mathcal{M}_{\rm big}))=\sum_{\substack{W\ \text{allowed}\\ \text{by}\ \mathcal{M}_{\rm big}}}\prod_{V\in W}(-\psi(\mathcal{M}(\mathcal{M}_{\rm big},V))) \tag{IV.77}$$

If $(Y_1,\dots,Y_{s_{\rm big}})$ is the polymer sequence of $\mathcal{M}_{\rm big}$, W is a subset of the power set $\mathcal{P}(\{1,\dots,s_{\rm big}\})$ of $\{1,\dots,s_{\rm big}\}$, made of disjoint subsets of $\{1,\dots,s_{\rm big}\}$. If $V=\{i_1,\dots,i_\alpha\}$ is an element of W, the polymer sequence of the elementary Mayer subconfiguration $\mathcal{M}(\mathcal{M}_{\rm big},V)$ of $\mathcal{M}_{\rm big}$ is $(Y_{i_1},\dots,Y_{i_\alpha})$. Its link sequence is the extracted sequence of links in $\mathcal{M}_{\rm big}$ with at least one leg located in $\{i_1,\dots,i_\alpha\}$. Besides, if a leg of such a link is not in this set we replace it by the of the quadruplet in $\mathfrak{L}(T(V))$ where it is listed. $T(V)$ here is the unique renormalization operation in the history of $\mathcal{M}_{\rm big}$ that lists exactly all these external legs of $\mathcal{M}(\mathcal{M}_{\rm big},V)$. The history of $\mathcal{M}(\mathcal{M}_{\rm big},V)$ is made by the elements T in the history of $\mathcal{M}_{\rm big}$ that contain a quadruplet (i,j,α,β) with i labeling a link having a leg located in V. A trivial relabeling is also needed.

To say that W is allowed by $\mathcal{M}_{\rm big}$ in (IV.77) means that each $\mathcal{M}(\mathcal{M}_{\rm big},V)$, $V\in W$, is indeed an AMC, and $\mathcal{M}_{\rm big}$ can be obtained from these elementary Mayer configurations by the concatenation procedure explained earlier, modulo a transformation by an element of $\mathfrak{G}(\mathcal{M}_{\rm big})$. This means in particular that the polymers of $\mathcal{M}_{\rm big}$ that are not in any $\mathcal{M}(\mathcal{M}_{\rm big},V)$, $V\in W$, are disjoint, These are called *actual* polymers in contrast with those of a $\mathcal{M}(\mathcal{M}_{\rm big},V)$ that are called *virtual* in $\mathcal{M}_{\rm big}$.

Now in order for equation (IV.56) to be satisfied it is enough to have for any non empty VAMC $\mathcal{M}_{\rm big}$ the equality

$$\sum_{\substack{W\ \text{allowed}\\ \text{by}\ \mathcal{M}_{\rm big}}}\prod_{V\in W}(-\psi(\mathcal{M}(\mathcal{M}_{\rm big},V)))=0 \tag{IV.78}$$

Note now that $W = \{\{1, \dots, s_{\text{big}}\}\}$ is allways allowed and therefore (IV.78) can be written as

$$\psi(\mathcal{M}_{\text{big}}) = \sum_{\substack{W \text{ strictly allowed} \\ \text{by } \mathcal{M}_{\text{big}}}} \prod_{V \in W} (-\psi(\mathcal{M}(\mathcal{M}_{\text{big}}, V))) \tag{IV.79}$$

"strictly" here means $W \neq \{\{1, \dots, s_{\text{big}}\}\}$.

Now (IV.79) is in fact the key for an inductive definition of the coefficients ψ. Indeed we can extend the equation (IV.78) to all AMC's and not only to the VAMC's. In that case (IV.79) allows to compute the coefficients $\psi(\mathcal{M})$ for any admissible Mayer configuration, by induction on the number of its polymers. It is easy to see that indeed such coefficients $\psi(\mathcal{M})$ will be symmetric under the action of $\mathfrak{G}(\mathcal{M})$ which justifies a posteriori the precedent derivation. We now have defined, for any AMC $\mathcal{M}$, the amplitude $\mathcal{A}(\mathcal{M})$, the symmetry factor $\sigma(\mathcal{M})$ and the Mayer coefficient $\psi(\mathcal{M})$ in such a way that

$$\mathcal{C}(\mathcal{M}) = \frac{\psi(\mathcal{M})\mathcal{A}(\mathcal{M})}{\sigma(\mathcal{M})} \tag{IV.80}$$

solves the equation (IV.56). Therefore it is straight-forward to compute

$$-\frac{\tilde{Z}(\Lambda, N)}{|\Lambda|} = -\frac{1}{|\Lambda|} \sum_{\mathcal{M} \text{ VAMC}} \frac{\psi(\mathcal{M})\mathcal{A}(\mathcal{M})}{\sigma(\mathcal{M})} \quad . \tag{IV.81}$$

The purpose of the next section is to prove that this expansion is *absolutely convergent uniformly in Λ and N.*

IV.4 The convergence of the expansion

IV.4.1 The tree structure of the Mayer coefficients

It is straight-forward to solve the recursion (IV.79) by

$$\psi(\mathcal{M}) = \sum_{\substack{\mathcal{F} \text{ strict forest} \\ \text{allowed by } \mathcal{M}}} (-1)^{\#(\mathcal{F})} \quad . \tag{IV.82}$$

Here, $\mathcal{F}$ is a forest of subsets of $\{1, \dots, s\}$ the polymer index set of $\mathcal{M}$. This means $\mathcal{F} \subset \mathcal{P}(\{1, \dots, s\})$ and for any V_1, $V_2 \in \mathcal{F}$ we have $V_1 \subset V_2$ or $V_2 \subset V_1$ or $V_1 \cap V_2 = \emptyset$. A strict forest means $\{1, \dots, s\} \notin \mathcal{F}$. Allowed by $\mathcal{M}$ means that if $V \in \mathcal{F} \cup \{\{1, \dots, s\}\}$ and W_V is the set of maximal elements of $\mathcal{F}$ strictly contained in V, then W_V is allowed by $\mathcal{M}(\mathcal{M}, V)$ according to the notations and definitions of the last section.

A few things remain to be proven in order to show the convergence of (IV.81). Among them, that the only AMC's $\mathcal{M}$ with non zero $\psi(\mathcal{M})$ are connected by overlap and do not contain a dangerous part. This will mean that from $\mathcal{A}(\mathcal{M})$ we will be able to get sufficient decrease in the horizontal and above all in the vertical direction to sum it in $\mathcal{D}(\Lambda, N)$.

But this will not be enough. Experts in the cluster and Mayer expansions might still view our claim of convergence as somewhat dubious. Indeed a sum over forests of subsets is a huge combinatorial sum. One would be ill advised to bound in (IV.82) the $(-1)^{\#(\mathcal{F})}$ simply by 1. One must find a way to profit by the numerous cancellations of this formula first. This is what we will do and the result will be a reduction of (IV.82) into a sum over graphs with no loops i.e. unions of disjoint trees on the set $\{1, \ldots, s\}$ labeling the polymers of $\mathcal{M}$.

It is noticeable that such a forest formula as (IV.82) bears some resemblance to Zimmermann's formula in perturbative renormalization. In fact an element V of $\mathcal{F}$ corresponds in our configuration $\mathcal{M}$, a kind of "super" Feynman graph, to a divergent subdiagram. The only major difference is that some V's might correspond to a vacuum subgraph. This is maybe the essential difference between perturbative and constructive theory. In the latter, vacuum graphs are not completely factorized from other two and four point graphs, for there is still a residual interaction due to the overlap of polymers.

First we show that for a CAMC of FAMC $\mathcal{M}$, $\psi(\mathcal{M}) \neq 0$ implies that $\mathcal{M}$ is connected with respect to junction. Recall that junction was the notion used to define the dangerous parts for the renormalization splittings, and it means that two polymers overlap or there is a link with a leg in each of them. Since

$$\psi(\mathcal{M}) = \sum_{\substack{W \text{ strictly} \\ \text{allowed by } \mathcal{M}}} \prod_{V \in W} (-\psi(\mathcal{M}(\mathcal{M}, V))) \tag{IV.83}$$

the allowedness of W meaning that $\mathcal{M}$ is built from the $(\mathcal{M}(\mathcal{M}, V))_{V \in W}$ by the cluster expansion and renormalization splitting of formula (IV.56) (modulo a reshuffling by $\tau \in \mathfrak{G}(\mathcal{M})$), and by our definition of dangerous parts as connected components for junction above some frequency, it is ipso facto required that $\mathcal{M}$ be connected for junction. Indeed $\mathcal{M}$ is itself the local part of a newly created dangerous part.

Next we show that the same statement holds for a VAMC $\mathcal{M}$. It is proven by induction on the number of its polymers. In

$$\psi(\mathcal{M}) = \sum_{\substack{W \text{ strictly} \\ \text{allowed by } \mathcal{M}}} \prod_{V \in W} (-\psi(\mathcal{M}(\mathcal{M}, V))) \tag{IV.84}$$

we see that either $\mathcal{M}(\mathcal{M}, V)$ is a VAMC thus by induction $\psi(\mathcal{M}(\mathcal{M}, V)) = 0$ unless it is connected, or it is a CAMC or a FAMC and the same holds because of the

previous argument. Now if $W_{\max} = \{V_1, \ldots, V_\rho\}$ is the set of polymer index subsets corresponding to maximal connected components for junction of $\mathcal{M}$ we have, if $\rho \geq 2$, the factorization

$$\sum_{\substack{W \text{ strictly} \\ \text{allowed by } \mathcal{M}}} \prod_{V \in W} (-\psi(\mathcal{M}(\mathcal{M}, V))) = \prod_{\mu=1}^{\rho} \left(\sum_{\substack{W_\mu \text{ allowed} \\ \text{by } \mathcal{M}(\mathcal{M}, V_\mu)}} \prod_{V \in W} (-\psi(\mathcal{M}(\mathcal{M}, V))) \right) = 0 \tag{IV.85}$$

because we do not require strict allowedness of W_μ with respect to $\mathcal{M}(\mathcal{M}, V_\mu)$, since V_μ is not equal to the full set of polymer indices of $\mathcal{M}$. From (IV.78) we deduce that each factor is 0.

Thus $\psi(\mathcal{M}) \neq 0$ implies $\rho = 1$, in other words that $\mathcal{M}$ is connected for junction.

The next step is to show that if there is a local part, i.e. a $T \in \mathcal{H}$ with status$(T) =$ loc, that corresponds to a connected component of $\mathcal{M}$ above $i(\underline{\Delta}(T))$, then $\psi(\mathcal{M}) = 0$ unless $\mathcal{M}$ is a CAMC or a FAMC and $T = T_{\text{out}}(\mathcal{M})$.

By T corresponding to a connected component above $i(\underline{\Delta}(T))$ we mean that when we consider polymers of $\mathcal{M}$ that are included in $\mathcal{D}_{\leq i(\underline{\Delta}(T))}$, for the notion of junction, there is a connected component R, not overlapping with a polymer of $\mathcal{M}$ that is not in $\mathcal{D}_{\leq i(\underline{\Delta}(T))}$, containing $U(T)$ and whose external legs are exactly those listed by $\mathfrak{L}(T)$.

This is done by induction on the number of polymers of $\mathcal{M}$. Let W be a strictly allowed set of disjoint polymer index subsets of $\mathcal{M}$ as in formula (IV.79). Suppose T was contained in the history of some $\mathcal{M}(\mathcal{M}, V)$, $V \in W$. Let then R_V be the connected component for junction of $U(T)$ in $\mathcal{M}(\mathcal{M}, V)$ above $i(\underline{\Delta}(T))$. Since $U(T) \subset R$ then $R_V \subset R$. As a result, the only external legs of R_V are those of R i.e. those listed in $\mathfrak{L}(T)$. Therefore T corresponds to a connected component above $i(\underline{\Delta}(T))$ in $\mathcal{M}(\mathcal{M}, V)$ and by induction $\psi(\mathcal{M}(\mathcal{M}, V)) = 0$ unless $T = T_{\text{out}}(\mathcal{M}(\mathcal{M}, V))$. But in this last case $V \subset R$.

Suppose now that T is not in the history of some $\mathcal{M}(\mathcal{M}, V)$ for $V \in W$, then T is a renormalization operation created to treat the dangerous parts of $\mathcal{M}$ at the last stage of building $\mathcal{M}$ from the elementary Mayer configurations $\mathcal{M}(\mathcal{M}, V)$, $V \in W$. From our definition of dangerous parts in section IV.3.2.4. it follows that a $V \in W$ is either disjoint from R or strictly contained in it.

Finally we have shown that for any W giving a non zero contribution in (IV.79), a $V \in W$ is either disjoint from R or a subset of R. Let us fix $W_{\text{out}} = \{V \in W | V \cap R = \emptyset\}$ and sum over $W_{\text{in}} = \{V \in W | V \subset R\}$. Since there is *no overlap* between a polymer in R and a polymer not in R, the restriction on W_{in} simply reduces to the allowedness with respect to the Mayer sub-configuration $\mathcal{M}(\mathcal{M}, R)$,

and thus if R is not the full polymer index set of $\mathcal{M}$ we have

$$\psi(\mathcal{M}) = \sum_{W_{\rm in}} \prod_{V \in W_{\rm in}} \psi(\mathcal{M}(\mathcal{M}, V)) \times \left(\sum_{\substack{W_{\rm out} \text{ allowed} \\ \text{by } \mathcal{M}(\mathcal{M},R)}} \prod_{V \in W_{\rm out}} \psi(\mathcal{M}(\mathcal{M}, V))) \right) = 0 \quad \text{(IV.86)}$$

since the last factor vanishes thanks to (IV.78) applied to $\mathcal{M}(\mathcal{M}, R)$.

If on the contrary R is the full polymer index set of $\mathcal{M}$ then $\mathcal{M}$ is a CAMC or a FAMC and $T = T_{\rm out}(\mathcal{M})$. This completes the induction step, and proves our statement.

As a consequence of the previous arguments, the sum over forests solving (IV.79) by (IV.82) is over forests $\mathcal{F}$ such that for every $V \in \mathcal{F}$, the Mayer subconfiguration $\mathcal{M}(\mathcal{M}, V)$ satisfies the previous connectedness properties.

Now we can prove that in such a forest, every $T \in \mathcal{H}$ such that status$(T) =$ loc corresponds to an element of V, i.e. there is $V \in \mathcal{F}$ such that $\mathcal{M}(\mathcal{M}, V)$ is a CAMC or a FAMC and $T_{\rm out}(\mathcal{M}(\mathcal{M}, V)) = T$.

Indeed, in the building process of $\mathcal{M}$, specified by $\mathcal{F}$, there is a $V \in \mathcal{F}$ such that T was created at stage V i.e. V is the minimal element of $\mathcal{F}$ such that T is in the history of the subconfiguration $\mathcal{M}(\mathcal{M}, V)$. By our rule for defining renormalization operations, T has to correspond to a dangerous part in V i.e. there is an $R \subset V$ such that R is a connected component of V for junction above $i(\underline{\Delta}(T))$, the external legs of R being those listed in $\mathfrak{L}(T)$. But by the previous argument $\psi(\mathcal{M}(\mathcal{M}, V)) = 0$ unless $\mathcal{M}(\mathcal{M}, V)$ is a CAMC or a FAMC with $T_{\rm out}(\mathcal{M}(\mathcal{M}, V)) = T$. The fact that now only the V's with $\psi(\mathcal{M}(\mathcal{M}, V)) \neq 0$ are taken into account in the forests expressing $\psi(\mathcal{M})$, proves the assertion.

Let now $\mathcal{F}$ be such a forest, we can split it into $\mathcal{F} = \mathcal{F}_{\rm cont} \cup \mathcal{F}_{\rm vac}$ in the following way.

$$\mathcal{F}_{\rm cont} \stackrel{\rm def}{=} \{V \in \mathcal{F} | \mathcal{M}(\mathcal{M}, V) \text{ is a CAMC or a FAMC}\} \quad , \qquad \text{(IV.87)}$$

$$\mathcal{F}_{\rm vac} \stackrel{\rm def}{=} \{V \in \mathcal{F} | \mathcal{M}(\mathcal{M}, V) \text{ is a VAMC}\} \quad . \qquad \text{(IV.88)}$$

To reduce the sum in (IV.82) we fix $\mathcal{F}_{\rm cont}$ and sum over $\mathcal{F}_{\rm vac}$. The first case we consider is when $\mathcal{M}$ is a VAMC.

Let S be the set of polymer indices that are not in any $V \in \mathcal{F}_{\rm cont}$. Since $\mathcal{M}$ is a VAMC we have $S \neq \emptyset$. We say that two indices s_1 and s_2 of S are *related* if there is a link with a leg in s_1 and another in s_2, or if there is a T in the history of $\mathcal{M}$ such that $\mathfrak{L}(T)$ contains a leg in s_1 and another in s_2. It is clear that an element of $\mathcal{F}_{\rm vac}$ cannot separate two related elements of S. Let $\overline{S}$ be the partition of S into connected components for this relation. We now introduce a graph $G_{\overline{S}}$ on $\overline{S}$ in the

following way. A pair $\{X_1, X_2\}$ with $X_1 \in \overline{S}$, $X_2 \in \overline{S}$, $X_1 \neq X_2$ belongs to $G_{\overline{S}}$ if and only if there is $s_1 \in X_1$ and $s_2 \in X_2$ such that $T_{s_1} \cap Y_{s_2} \neq \emptyset$.

Let us define

$$\overline{\mathcal{F}}_{\rm vac} \stackrel{\rm def}{=} \{B \subset \overline{S} | \exists V \in \mathcal{F}_{\rm vac}, \forall X \in \overline{S}, (X \in B \Leftrightarrow X \subset V)\} \quad , \tag{IV.89}$$

and

$$\breve{\mathcal{F}}_{\rm vac} \stackrel{\rm def}{=} \{V \in \mathcal{F}_{\rm vac} | \forall V' \in \mathcal{F}_{\rm cont}, V \not\subset V'\} \quad . \tag{IV.90}$$

It is not difficult to see that $\overline{\mathcal{F}}_{\rm vac}$ fully determines $\breve{\mathcal{F}}_{\rm vac}$ and vice versa. Besides, if we fix in (IV.82) $\mathcal{F}_{\rm cont}$ and and $\mathcal{F}_{\rm vac} \backslash \breve{\mathcal{F}}_{\rm vac}$, the remaining sum on $\breve{\mathcal{F}}_{\rm vac}$ is in fact a sum over $\overline{\mathcal{F}}_{\rm vac}$ with the only constraint that the elements of $\overline{\mathcal{F}}_{\rm vac}$ are connected subgraghs of $G_{\overline{S}}$, and that $\overline{\mathcal{F}}_{\rm vac}$ cuts all pairs of $G_{\overline{S}}$. This means that if $\{X_1, X_2\} \in G_{\overline{S}}$, there exists a $B \in \overline{\mathcal{F}}_{\rm vac}$ such that $X_1 \in B$ and $X_2 \notin B$. Now we have to reduce the sum

$$\Theta \stackrel{\rm def}{=} \sum_{\overline{\mathcal{F}}_{\rm vac}} (-1)^{\#(\overline{\mathcal{F}}_{\rm vac})} \tag{IV.91}$$

to a sum over trees.

Let us number the elements of $\overline{S}$ as $\{X_1, \ldots, X_\rho\}$ by increasing order of $\min\{s | s \in X\}$ for $X \in \overline{S}$. Consider in (IV.91) the sum over the $\overline{\mathcal{F}}_{\rm vac}$ such that X_1 is in some $B \in \overline{\mathcal{F}}_{\rm vac}$. Let us fix B_1 the maximal such B in $\overline{\mathcal{F}}_{\rm vac}$, and sum on $\overline{\mathcal{F}}_{\rm out}$ the part of $\overline{\mathcal{F}}_{\rm vac}$ that does not intersect B_1. Let $B_2, \ldots, B_\mu$ be the connected components of $\overline{S} \backslash B_1$ for $G_{\overline{S}}$ restricted to $\overline{S} \backslash B_1$. Note that there exist at least one of them since $B_1 \neq \overline{S}$ because the forests $\mathcal{F}$ in (IV.82) do not contain the full set of polymer indices of $\mathcal{M}$.

Now the conditional sum

$$\sum_{\overline{\mathcal{F}}_{\rm out}} (-1)^{\#(\overline{\mathcal{F}}_{\rm out})} \tag{IV.92}$$

is zero since we can add or substract from $\overline{\mathcal{F}}_{\rm out}$ the set B_2, for instance, without any constraint. Indeed the only pairs of $G_{\overline{S}}$ crossing B_2, link it to B_1 and thus are *already cut* by $B_1 \in \overline{\mathcal{F}}_{\rm vac}$.

This argument shows a first reduction

$$\Theta = \sum_{\substack{\overline{\mathcal{F}}_{\rm vac} \\ X_1 \notin \cup \overline{\mathcal{F}}_{\rm vac}}} (-1)^{\#(\overline{\mathcal{F}}_{\rm vac})} \tag{IV.93}$$

Now consider $X_{i_1}, \ldots, X_{i_n}$, $2 \leq i_1 < \ldots < i_n \leq \rho$, the nearest neighbors of X_1 in the graph $G_{\overline{S}}$, they are necessarily contained in maximal blocks $B_1, \ldots, B_\mu$ of $\overline{\mathcal{F}}_{\rm vac}$. Now let us fix $B_1, \ldots, B_\mu$ as well as the part of $\overline{\mathcal{F}}_{\rm vac}$ contained in one of them and sum

over $\overline{\mathcal{F}}_{\rm out}$ the part of $\overline{\mathcal{F}}_{\rm vac}$ that does not intersect them. If $\{X_1\} \cup B_1 \cup \ldots \cup B_\mu \neq \overline{S}$ then

$$\sum_{\overline{\mathcal{F}}_{\rm out}} (-1)^{\#(\overline{\mathcal{F}}_{\rm out})} = 0 \tag{IV.94}$$

by the same argument as before. Indeed the pairs of $G_{\overline{S}}$ coming out of a connected component of $\overline{S}\backslash(\{X_1\} \cup B_1 \cup \ldots \cup B_\mu)$ are already cut by $B_1, \ldots, B_\mu$. Therefore

$$\Theta = \sum_{\{B_1,\ldots,B_\mu\}} (-1)^\mu \times \prod_{\nu=1}^{\mu} \left(\sum_{\overline{\mathcal{F}}_{{\rm vac},\nu}} (-1)^{\#(\overline{\mathcal{F}}_{{\rm vac},\nu})} \right) \tag{IV.95}$$

where $\{B_1, \ldots, B_\mu\}$ is a partition of $\overline{S}\backslash\{X_1\}$ by sets connected by $G_{\overline{S}}$, and each sum over $\overline{\mathcal{F}}_{{\rm vac},\nu}$ is the analog of Θ, when we replace $\overline{S}$ by B_ν and $G_{\overline{S}}$ by B_ν and $G_{\overline{S}}$ by its restriction to B_ν.

The only difference is that when we apply again the procedure inside some B_ν, we start, as an analog of X_1 in B_ν, from the element of $\overline{S}\backslash\{X_1\}$ that contains the smallest polymer index s such that Y_s overlaps with some polymer of the previous X_1.

Now (IV.95) wants iteration. It is straight-forward to solve it as

$$\Theta = (-1)^{\#(\overline{S})-1} \times \sum_{\mathfrak{t}} 1 \tag{IV.96}$$

where $\mathfrak{t}$ is a tree connecting $\overline{S}$ satisfying the following properties.

- We require $\mathfrak{t} \subset G_{\overline{S}}$.

- Let X_1 be the root. Let for any $X_i \in \overline{S}$, $d_{\mathfrak{t}}(X_i)$ be the *depth* of X_i i.e. the minimal number of links in $\mathfrak{t}$ to go from X_i to X_1. We define also if $X_i \neq X_1$, the ancestor $\mathrm{an}(X_i)$ of X_i, to be the unique X_j with $d_{\mathfrak{t}}(X_i) - 1$ to which X_i is connected by $\mathfrak{t}$. We define also for any $X_i \in \overline{S}$ the branch growing from X_i, $\mathrm{br}(X_i)$ as the connected component of X_i with respect to the graph $\mathfrak{t}\backslash\{\{X_i, \mathrm{an}(X_i)\}\}$ if $X_i \neq X_1$ and as $\overline{S}$ if $X_i = X_1$. We now require from $\mathfrak{t}$ that for any $X_i \neq X_1$, if $X_j \in \mathrm{br}(X_i)$ and $\{X_i, X_j\} \in G_{\overline{S}}$ then $j \geq i$.

This choice is quite arbitrary but it succeeds in writing Θ as a sum over trees which is our modest goal. This will provide an acceptable bound on the Mayer coefficients. The way we proceed from (IV.95) is simply to identify $B_1, \ldots, B_\mu$ as the branchs of a tree growing from X_1. To keep track of $B_1, \ldots, B_\mu$ one needs to choose a link of $G_{\overline{S}}$ for each of them. We do that by taking for each B_ν the link $\{X_1, X_{j_\nu}\}$ where j_ν si the smallest index j such that $X_j \in B_\nu$ and $\{X_1, X_j\} \in G_{\overline{S}}$. The corresponding X_{j_ν} will be the analog of root X_1, inside B_ν to continue the induction.

What we have done up to now is to reduce the sum over $\breve{\mathcal{F}}_{\rm vac}$ to a sum over trees in case $\mathcal{M}$ is a VAMC.

Let us now suppose $\mathcal{M}$ is a CAMC or a FAMC. We consider in the same way as before S to be the set of polymer indices that are not in any $V \in \mathcal{F}_{\rm cont}$. If $S = \emptyset$ there is nothing like $\breve{\mathcal{F}}_{\rm vac}$ to sum about at this stage. If $S \neq \emptyset$, we consider also $\overline{S}$ the set of connected components of S for the relation introduced before, taking into account relations via the external legs of $\mathcal{M}$. As a consequence if some polymers of S contain a leg from a link cut by $T_{\rm out}(\mathcal{M})$, they must belong to the same element of $\overline{S}$. If such a component exists we call it X_1 and number the other components $X_2, \ldots, X_\rho$ following the same, somewhat arbitrary, rule as before. It is clear that we can thus reduce the sum over $\breve{\mathcal{F}}_{\rm vac}$, the part of $\mathcal{F}_{\rm vac}$ made by V's that are not contained in any element of $\mathcal{F}_{\rm cont}$, to a sum over trees, as we did in the first case.

What happens if there is no such component X_1? This implies that the external legs of $\mathcal{M}$ are already cut by T's in the history of $\mathcal{M}$ corresponding to some maximal elements of $\mathcal{F}_{\rm cont}$. Let Q be the set of polymer indices of $\mathcal{M}$ that are not within the union of the maximal elements of $\mathcal{F}_{\rm cont}$ whose relative external legs are external legs of $\mathcal{M}$. Then Q is not equal to the full polymer index set of $\mathcal{M}$ and any element of $\breve{\mathcal{F}}_{\rm vac}$ is contained in Q. Besides $S \neq \emptyset$ implies $Q \neq \emptyset$. But now since we can add or substract from $\breve{\mathcal{F}}_{\rm vac}$ the set Q itself, without constraint, the sum over $\breve{\mathcal{F}}_{\rm vac}$ vanishes, by an argument becoming familiar now.

We have now treated all cases in summing over $\breve{\mathcal{F}}_{\rm vac}$. The next step is simply to iterate the procedure to sum inside each $V \in \mathcal{F}_{\rm cont}$ over $\mathcal{F}^V_{\rm vac}$, the subforest of $\mathcal{F}_{\rm vac}$ made by subsets that are strictly contained in V but not contained in any $V' \in \mathcal{F}_{\rm cont}$ with V' strictly in V. We proceed from the maximal V's of $\mathcal{F}_{\rm cont}$ to the minimal ones. The resulting sum over $\mathcal{F}_{\rm vac}$ can be expressed as a sum over an ordinary graph without loops on the polymer index set of $\mathcal{M}$. To formulate the result we introduce the notion of *Mayer graph adapted to the counter-term forest* $\mathcal{F}_{\rm cont}$.

A Mayer graph $\mathfrak{m}$ adapted to $\mathcal{F}_{\rm cont}$ is an ordinary graph on the polymer index set of $\mathcal{M}$ i.e. a set of pairs, with the following properties.

1- For any pair $\{s_1, s_2\}$ in $\mathfrak{m}$, $Y_{s_1} \cap Y_{s_2} \neq \emptyset$.

2- We say that two polymer indices s_1 and s_2 are connected if there is a leg in s_1 and another in s_2 that are listed in the same link of $\mathcal{M}$ or the same $\mathfrak{L}(T)$ fot T in the history of $\mathcal{M}$, or if $\{s_1, s_2\} \in \mathfrak{m}$. If $\mathcal{M}$ is a CAMC or a FAMC, we also consider the elements of $\mathrm{up}(T_{\rm out}(\mathcal{M}))$ to be connected together. Now we ask that for this notion of connectedness, $\mathcal{M}$ is entierly connected. we also ask that for any $V \in \mathcal{F}_{\rm cont}$ the Mayer subconfiguration $\mathcal{M}(\mathcal{M}, V)$ is entierly connected for the previous notion after removing from $\mathfrak{m}$ the pairs that are not contained in V.

3- The third requirement is inductive. It is checked as follows. There are two cases to consider at the beginning.

First case: $\mathcal{M}$ is a VAMC. We define S as before as well as X_1. We ask that no pair of $\mathfrak{m}$ is internal to X_1. We consider then the connected components $K_1, \ldots, K_\tau$ of the complement of S in the polymer index set of $\mathcal{M}$ for the same notion of connectedness as in 2, but after removing from $\mathfrak{m}$ all pairs that go out of X_1. Now for any K_ν we ask the following. If K_ν is in $\mathcal{F}_{\rm cont}$, then there must be no pair of $\mathfrak{m}$ joining X_1 to K_ν. If K_ν is not in $\mathcal{F}_{\rm cont}$, then no link of $\mathfrak{m}$ must go out of K_ν, and there is a single pair of $\mathfrak{m}$ going out of K_ν. This pair must be of the form $\{s_1, s_2\}$ with $S_1 \in X_1$, $s_2 \in K_\nu$ and (s_2, s_1) minimal in lexicographic order among all pairs of polymer indices such that $s_1 \in X_1$, $s_2 \in K_\nu$ and $Y_{s_1} \cap Y_{s_2} \neq \emptyset$.

Second case: $\mathcal{M}$ is a CAMC or a FAMC. S is defined as before. If $S = \emptyset$ then the components $K_1, \ldots, K_\tau$ are by definition the maximal elements of $\mathcal{F}_{\rm cont}$. We require then that no pair of $\mathfrak{m}$ goes out of some K_ν. If $S \neq \emptyset$ then X_1 is defined as before to be the component of $\mathrm{up}(T_{\rm out}(\mathcal{M}))$ and again we define $K_1, \ldots, K_\tau$ as in the first case and we ask the same requirements.

Now after checking these first conditions on $\mathcal{M}$ we do the same check for the Mayer subconfigurations $\mathcal{M}(\mathcal{M}, K_\nu)$ with, for each K_ν, $\mathfrak{m}$ replaced by the set of its pairs that are internal to K_ν. We continue inductively the check inside each K_ν. The only difference with the initial step is in the definition of "X_1" inside some K_ν such that $\mathcal{M}(\mathcal{M}, K_\nu)$ is a VAMC. We choose the element of "$\overline{S}$" inside K_ν that contains the only $s \in K_\nu$ such that there exists an $s' \notin K_\nu$ with $\{s, s'\} \in \mathfrak{m}$.

This completes the definition of a Mayer graph adapted to $\mathcal{F}_{\rm cont}$. It is easy to see that it forces $\mathfrak{m}$ to be a graph without loops on the polymer index set of $\mathcal{M}$, i.e. a union of disjoint trees. This is what we call a forest graph in graph theory and in [AR2], but this terminology here is troublesome since it may create a confusion with the forests of subsets we consider here and that are more familiar to the renormalization theory.

Up to now we have shown that

$$\psi(\mathcal{M}) = (-1)^{\mathrm{loc}(\mathcal{M})} \times \sum_{\mathcal{F}_{\rm cont}} \sum_{\substack{\mathfrak{m} \text{ adapted} \\ \text{to } \mathcal{F}_{\rm cont}}} (-1)^{\#(\mathfrak{m})} \tag{IV.97}$$

where $\mathrm{loc}(\mathcal{M})$ is the number of T's in the history of $\mathcal{M}$ with $\mathrm{status}(T) = \mathrm{loc}$, it is also the cardinal of $\mathcal{F}_{\rm cont}$. We say that $\mathfrak{m}$ is simply a Mayer graph on $\mathcal{M}$ if there exists a forest $\mathcal{F}_{\rm cont}$ such that $\mathfrak{m}$ is adapted to $\mathcal{F}_{\rm cont}$. Note that the subsets K_ν defined in checking the property 3 form the forest $\mathcal{F}_{\rm vac}$ associated to $\mathfrak{m}$. We denote by $\mathcal{F}(\mathfrak{m}, \mathcal{F}_{\rm cont})$ the total forest made by the union of this $\mathcal{F}_{\rm vac}$ with $\mathcal{F}_{\rm cont}$.

The only forests that survive in (IV.82) after the procedure we used to exploit the numerous cancellations are that of the form $\mathcal{F}(\mathfrak{m}, \mathcal{F}_{\rm cont})$.

We are not finished yet. One has to notice the important fact that given a Mayer graph $\mathfrak{m}$, there is a *unique* $\mathcal{F}_{\rm cont}$ such that $\mathfrak{m}$ is adapted to it.

Indeed, suppose $\mathcal{M}$ is a VAMC, let $S_{\rm low}$ be the smallest index of polymer in $\mathcal{M}$ that maximizes $i_{\max}(Y_s)$ where $i_{\max}(Y) \stackrel{\rm def}{=} \max\{i | Y \cap \mathcal{D}_i \neq \emptyset\}$.

Then $s_{\rm low}$ cannot be in an element of $\mathcal{F}_{\rm cont}$ therefore it is in the set S defined before. Besides S can be recovered as the connected component of $s_{\rm low}$ due to connection made by an $\mathfrak{L}(T)$, an ${\rm up}(T)$, a pair of $\mathfrak{m}$ or a link of $\mathcal{M}$ except those of ${\rm cut}(T)$ for some T in the history of $\mathcal{M}$ such that ${\rm status}(T) = {\rm loc}$ and T is maximal for that property with respect to the partial ordering $\preceq$ introduced in section 3.1. Now let us consider for each such T, with an external leg in S, the connected component of ${\rm up}(T)$ for the same notion of connectedness. The collection of these connected components gives us the maximal elements of $\mathcal{F}_{\rm cont}$.

Suppose $\mathcal{M}$ is a FAMC or a VAMC. If ${\rm up}(T_{\rm out}(\mathcal{M}))$ is a union of ${\rm up}(T)$'s for $T \prec T_{\rm out}(\mathcal{M})$, and ${\rm status}(T) = {\rm loc}$ then necessarily $S = \emptyset$. Besides when we consider the maximal such T's for $\preceq$ and the connected components of their ${\rm up}(T)$'s for the above mentioned connectivity definition after removing ${\rm up}(T_{\rm out}(\mathcal{M}))$ we obtain again the maximal elements of $\mathcal{F}_{\rm cont}$.

Finally in the remaining case, there exists an element of ${\rm up}(T_{\rm out}(\mathcal{M}))$ that is not in ${\rm up}(T)$ for T maximal for $\preceq$ with $T \prec T_{\rm out}(\mathcal{M})$ and ${\rm status}(T) = {\rm loc}$. Then S is again the connected component, after removing ${\rm up}(T_{\rm out}(\mathcal{M}))$ of the union of such elements. we then recover the maximal elements of $\mathcal{F}_{\rm cont}$ as the connected components, after removing ${\rm up}(T_{\rm out}(\mathcal{M}))$, of the maximal T's with ${\rm status}(T) = {\rm loc}$ and $T \prec T_{\rm out}(\mathcal{M})$ that have an external leg in S.

The inductive continuation of this procedure inside each of these maximal elements of $\mathcal{F}_{\rm cont}$ allows us to reconstruct $\mathcal{F}_{\rm cont}$ entierly from the knowledge of $\mathfrak{m}$ and the structure of $\mathcal{M}$. Therefore we denote this unique $\mathcal{F}_{\rm cont}$ by $\mathcal{F}_{\rm cont}(\mathfrak{m})$ and pose $\mathcal{F}(\mathfrak{m}) \stackrel{\rm def}{=} \mathcal{F}(\mathfrak{m}, \mathcal{F}_{\rm cont}(\mathfrak{m}))$.

Finally

$$\psi(\mathcal{M}) = \sum_{\mathfrak{m} \text{ Mayer graph}} (-1)^{\#(\mathcal{F}(\mathfrak{m}))} \tag{IV.98}$$

i.e. we have reduced the Mayer coefficients to a sum of factors ± 1 indexed by ordinary graphs with no loops, whose links $\{s_1, s_2\}$ are such that Y_{s_1} and Y_{s_2} overlap. This is the analog of Rota's theorem in our context. This is what the Mayer expansion is all about in constructive theory.

IV.4.2 The bounds

This section will be a bit sketchy, for the bounds on the polymer amplitudes, in a slightly different setting, were explained in great detail in [AR2]. Besides, the control of the flow of the effective constants by extracting first orders of perturbation theory does not differ much from the well understood perturbative treatment since

our expansion is here to control the remainder terms. For the study of the flow we refer to [GK, FMRS3, R1, BDH2].

In our context we need simply an ansatz like

$$0 \leq \lambda_i \leq \lambda \tag{IV.99}$$

$$0 < \lambda_i^{-1} \leq \lambda^{-1} + \mathcal{O}(1)(\log M^i) \tag{IV.100}$$

$$|\delta\mu_i| \leq \mathcal{O}(1)\lambda_i M^{-2i} \tag{IV.101}$$

$$|\delta\epsilon_i| \leq \mathcal{O}(1)(\lambda_i)^2 \tag{IV.102}$$

where $\lambda_i \stackrel{\text{def}}{=} \lambda^i_{4,(0,0,0,0)}$, $\mu_i \stackrel{\text{def}}{=} \lambda^i_{2,(0,0)}$, $\epsilon_i = \lambda^i_{2(\gamma,\gamma)}$ for any multi-index γ with $|\gamma| = 1$, $\delta\lambda_i \stackrel{\text{def}}{=} \lambda_i - \lambda_{i+1}$, $\delta\mu_i \stackrel{\text{def}}{=} \mu_i - \mu_{i+1}$ and $\delta\epsilon_i \stackrel{\text{def}}{=} \epsilon_i - \epsilon_{i+1}$. To prove such a statement one must bound the expansion (IV.29) for the variations $\delta\lambda_i$, $\delta\mu_i$ and $\delta\epsilon_i$ successively for all values of i starting with $i = 0$. This bound is done in a way similar to that for the expansion (IV.81).

Note that our definition of the coefficients $\mathcal{C}(\mathcal{M})$ verifies the translation invariance and face symmetry properties needed at the early stages of our expansion scheme. Besides, it is clear that the only effective constants that are nonzero are the ones listed above. Indeed one can prove that by induction on i. First, all $\lambda^i_{e,(\gamma_1,\ldots,\gamma_e)}$ with e odd vanish since the induction hypothesis compels each polymer amplitude, from the high frequencies with an odd number of fields, to vanish by the $\phi \leftrightarrow -\phi$ symmetry. There remains the $\phi\partial_\mu\phi$, $\partial_\mu\phi\partial_\nu\phi$ and $\phi\partial_\mu\partial_\nu\phi$ terms. The $\phi\partial_\mu\partial_\nu\phi$ terms do not flow because of the systematic integration by part trick and the introduction of the FAMC counterterms. For the $\phi\partial_\mu\phi$ and $\partial_\mu\phi\partial_\nu\phi$ terms with $\mu \neq \nu$, a space reflection in the direction μ shows that the counterterms of this kind cancel each other, so that there remains the $\partial_\mu\phi\partial_\mu\phi$ (μ is not summed over here). Furthermore, the latter, for different values of μ, have equal effective constants by space rotational symmetry with respect to rotations that exchange the coordinate axes. Indeed the size of Λ is the same, i.e. L, in every direction.

The bound on a polymer amplitude $\mathcal{A}(Y, \mathcal{J})$ is done by the method of [AR2] with only a few minor changes. In proposition 2 of [AR2] the only difference is in getting small factors from the interaction for type II2.2.a cubes, in a large field block. For this we have to compare the ϕ^4 and the ϕ^2 terms. The $\partial\phi\partial\phi$ term is positive thanks to the ϵ protection incorporated in section IV.2 and to the ansatz (IV.102). Therefore it does not spoil the picture. We now have to get a lower bound on the expression

$$\int_{\tilde{\Delta}} dx \ \left(\lambda_{i_B(\tilde{\Delta})}\phi^B(x)^4 + \mu_{i_B(\tilde{\Delta})}\phi^B(x)^2\right) \tag{IV.103}$$

appearing in the interaction. Besides we know that

$$\int_{\tilde{\Delta}} dx \ \lambda_{i_B(\tilde{\Delta})}\phi^B(x)^4 \geq K_{13}\lambda^{-(1+\epsilon_1)}_{i_B(\tilde{\Delta})} \quad . \tag{IV.104}$$

But by the Cauchy-Schwartz inequality

$$\int_{\tilde{\Delta}} dx \ \left(\lambda_{i_B(\tilde{\Delta})}\phi^B(x)^4 + \mu_{i_B(\tilde{\Delta})}\phi^B(x)^2\right) \tag{IV.105}$$

$$\geq \int_{\tilde{\Delta}} dx \ \lambda_{i_B(\tilde{\Delta})}\phi^B(x)^4 - |\mu_{i_B(\tilde{\Delta})}|.|\tilde{\Delta}|^{\frac{1}{2}}.\left(\int_{\tilde{\Delta}} dx \ \lambda_{i_B(\tilde{\Delta})}\phi^B(x)^4\right)^{\frac{1}{2}} \tag{IV.106}$$

$$\geq \lambda_i A - |\mu_i| M^{2(i-1)} A^{\frac{1}{2}} \tag{IV.107}$$

with $i = i_B(\tilde{\Delta})$ and $A = \int_{\tilde{\Delta}} dx \ \lambda_{i_B(\tilde{\Delta})}\phi^B(x)^4 \geq K_{13}\lambda_i^{-(1+\epsilon_1)}$. Since we are considering the critical theory obtained by solving $\mu_N = 0$ i.e. fixing the effective mass at the last scale to be zero, we have $\mu_i = \sum_{j\geq i}\delta\mu_i$ and thus by the ansatz $|\mu| \leq \mathcal{O}(1)\lambda_i M^{-2i}$. Therefore for some constant K

$$\lambda_i A - |\mu_i| M^{2(i-1)} A^{\frac{1}{2}} \geq \lambda_i (A - K.A^{\frac{1}{2}}) \geq \frac{1}{2} K_{13}\lambda_i^{-\epsilon_1} \tag{IV.108}$$

if we take λ_i small enough for

$$\frac{1}{2} K_{13}^{\frac{1}{2}} \lambda_i^{-\frac{1+\epsilon_1}{2}} \geq K \tag{IV.109}$$

to hold. This is obtained by merely taking the initial λ small since the study of the flow of effective constants shows easily that asymptotically $\lambda_i \sim \text{Cst} \times \frac{\lambda}{i}$. Finally we obtain from the exponential of the interaction the small factor $\exp(-\frac{1}{2}K_{13}\lambda_i^{-\epsilon_1})$.

Another modification is in the domination of low momentum fields. One has to dominate $\partial\phi(x)$ fields. This can be bounded by a $(\frac{1}{|\underline{\Delta}|}\int_{\underline{\Delta}}\phi(x)^4 dx)^{\frac{1}{4}}$ term plus a fluctuation containing double gradients by the same trick as in lemma 19 of [AR2].

Indeed if $\underline{\Delta}$ is the averaging cube, we write $\partial\phi(x) = \int dz \ \partial\phi(x)\delta_{\underline{\Delta}}(z)$ where $\delta_{\underline{\Delta}}$ is a scaled smooth function with an integral equal to 1, defined as in lemma 19 of [AR2], i.e.

$$\delta_{\underline{\Delta}} \stackrel{\text{def}}{=} M^{-4i(\underline{\Delta})}\delta_0((y-\zeta)M^{-i(\underline{\Delta})}) \tag{IV.110}$$

where ζ is a corner of $\underline{\Delta}$ and δ_0 is a fixed function. Then we interpolate $\partial\phi$ between x and z. The fluctuation term gives double-gradients and behaves well with respect to local Gaussian factorials. The equal position term is bounded thanks to an integration by parts and a Holder inequality:

$$\left|\int dz \ \partial\phi(z)\delta_{\underline{\Delta}}(z)\right| = \left|\int dz \ \phi(z)\partial\delta_{\underline{\Delta}}(z)\right| \tag{IV.111}$$

$$\leq \left(\int dz \ \phi(z)^4\right)^{\frac{1}{4}} \times \left(\int dz \ |\partial\delta_{\underline{\Delta}}(z)|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq \left(\int dz \ \phi(z)^4\right)^{\frac{1}{4}} \times M^{-2i(\underline{\Delta})} \ . \tag{IV.112}$$

One recovers finally a scaling $M^{-2i(\underline{\Delta})}$ for this $\partial\phi$ factor, where $\underline{\Delta}$ is the the averaging cube.

Finally remark that when dominating a low momentum leg ϕ_i of a vertex localized at scale j we get a loss $\lambda_i^{\frac{1}{4}}.\lambda_j^{-\frac{1}{4}} \sim (\frac{j}{i})^{-\frac{1}{4}}$ that can be beaten by extracting a fraction of the power counting factor $M^{-\epsilon(j-i)}$ of the leg ϕ_j.

We now explain how to sum over $(Y_1, \ldots, Y_k)$ the polymer sequence of $\mathcal{M}$ and the Mayer graph $\mathfrak{m}$ when we restrict Y_1 to contain a fixed box $\Delta_{\rm org}$. This condition comes from the $\frac{1}{|\Lambda|}$ factor in the expansion of the free energy per volume. It is simply a condition to break the translation invariance degeneracy.

We suppose we have a decay factor $\mathcal{T}_\epsilon(Y_1, \ldots, Y_k)$ defined in the following way

$$\mathcal{T}_\epsilon(Y_1, \ldots, Y_k) \stackrel{\rm def}{=} \prod_{i=1}^{k} \left(u(\lambda)^{\#(Y_i)}.\mathcal{T}_\epsilon(y_i)\right) \times \mathcal{T}_\epsilon^{\rm ext}(Y_1, \ldots, Y_k) \tag{IV.113}$$

where $u(\lambda)$ is a small factor i.e. a function of λ such that $u(\lambda) \to 0$ if $\lambda \to 0$, and $\mathcal{T}_\epsilon(Y_i)$ is the decay factor defined in [AR2]. To define $\mathcal{T}_\epsilon^{\rm ext}(Y_1, \ldots, Y_k)$, we consider successively all $i \leq i_{\max}(Y_1, \ldots, Y_k) \stackrel{\rm def}{=} \max\{i | \mathcal{D}_i \cap (Y_1 \cup \ldots \cup Y_k) \neq \emptyset\}$ and define the projection on scale i, i.e. by the previously introduced function pr_i, of the polymers that are contained in $\mathcal{D}_{\leq i}$. Two cubes of ths projection are connected if they are the projections of two cubes belonging to the same polymer. We now consider the connected components of the projection on scale i and if there is no cube, of a polymer that is not contained in $\mathcal{D}_{\leq i}$, that projects by pr_i on some component, we say that the latter is a dangerous component at scale i. Now by definition

$$\mathcal{T}_\epsilon^{\rm ext}(Y_1, \ldots, Y_k) \stackrel{\rm def}{=} \prod_{i < i_{\max}(Y_1, \ldots, Y_k)} \; \prod_{X \text{ dangerous component at scale } i} M^{-(4+\epsilon)} \quad . \tag{IV.114}$$

Besides we suppose that $Y_1, \ldots, Y_k$ are such that at scale $i_{\max}(Y_1, \ldots, Y_k)$ there is only one component for the projection.

We now show that provided λ is small enough

$$\sum_{k \geq 1} \sum_{(Y_1, \ldots, Y_k)} \sum_{\mathfrak{m}} \mathcal{T}_\epsilon(Y_1, \ldots, Y_k) < +\infty \quad , \tag{IV.115}$$

uniformly in N and Λ. We proceed by constructing a rooted tree connecting the objects we are considering and show that for every link on the tree there is a corresponding summable decay factor.

Given k, $\mathfrak{m}$ and $Y_1, \ldots, Y_k$, we consider $(k+1)$ copies $\mathcal{D}(1), \ldots \mathcal{D}(k), \mathcal{D}(\mathrm{ext})$ of $\mathcal{D}(\Lambda, N)$, the set of boxes containing the polymers. Each Y_i is viewed as a subset of $\mathcal{D}(i)$. We consider

$$\Omega_i \stackrel{\rm def}{=} \{\Delta \in \mathcal{D}(i) | i(\Delta) \leq i_{\max}(Y_i), \exists \Delta' \in Y_i, \Delta' \subset \Delta\} \tag{IV.116}$$

and also

$$\Omega_{\rm ext} \stackrel{\rm def}{=} \{\Delta \in \mathcal{D}({\rm ext}) | i(\Delta) \leq i_{\max}(Y_1, \ldots, Y_k), \exists \Delta' \in (Y_1 \cup \ldots \cup Y_k), \Delta' \subset \Delta\} \quad . \tag{IV.117}$$

We know from [AR2] (proposition 1) that using a fraction $(1 + M^{-j} d_2^{\Lambda}(\Delta, \Delta'))^{-\frac{\epsilon}{3}}$ of the decay of the horizontal links $\{\Delta, \Delta'\}$ in an admissible forest graph in $\mathcal{T}_\epsilon(Y_i)$, as well as a fraction $M^{-\frac{\epsilon}{3}}$ of the decay factor associated to a component at some scale of Y_i we can collect a factor $M^{-\kappa} < 1$ for each cube of Ω_i. Let G_i be the graph of Ω_i whose links are all pairs $\{\Delta_1, \Delta_2\}$ in Ω_i such that Δ_1 is just above Δ_2. We thus have a factor $M^{-\kappa}$ for each link in G_i. Let $\mathcal{F}_i$ be an admissible forest graph of horizontal links for Y_i. We consider also $G_{\rm ext}$ the graph of $\Omega_{\rm ext}$ made by all pairs $\{\Delta_1, \Delta_2\}$ in $\Omega_{\rm ext}$ with Δ_1 just above Δ_2. We have to show that for any such link we can extract from $\mathcal{T}_\epsilon(Y_1, \ldots, Y_k)$ a factor $M^{-\kappa}$.

For any link $\{\Delta, \Delta'\}$ in $\mathcal{F}_{\rm ext} \stackrel{\rm def}{=} \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$ viewed as a graph on $\Omega_{\rm ext}$ we can extract from $\mathcal{T}_\epsilon(Y_1) \ldots \mathcal{T}_\epsilon(Y_k)$ a factor $(1 + M^{-j} d_2^{\Lambda}(\Delta, \Delta'))^{-\frac{\epsilon}{3}}$ with $j = i(\Delta) = i(\Delta')$. Let $j < i_{\max}(Y_1, \ldots, Y_k)$ and X be a connected component of $\{\Delta \in \Omega_{\rm ext} | i(\Delta) \leq j\}$ defined by the graph obtained from $G_{\rm ext} \cup \mathcal{F}_{\rm ext}$ by keeping the links $\{\Delta, \Delta'\}$ with $i(\Delta) \leq j$, and $i(\Delta) \leq j$. We have to extract a factor $M^{-\frac{\epsilon}{3}}$ for each such component.

Let Y_i a polymer such that $X \cap Y_i \neq \emptyset$. then the component at scale j of some $\Delta \in X \cap Y_j$ for Y_i equipped with $\mathcal{F}_i$ is entierly contained in X. If $j < i_{\max}(Y_i)$, then $\mathcal{T}_\epsilon(Y_i)$ still contains a factor $M^{-(4+\frac{2\epsilon}{3})}$ associated to this component. We can extract from it $M^{-\frac{\epsilon}{3}}$ and associate it to X. If for all Y_i with $Y_i \cap X \neq \emptyset$ we have $J \geq i_{\max}(Y_i)$, then X corresponds to a dangerous component at scale j as defined when introducing $\mathcal{T}_\epsilon(Y_1) \ldots \mathcal{T}_\epsilon(Y_k)$. Therefore we have from the latter a factor $M^{-(4+\epsilon)}$ for X from which we can extract an $M^{-\frac{\epsilon}{3}}$.

We have shown thus that to every component X there can be associated a factor $M^{-\frac{\epsilon}{3}}$, besides for every link $\{\Delta, \Delta'\}$ of $\mathcal{F}_{\rm ext}$ we have a factor $(1 + M^{-j} d_2^{\Lambda}(\Delta, \Delta'))^{-\frac{\epsilon}{3}}$. Now from the proof of Proposition 1 in [AR2] we can use these two kinds of factors to obtain an $M^{-\kappa} < 1$ for any cube of $\Omega_{\rm ext}$ and thus for every link of $G_{\rm ext}$.

Let Ω be the disjoint union of $\Omega_1, \ldots, \Omega_k, \Omega_{\rm ext}$. Let $G \stackrel{\rm def}{=} G_1 \cup \ldots \cup G_k \cup G_{\rm ext}$ viewed as a graph on Ω. Recall that $\mathfrak{m}$ is a graph without loops of $\{1, \ldots, k\}$ such that $\{i_1, i_2\} \in \mathfrak{m}$ implies $Y_{i_1} \cap Y_{i_2} \neq \emptyset$. Let $\overline{\mathfrak{m}}$ be also a graph without loops of overlap relations obtained from $\mathfrak{m}$ by adding as much links as necessary such that connected components of $\overline{\mathfrak{m}}$ in $\{1, \ldots, k\}$ are the connected components obtained by taking into account all the overlap relations between $Y_1, \ldots, Y_k$. Let also $\mathfrak{t}_{\mathfrak{m}}$ be a graph on Ω obtained by choosing for every link $\{i_1, i_2\}$ of $\overline{\mathfrak{m}}$ a realization "$\{\Delta_1, \Delta_2\}$" where $\Delta_1 \in \Omega_{i_1}$, $\Delta_2 \in \Omega_{i_2}$ and Δ_1, Δ_2 are in fact the same box Δ in $\mathcal{D}(\Lambda, N)$ with $\Delta \in Y_1 \cap Y_2$.

Let now for each Ω_i, $\mathfrak{t}_i$ be a tree such that $\mathcal{F}_i \subset \mathfrak{t}_i \subset G_i \cup \mathcal{F}_i$, which connects Ω_i and such that for any component X of Ω_i at some scale i, the links of $\mathfrak{t}_i$ inside

X form a tree connecting X. The construction of such a $\mathfrak{t}_i$ is easy, we refer to the proof of Proposition 1 in [AR2] for the details.

We now have to construct on Ω a connecting tree $\mathfrak{t}$ with $\mathfrak{t}_{\mathfrak{m}} \cup \mathfrak{t}_1 \cup \ldots \cup \mathfrak{t}_k \subset \mathfrak{t}$ and $\mathfrak{t} \subset \mathfrak{t}_{\mathfrak{m}} \cup \mathfrak{t}_1 \cup \ldots \cup \mathfrak{t}_k \cup G_{\rm ext} \cup H \stackrel{\rm def}{=} \mathfrak{G}$ where H is the set of all links $\{\Delta_1, \Delta_2\}$ in Ω with $\Delta_1 \in \mathcal{D}(\text{ext})$, Δ_2 is in some Y_i viewed inside Ω_i and Δ_1, Δ_2 are in fact the same box of $\mathcal{D}$.

Let $j \leq i_{\max}(Y_1, \ldots, Y_k)$, we define the graph

$$\mathfrak{G}_j \stackrel{\rm def}{=} \mathfrak{t}_{\mathfrak{m}} \cup \mathfrak{t}_1 \cup \ldots \cup \mathfrak{t}_k \cup \{\{\Delta, \Delta'\} \in G_{\rm ext} \cup H | i(\Delta) \leq j, i(\Delta') \leq j\} \quad . \tag{IV.118}$$

If we pose $\mathfrak{G}_{-1} \stackrel{\rm def}{=} \mathfrak{t}_{\mathfrak{m}} \cup \mathfrak{t}_1 \cup \ldots \cup \mathfrak{t}_k$, we obviously have

$$\mathfrak{G}_{-1} \subset \mathfrak{G}_0 \subset \mathfrak{G}_1 \subset \cup \ldots \cup \mathfrak{G}_{i_{\max}(Y_1,\ldots,Y_k)} = \mathfrak{G}. \tag{IV.119}$$

We construct $\mathfrak{t}$ in such a way that for any j, $-1 \leq j \leq i_{\max}(Y_1, \ldots, Y_k)$, $\mathfrak{t} \cap \mathfrak{G}_j$ has the same connected components as $\mathfrak{G}_j$. We start with the forest graph $\mathfrak{G}_{-1}$. Suppose $-1 \leq j < i_{\max}(Y_1, \ldots, Y_k)$ and we have constructed $\mathfrak{t} \cap \mathfrak{G}_j$. We just have to add to $\mathfrak{t}$ links from $\mathfrak{G}_{j+1} \backslash \mathfrak{G}_j$ in such a way as we connect together the components of $\mathfrak{G}_{j+1}$, without making loops. At the end of the induction we have constructed $\mathfrak{t} \cap G_{i_{\max}(Y_1,\ldots,Y_k)}$ i.e. $\mathfrak{t}$ completely.

Since we supposed when defining $\mathcal{T}_\epsilon(Y_1, \ldots, Y_k)$ that there is only one component of the projection at scale $i_{\max}(Y_1, \ldots, Y_k)$, it is easy to see that $\mathfrak{G}$ connects Ω and so does $\mathfrak{t}$. this shows also that $\mathfrak{t}$ is indeed a tree i.e. is connected.

Now let us consider $\Delta_{\rm org} \in Y_1$ viewed in $\Omega_1 \subset \Omega$ to be the *root* of $\mathfrak{t}$. Each link of $\mathfrak{t}$ can be *oriented towards the root* i.e. for each $l = \{\Delta_1, \Delta_2\}$ we define $f(l)$ (like father) to be the element of l which is closest to the root, in $\mathfrak{t}$, and $s(l)$ (like son) to be the remotest, as we did in [AR2]. The direction of l is from $s(l)$ to $f(l)$. The summation procedure on the location of the boxes in Ω begins with the leafs of $\mathfrak{t}$ and progresses towards the root $\Delta_{\rm org}$, i.e. for each l we sum over $s(l)$ knowing $f(l)$. Taking into account this orientation of the links we have to extract a summable decay factor for each of them.

If $l \in \mathfrak{t}$ is a link in H or in $\mathfrak{t}_{\mathfrak{m}}$ the factor 1 will be enough. If $l = \{\Delta_1, \Delta_2\}$ in $\mathfrak{t}$ is a link of $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$, we have still a decay factor $(1 + M^{-i(\Delta_1)} d_2^{\Lambda}(\Delta_1, \Delta_2))^{-(4+\frac{\epsilon}{3})}$ which is enough. Now the remaining links of $\mathfrak{t}$ are in G i.e. are links between vertical nearest neighbors. There are two cases to consider.

First, l *goes upward*, i.e. we sum the big cube knowing the small one. In that case we dispose of a factor $M^{-\kappa}$ which is enough.

The second case is more delicate, i.e. when the link l *goes downward* that is when we sum the small cube inside the big one. One has to extract for such links a factor $M^{-(4+\frac{\epsilon}{3})}$. This is what we explain now.

Let l be such a downward link and let $j = i(s(l))$. The first case we consider is when $s(l) \in \Omega_i$, $1 \leq i \leq k$. Let X be the connected component of $s(l)$ for the graph $\mathfrak{G}_{i,j}$ obtained from $\mathcal{F}_i \cup \mathfrak{G}_i$ by keeping only the links $\{\Delta, \Delta'\}$ with $i(\Delta) \leq j$ and $i(\Delta') \leq j$. By construction of $\mathfrak{t}_i$, as done in [AR2], $\mathfrak{t}_i \cap \mathfrak{G}_{i,j}$ is a tree connecting X. Clearly $f(l) \notin X$ i.e. l goes out of X. Indeed X corresponds to the support of a subtree of $\mathfrak{t}$, and in a big tree that is oriented towards the root there can be at most one link going out of a subtree. This is intuitively true, but one can find a formal proof of this statement in [AR2] (lemma 13).

Necessarily we have $j < i_{\max}(Y_i)$ and since then X corresponds to a component of Y_i at scale j, $\mathcal{T}_\epsilon(Y_i)$ provides for it a factor $M^{-(4+\frac{\epsilon}{3})}$ we can now allocate to the link l. We have actually checked that we do not risk allocating it twice. Each time we perform this allocation we say that the component X was *used*.

The second case is when $s(l) \in \Omega_{\text{ext}}$. Let $\mathcal{X}$ be the connected component of $s(l)$ for the graph $\mathfrak{G}_j$. Since the cubes Δ of Ω_{ext} with $i(\Delta) > j$ are isolated by $\mathfrak{G}_j$ and $f(l)$ is one of them, l goes out of $\mathcal{X}$.

Now let us consider all components X, of all polymer, at scale j, as defined in the first case, that intersect $\mathcal{X}$. Clearly, since $\mathfrak{G}_{i,j} \subset \mathfrak{G}_j$ for each i, $1 \leq i \leq k$, these components X are entierly contained in $\mathcal{X}$. Suppose there is such a component X in $\mathcal{X}$ that was not used for the allocation in the first case . We can allocate the corresponding factor $M^{-(4+\frac{\epsilon}{3})}$ in $\mathcal{T}_\epsilon(Y_1) \dots \mathcal{T}_\epsilon(Y_k)$ to the link l. It is easy to see that we cannot allocate in this way the same factor twice. Indeed $\mathcal{X}$ corresponds to a subtree $\mathfrak{t}(\mathcal{X})$ of $\mathfrak{t}$. Note that the orientation of $\mathfrak{t}(\mathcal{X})$ induced from $\mathfrak{t}$ is the same as the "internal" orientation of $\mathfrak{t}(\mathcal{X})$ when we choose $s(l)$ as a root.

Finally suppose that there are no unused components X in $\mathcal{X}$. Let $\mathcal{X}^0_{\text{ext}} \stackrel{\text{def}}{=} \mathcal{X} \cap \Omega_{\text{ext}}$,

$$\mathcal{X}^0_{\text{down}} \stackrel{\text{def}}{=} \{\Delta \in \mathcal{X} \cap (\Omega_1 \cup \dots \cup \Omega_k) | i(\Delta) > j\} \tag{IV.120}$$

and

$$\mathcal{X}^0_{\text{up}} \stackrel{\text{def}}{=} \{\Delta \in \mathcal{X} \cap (\Omega_1 \cup \dots \cup \Omega_k) | i(\Delta) \leq j\} \tag{IV.121}$$

so that $\mathcal{X}$ is the disjoint union of $\mathcal{X}^0_{\text{ext}}$, $\mathcal{X}^0_{\text{down}}$ and $\mathcal{X}^0_{\text{up}}$. The above mentioned components X form a partition of $\mathcal{X}^0_{\text{up}}$. Since the root of $\mathfrak{t}(\mathcal{X})$ is $s(l) \in \mathcal{X}_{\text{ext}}$ and $\mathcal{X}^0_{\text{up}} \neq \emptyset$, we have links going out of $\mathcal{X}^0_{\text{up}} \cup \mathcal{X}^0_{\text{down}}$. Such a link can only be one of H linking a cube of some component $X \subset \mathcal{X}^0_{\text{up}}$ to the corresponding cube in $\mathcal{X}^0_{\text{ext}}$. Since there is at most one link going out of X, there cannot be any downward link going out of X. However we supposed that there are no unused components in $\mathcal{X}$ this means that $X \subset \Omega_i$ does not have the corresponding factor $M^{-(4+\frac{\epsilon}{3})}$ coming from $\mathcal{T}_\epsilon(Y_i)$. The only possibility for that is that $X = \Omega_i$ i.e. $j \geq i_{\max}(Y_i)$.

Let $\mathcal{X}^1_{\text{ext}}$ be the union of $\mathcal{X}^0_{\text{ext}}$ with these components X, and $\mathcal{X}^1_{\text{up}} \stackrel{\text{def}}{=} \mathcal{X}^0_{\text{up}} \setminus (\mathcal{X}^0_{\text{up}} \cap \mathcal{X}^1_{\text{ext}})$ and $\mathcal{X}^1_{\text{down}} \stackrel{\text{def}}{=} \mathcal{X}^0_{\text{down}}$. Now either $\mathcal{X}^1_{\text{up}} = \emptyset$, in that case $\mathcal{X}^1_{\text{down}} = \emptyset$ also and all polymers Y_i intersecting $\mathcal{X}$ satisfy $j \geq i_{\max}(Y_i)$ and therefore $\mathcal{X}$ corresponds to a

dangerous component at scale j of $Y_1, \ldots, Y_k$ and $\mathcal{T}_\epsilon^{\rm ext}(Y_1, \ldots, Y_k)$ provides for it a factor $M^{-(4+\frac{\epsilon}{3})}$; or $\mathcal{X}^1_{\rm up} \neq \emptyset$.

In that case the situation is the same as before. That is there exist links going out of $\mathcal{X}^1_{\rm up} \cup \mathcal{X}^1_{\rm down}$ and they are either overlap links from H or from $\mathfrak{t}_{\mathfrak{m}}$.

They must then go out from components X in $\mathcal{X}^1_{\rm up}$. As before since there is no factor $M^{-(4+\frac{\epsilon}{3})}$ coming from $\mathcal{T}_\epsilon(Y_1) \ldots \mathcal{T}_\epsilon(Y_k)$ for these components, the only possibility for such a $X \subset \Omega_i$ is $X = \Omega_i$ i.e. $j \geq i(Y_i)$. We repeat the operation by constructing $\mathcal{X}^2_{\rm ext}$ as the union of those components with $\mathcal{X}^1_{\rm ext}$, and by constructing $\mathcal{X}^2_{\rm up}$ as well as $\mathcal{X}^2_{\rm down}$. We construct thus a sequence $(\mathcal{X}^\alpha_{\rm ext}, \mathcal{X}^\alpha_{\rm up}, \mathcal{X}^\alpha_{\rm down})_{0 \leq \alpha \leq \beta}$ until $\mathcal{X}^\beta_{\rm up} = \emptyset$, which entails $\mathcal{X}^\alpha_{\rm ext} = \mathcal{X}$.

We prove in this way that every polymer Y_i that intersects $\mathcal{X}$ satisfies $j \geq i_{\max}(Y_i)$ and so $\mathcal{T}_\epsilon^{\rm ext}(Y_1, \ldots, Y_k)$ provides a factor $M^{-(4+\frac{\epsilon}{3})}$ for the dangerous component $\mathcal{X}$. Again this allocation cannot be made twice since there can be at most one link l of $\mathfrak{t}$ going out of $\mathcal{X}$, by construction of $\mathfrak{t}$.

We have finally shown that to every oriented link of $\mathfrak{t}$ there is a summable decay factor associated.

Now we perform the same pruning procedure to $\mathfrak{t}$ as in [AR2]. We erase all cubes Δ in Ω that do not belong to a polymer, have coordinance at most two in $\mathfrak{t}$ and such that the links of $\mathfrak{t}$ connected to them either go both downward or both upward. Finally we cut off all the leafs of the resulting tree that terminate by a cube Δ that is not a box of some Y_i viewed in Ω_i.

As was shown in [AR2] the cubes that remain in some Ω_i, and are not in Y_i are not more than $\#(Y_i) - 1$. Likewise the total number of the remaining cubes in $\Omega_{\rm ext}$ is $k_{\rm ext} \leq 2\#(Y_1 \cup \ldots \cup Y_k) - 1$. Let $\hat{\mathfrak{t}}$ be the tree obtained from $\mathfrak{t}$ after the pruning procedure and $\hat{\mathfrak{t}}_i$ the subtree obtained by pruning $\mathfrak{t}_i$ for $1 \leq i \leq k$. Let us introduce a sum over numberings of the remaining cubes in $\Omega_{\rm ext}$ and divide by $\frac{1}{k_{\rm ext}!}$.

Let us now contract each $\hat{\mathfrak{t}}_i$ to a point, and view $\hat{\mathfrak{t}}$ as a tree $\tilde{\mathfrak{t}}$ on the disjoint union $\{1, \ldots, k_{\rm ext}\} + \{1, \ldots, k\}$. $\tilde{\mathfrak{t}}$ is indeed a tree since each $\hat{\mathfrak{t}}_i$ is a subtree of $\hat{\mathfrak{t}}$. The root of $\tilde{\mathfrak{t}}$ is the label 1 in $\{1, \ldots, k\}$. For each i, $2 \leq i \leq k$ we denote by Δ_i the cube from which a link of $\hat{\mathfrak{t}}$ goes out from Ω_i. By convention $\Delta_1 \stackrel{\rm def}{=} \Delta_{\rm org}$. Each Y_i will be summed over with constraint that $\Delta_i \in Y_i$ thanks to the tree decay corresponding to $\hat{\mathfrak{t}}_i$. The proof of property 1 in [AR2] shows that this sum is convergent and bounded by a constant independent of Λ and N.

We now have since $u(\lambda)$ can be taken arbitrarily small

$$\sum_{k \geq 1} \sum_{(Y_1, \ldots, Y_k)} \sum_{\mathfrak{m}} \mathcal{T}_\epsilon(Y_1, \ldots, Y_k) \leq \sum_{k \geq 1} \sum_{k_{\rm ext} \geq 0} \frac{u(\lambda)^{\frac{k + k_{\rm ext}}{6}}}{k! k_{\rm ext}!}$$

$$\sum_{\tilde{\mathfrak{t}}} \sum_{\mathfrak{m}} \sum_{\Delta_2, \ldots, \Delta_k} D(\tilde{\mathfrak{t}}) \times \prod_{i=1}^{k} D(\Delta_i, Y_i) \qquad \text{(IV.122)}$$

where $D(\tilde{\mathfrak{t}})$ is the product of the decay factors of the links in the oriented tree $\tilde{\mathfrak{t}}$, and $D(\Delta_i, Y_i)$ is $u(\lambda)^{\frac{\#(Y_i)}{3}}$ times the supremum over the trees $\hat{\mathfrak{t}}_i$, oriented by the choice of root Δ_i, of the product of the decay factors of their links. From [AR2] we know that

$$\sum_{Y_i | \Delta_i \in Y_i} D(\Delta_i, Y_i).e^{\#(Y_i)} \tag{IV.123}$$

is bounded by a constant K, provided λ is small.

We now sum over the coordinances $d_1, \ldots, d_{k_{\rm ext}}$, $d'_1, \ldots, d'_k$ of the vertices of $\tilde{\mathfrak{t}}$ and then over $\tilde{\mathfrak{t}}$. From Cayley's theorem we know that the number of such trees, once the coordinances are fixed is

$$\frac{(k + k_{\rm ext} - 2)!}{(d_1 - 1)! \ldots (d_{k_{\rm ext}} - 1)!)(d'_1 - 1)! \ldots (d'_k - 1)!} \quad . \tag{IV.124}$$

Now as explained in [R1] in the section on the convergence of the Mayer expansion, we sum over the Y_i and the cubes $\Delta_1, \ldots, \Delta_k$ as well as the location of the numbered cubes $\Delta^1, \ldots, \Delta^{k_{\rm ext}}$ of $\Omega_{\rm ext}$, starting from the leafs of $\tilde{\mathfrak{t}}$ and progressing towards the root.

Due to the pruning procedure, the leafs of $\tilde{\mathfrak{t}}$ are polymer indices in $\{1, \ldots, k\}$. Let i be such a leaf, the sum over Y_i knowing Δ_i costs a factor K. Suppose we have performed all the summations on the vertices in the branch of $\tilde{\mathfrak{t}}$ attached to some vertex i. A few cases must be considered.

If $i \in \{1, \ldots, k\}$ then the sum over each index j whose ancestor in $\tilde{\mathfrak{t}}$ is i involves a choice of a box in Y_i i.e. we recover a factor $|Y_i|^{d_i - 1}$. But

$$\frac{1}{(d_i - 1)!} \sum_{Y_i | \Delta_i \in Y_i} D(\Delta_i, Y_i) |Y_i|^{d_i - 1} \leq \sum_{Y_i | \Delta_i \in Y_i} D(\Delta_i, Y_i) e^{\#(Y_i)} \leq K \tag{IV.125}$$

If $i \in \{1 \ldots, k_{\rm ext}\}$, then the sum over the vertices j in the nearest descendence of i do not involve a choice of cube of reference since it must be Δ^i. But we have now to perform the sum over Δ^i knowing the ancestor i' of i. Either $i \in \{1, \ldots, k\}$ in that case we have a factor $|Y_{i'}|$ that will be taken into account, as in the first case, when summing over $Y_{i'}$, or $i' \in \{1, \ldots, k_{\rm ext}\}$ and then $D(\tilde{\mathfrak{t}})$ provides a summable decay factor to sum over Δ^i knowing $\Delta^{i'}$ and the result is a constant K'. Therefore the right hand side of (IV.122) is bounded by

$$S \stackrel{\rm def}{=} \sum_{k, k_{\rm ext}} \frac{u(\lambda)^{\frac{k + k_{\rm ext}}{6}}}{k! k_{\rm ext}!} \sum_{\substack{d_1, \ldots, d_{k_{\rm ext}} \\ d'_1, \ldots, d'_k}} (k + k_{\rm ext} - 2)! \times K^k \times K'^{k_{\rm ext}} \times 2^{k-1} \quad . \tag{IV.126}$$

Note that $\mathfrak{m}$ being a subtree of $\tilde{\mathfrak{t}}$ between elements of $\{1, \ldots, k\}$ and there can be at most $k - 1$ such links in $\tilde{\mathfrak{t}}$, the sum over $\mathfrak{m}$ inside $\tilde{\mathfrak{t}}$ costs a factor 2^{k-1}.

Now since

$$d_1 + \cdots + d_{k_{\rm ext}} + d'_1 + \cdots + d'_k = 2(k + k_{\rm ext}) - 2 \tag{IV.127}$$

we have

$$u(\lambda)^{\frac{k+k_{\rm ext}}{12}} \leq u(\lambda)^{\frac{d_1}{24}} \dots u(\lambda)^{\frac{d_{k_{\rm ext}}}{24}} . u(\lambda)^{\frac{d'_1}{24}} \dots u(\lambda)^{\frac{d'_k}{24}} \tag{IV.128}$$

and therefore

$$S \leq \sum_{k,k_{\rm ext}} u(\lambda)^{\frac{k+k_{\rm ext}}{6}} \times 4^{k+k_{\rm ext}} \times K^k \times K'^{k_{\rm ext}} \times \left(\frac{u(\lambda)^{\frac{1}{24}}}{1 - u(\lambda)^{\frac{1}{24}}} \right)^{k+k_{\rm ext}} \leq K'' \tag{IV.129}$$

for some constant K'' independent of Λ and N provided λ and thus $u(\lambda)$ is small enough.

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